# 1 Blowup

## 1.1 Example

The curve $y^2 = x^3 + x^2$ has a singularity (double point) at the origin, but it feels as if nothing is wrong there: if one traces the curve then the curve appears nice and smooth - it is just that we visit the same point twice.

So, with an additional coordinate $t$ (time) the singularity is gone. If we study curves up to birational equivalence, then the new coordinate should be a rational function of the old coordinates, and $t = y/x$ works: the two passes through the origin have $t = 1$ and $t = -1$.

Now the pair of equations $y^2 = x^3 + x^2$, $y = tx$ in $(x, y, t)$-space defines the union of a lifted version of the planar $y^2 = x^3 + x^2$, and the vertical line $x = y = 0$. Substitute $y = tx$ in the first equation and divide by $x^2$ to get the pair $t^2 = x + 1$, $y = tx$ defining a smooth space curve, birationally equivalent to the original curve. The maps are $(x, y, t) \mapsto (x, y)$ and $(x, y) \mapsto (x, y, y/x)$, well-defined and inverses of each other outside the origin.

## 1.2 Blowing up a point in affine space

This can be done more generally. Given affine space with coordinates $(x_1, ..., x_n)$, introduce new projective coordinates $(t_1, ..., t_n)$ restricted by $x_i t_j = x_j t_i$. This defines a subvariety of $\mathbf{A}^n \times \mathbf{P}^{n-1}$. The map $\phi$ given by $(x_1, ..., x_n, t_1, ..., t_n) \mapsto (x_1, ..., x_n)$ is 1-1 outside the inverse image of the origin, with inverse $(x_1, ..., x_n) \mapsto (x_1, ..., x_n, x_1, ..., x_n)$. (At the origin this inverse is not defined since a projective point cannot have all coordinates zero.) The inverse image of the origin is the entire projective space $\{0\} \times \mathbf{P}^{n-1}$.

This variety is irreducible, since the inverse image of (the irreducible variety) $\mathbf{A}^n \setminus \{0\}$ is dense.

## 1.3 Blowing up a point in a subvariety of affine space

Now if $Y$ is a subvariety of $\mathbf{A}^n$, then the blow-up of $Y$ at the origin is by definition the closure in $\mathbf{A}^n \times \mathbf{P}^{n-1}$ of $\phi^{-1}(Y \setminus \{0\})$.

(The example we started with is the special case where $n = 2$ and instead of projective coordinates $(s, t)$ with $sy = xt$ we used the affine part with $s = 1$.)

## 1.4 Blowing up a subvariety of a variety

If $X$ is a variety with coordinates $x_1, ..., x_m$ and $Y$ is a subvariety with ideal $I(Y)$ generated by functions $f_1, ..., f_n$, then we have a map

$$(x_1, ..., x_m) \mapsto (x_1, ..., x_m, f_1(x_1, ..., x_m), ..., f_n(x_1, ..., x_m))$$

from $X \setminus Y$ into $\mathbf{A}^m \times \mathbf{P}^{n-1}$. Now the blowup of $X$ along $Y$ is defined as the closure in $\mathbf{A}^m \times \mathbf{P}^{n-1}$ of the image of this map.

## 1.5 Resolution of singularities

It turns out to be possible to show that an arbitrary algebraic variety is birationally equivalent to a smooth one (i.e., one without singularities), and that nonsingular model can be reached by a series of blowups.

**Theorem 1.1** (Hironaka) *Let $X$ be any variety over a field of characteristic zero. Then there exists a variety $Y$ and a regular map $\phi : Y \to X$ that is a birational equivalence.*

The analog of this theorem in char $p$ is open, but has been proved for curves and surfaces.

# 2 Quadratic transformations

Blowup has the disadvantage that the dimension goes up. For the case of curves in the plane one can find a reasonably good model while staying in the plane.

The transformation used is the map $\phi$ given (in projective coordinates) by $(X, Y, Z) \mapsto (YZ, XZ, XY)$. It is defined on $\mathbf{P}^2 \backslash \{P, P', P''\}$ where $P = (0, 0, 1)$, $P' = (0, 1, 0)$ and $P'' = (1, 0, 0)$. Let $U$ be the open set $\mathbf{P}^2 \setminus V(XYZ)$. Then $\phi$ is 1-1 on $U$, and its own inverse. Hence $\phi$ is a birational isomorphism. It maps the line $Z = 0$ to the point $P$, so $\phi^{-1}$ looks like a blowup: it blows up the point $P$ into a line.

**Theorem 2.1** *Any curve in the plane can be brought into a form where the only singularities are ordinary multiple points.*

(An ordinary multiple point is a multiple point without multiple tangent.)

