1 Projective space

The ordinary plane $\mathbf{R}^2$ has the problem that parallel lines do not meet. So, introduce *'ideal'* points at infinity, and an ideal line containing those points, to get a *projective plane* in which two points always determine a unique line, and two lines always determine a unique point.

A model of this projective plane is found in $\mathbf{R}^3$ by taking the lines through the origin as projective points. The original plane can be taken to be the plane $Z = 1$. Each line $\langle(x, y, z)\rangle$ with $z \neq 0$ hits this plane in a unique point, namely $(x/z, y/z, 1)$, the 'old point' $(x/z, y/z)$. The lines $\langle(x, y, 0)\rangle$ correspond to the ideal points, the plane $Z = 0$ to the ideal line (the 'line at infinity').

Now generalize to arbitrary fields and dimensions.

Let V be a vector space over a field F. The associated projective space PV is the lattice of subspaces of V (where incidence is inclusion). Projective points, lines, planes, ..., i-spaces, ... are d-dimensional linear subspaces of V with $d = 1$, 2, 3, ..., $i+1$, ... We shall always use vector space dimension, never projective dimension. Points will be projective points - elements of V are called vectors.

When V is coordinatized, a projective point is a 1-space $\{(\lambda x_1, ..., \lambda x_n) | \lambda \in F\}$ and we can say that this point has coordinates $(x_1, ..., x_n)$ provided we agree that $(x_1, ..., x_n)$ and $(\alpha x_1, ..., \alpha x_n)$ denote the same point (for $\alpha \in F$, $\alpha \neq 0$). Such coordinates are called *homogeneous coordinates*.

We would like to describe sets of points by an equation $f(X_1, ..., X_n) = 0$. For this to be meaningful, one wishes $f(x_1, ..., x_n) = 0$ iff $f(\alpha x_1, ..., \alpha x_n) = 0$ (for $\alpha \in F$, $\alpha \neq 0$). This leads us to consider *homogeneous polynomials*, that is, polynomials in which the total degree of each term is the same (say d). These satisfy our restraint, since $f(\alpha x_1, ..., \alpha x_n) = \alpha^d f(x_1, ..., x_n)$.

2 Chevalley-Warning

Below we shall need that a quadric on a projective plane over a finite field F has at least one point. This is a very special case of the following.

Theorem 2.1 *Given homogeneous polynomials $f_1, ..., f_m$ over F (of respective degrees $d_1, ..., d_m$) in the variables $X_1, ..., X_n$, where $\sum d_i < n$. Then there is a common zero $(x_1, ..., x_n) \neq (0, ..., 0)$.*

Proof: Consider
$$\sum_{x \in F^n} \prod_i (1 - f_i(x)^{q-1}).$$
It is the total number of solutions (considered as an element of F) of $f_1 = ... = f_m = 0$. But $\sum_{z \in F} z^j = 0$ unless j is a nonzero multiple of $q - 1$, so in the expansion of the above sum only terms of degree at least $n(q-1)$ contribute, but there are no such terms. Consequently, the number of solutions is 0 (mod char$F$). Since $(0, ..., 0)$ is a solution, there are also other solutions. □

It follows that a quadric in dimension at least 3 is nonempty.

3 Quadrics over finite fields

A homogeneous polynomial of degree one (a linear form) defines a hyperplane. A homogeneous polynomial of degree two (a quadratic form) defines a quadric. We want to classify quadrics in projective spaces over finite fields.

3.1 Quadratic forms

Let us first describe quadratic forms in a coordinate-free way. A *quadratic form* is a map $Q : V \to F$ such that $Q(\lambda u) = \lambda^2 Q(u)$ (for $\lambda \in F$, $u \in V$) and having the property that B defined by $B(u, v) = Q(u + v) - Q(u) - Q(v)$ is a bilinear form. We have $B(u, u) = 2Q(u)$, so if $\text{char} F \neq 2$ then we can retrieve Q from B. (But if $\text{char} F = 2$, then many quadratic forms yield the same bilinear form.)

3.2 Nondegeneracy

Two points $\langle u \rangle$ and $\langle v \rangle$ are called *orthogonal* if $B(u, v) = 0$. If U is a subspace of V, then $U^\perp$ denotes the subspace of V consisting of the vectors orthogonal to each element of U. The bilinear form B is called *nondegenerate* if $V^\perp = 0$. The quadratic form Q is called *nondegenerate* if Q does not vanish on any point in $V^\perp$.

3.3 Orthogonal direct sums

We write $V = U \perp W$ if $V = U \oplus W$ (that is, $V = U + W$ and $U \cap W = 0$) and $B(u, w) = 0$ for all $u \in U$, $w \in W$. If this is the case, then Q is determined by its restrictions to U and W. Indeed, $Q(u + w) = Q(u) + Q(w)$. And conversely, given arbitrary quadratic forms Q_U on U and Q_W on W, the formula $Q(u+w) = Q_U(u)+Q_W(w)$ defines a quadratic form Q on $U \oplus W$, and for this form $V = U \perp W$.

3.4 Classification

Thus, it suffices to classify pairs (V, Q) where V cannot be decomposed as orthogonal direct sum. In particular, we may suppose that $V^\perp = 0$ (unless $V = V^\perp$ is of dimension 1 and we have $Q = 0$).

Now suppose $\dim V \geq 3$. Then by Chevalley-Warning there is a point $\langle u \rangle$ with $Q(u) = 0$. Since $V^\perp = 0$ there is a vector w with $B(u, w) = \lambda \neq 0$. With $v = w + \beta u$ we find $Q(v) = Q(w) + \beta\lambda$, so for $\beta = -Q(w)/\lambda$ we have $Q(v) = 0$. After scaling v we may assume that $B(u, v) = 1$. Thus, we have found a line $H = \langle u \rangle + \langle v \rangle$ with quadratic form defined by $Q(u) = Q(v) = 0$ and $B(u, v) = 1$. (With u and v as unit vectors, the form is X_1X_2.) Such a line is called a *hyperbolic line*. Clearly, H is nondegenerate, and we have $V = H \perp H^\perp$ where also $H^\perp$ is nondegenerate. In this way we can peel off hyperbolic lines until the dimension n of V has become less than 3.

If $n = 0$ then PV has no points and $Q = 0$.

If $n = 1$ then PV has a single point and the number of nonisomorphic possibilities equals $1+|F^*/F^{*2}|$. Indeed, if the single point is $\langle u \rangle$, then changing u by a constant α changes $Q(u)$ by α^2, so either $Q(u) = 0$ (u is *singular*) or $Q(u)$ lies in one of the cosets of F^{*2} in F^*. For finite fields this means that either char$F = 2$ and there are just two possibilities: $Q(u)$ vanishes or not, or char$F$ is odd, and there are three possibilities: $Q(u)$ is zero, a square or a nonsquare.

If $n = 2$ and V is not a hyperbolic line, then Q never vanishes on PV, and the line is called an *elliptic line*.

Thus: V is the orthogonal direct sum of its radical $V^\perp$, a number of hyperbolic lines, and perhaps a single point or an elliptic line.

By change of coordinates $H \perp P$ can be transformed into $H \perp P'$ where $Q(P)$ consists of squares, and $Q(P')$ of nonsquares. So, for odd $n > 1$ the type of the point occurring in the decomposition on a nondegenerate quadratic form is irrelevant. Also, any two elliptic lines are equivalent. The parity of the number of elliptic lines occurring in a decomposition is an invariant (found from the determinant of Q).

Thus: over a finite field we have in even dimension $n > 0$ precisely two types of nondegenerate quadrics: *hyperbolic quadrics* that are an orthogonal direct sum of hyperbolic lines, and *elliptic quadrics*, that are not. In odd dimension $n > 1$ there is only one type of nondegenerate quadric.

The maximal totally singular subspaces (subspaces where Q vanishes identically) have dimension m for a hyperbolic quadric in $n = 2m$ dimensions, and $m - 1$ for an elliptic quadric in $n = 2m$ dimensions, and m for a nondegenerate quadric in $n = 2m + 1$ dimensions.

The (vector space) dimension of the maximal totally singular subspaces is called the *Witt index*.