We give a full proof.

Let $C$ be an irreducible curve, defined by the equation $F(X, Y, Z) = 0$. Let the points $P, P', P''$ have multiplicities $m, m', m''$ on $C$, respectively. Then the image $C^\phi$ under $\phi$ is given by the equation $F^\phi(X, Y, Z) = 0$, where

$$F^\phi(X, Y, Z) = F(YZ, XZ, XY)/(X^{m''} Y^{m'} Z^m).$$

Indeed, by definition of multiplicity, if $P$ has multiplicity $m$ on $C$, and $F$ has degree $n$, then $F(X, Y, Z) = F_m(X, Y)Z^{n-m} + ... + F_n(X, Y)$, with $F_j(X, Y)$ homogeneous of degree $j$. Now

$$F(YZ, XZ, XY) = F_m(YZ, XZ)(XY)^{n-m} + ... + F_n(YZ, XZ)$$

has precisely $m$ factors $Z$. This shows that $F^\phi$ is a polynomial, homogeneous of degree $2n - m - m' - m''$, and this polynomial is irreducible (since $F$ is).

The point $P$ has multiplicity $n - m' - m''$ on $C^\phi$:
Indeed, expanding $F^\phi$ in powers of $Z$ we find

$$F^\phi(X, Y, Z) = F_m(Y, X)X^{n-m-m''} Y^{n-m-m'} + ... + F_n(Y, X)X^{-m''} Y^{-m'} Z^{n-m}$$

with highest $Z$-exponent in the last term.

We have $(F^\phi)^\phi = F$.
Indeed, $F^{\phi\phi}(X, Y, Z) = F^\phi(YZ, XZ, XY)/(X^{n-m-m'} Y^{n-m-m''} Z^{n-m'-m''}) = F(XYZX, XYZY, XYZZ)/(X^{n-m-m'} Y^{n-m-m''} Z^{n-m'-m''} (XY)^m (XZ)^{m'} (YZ)^{m''}) = F(X, Y, Z)$.

That settles the relation between $F$ and $F^\phi$. The goal is to choose coordinates in such a way that things improve around $P$ and do not get worse elsewhere.

The quadratic transformation is 1-1 outside the triangle $V(XYZ)$, so nothing happens there. In order to control what happens near the corners of the triangle we must choose the triangle in *good position.* That is, by definition, choose them such that none of the three lines $X = 0$, $Y = 0$, $Z = 0$ is tangent to $C$ at $P$, $P'$ or $P''$.

If the triangle is in good position with respect to $C$, then it is also in good position with respect to $C^\phi$.

Indeed, the line $Z = 0$ is tangent to $C^\phi$ at $P' = (0, 1, 0)$ iff

$$I(P', C^\phi \cap Z) > m_{P'}(C^\phi),$$

i.e.,

$$I(P', F_m(Y, X)X^{n-m-m''}Y^{n-m-m'} \cap Z) > n - m - m'',$$

i.e.,

$$I(P', F_m(Y, X) \cap Z) > 0,$$

i.e., $F_m(1, 0) = 0$, i.e., $Y = 0$ is a tangent to $C$ at $P = (0, 0, 1)$, which it is not.

For a triangle in good position we have control over the 'blowup' of $P$: Suppose $C^\phi$ meets the line $Z = 0$ in the points $P_1, ..., P_s$ distinct from $P', P''$ (and possibly also in $P', P''$). Then for each $i$ the multiplicity of $P_i$ as a point of $C^\phi$ is bounded by $m_{P_i} \leq I(P_i, C^\phi \cap Z)$, and the sum of these intersection numbers equals $m$.

Indeed $\sum_i I(P_i, C^\phi \cap Z) = \sum_i I(P_i, F_m(Y, X) \cap Z) = m$. (The powers of $X$ and $Y$ in $I(P_i, F_m(Y, X)X^{n-m-m''}Y^{n-m-m'} \cap Z)$ disappear because $P_i$ is distinct from $P'$ and $P''$ and hence does not lie on $X = 0$ or $Y = 0$. By the assumption of good position, the points $P'$ and $P''$ do not occur in the sum. Since the field is algebraically closed and $F_m$ has degree $m$, we find $m$ roots.)

So far, everything was symmetric in $P, P', P''$. Introduce asymmetry now. The triangle is in *excellent position* if it is in good position, and moreover $Z = 0$ intersects $C$ in $n$ distinct points different from $P', P''$ and the lines $X = 0$ and $Y = 0$ intersect $C$ in $n-m$ distinct points different from $P, P', P''$. (That is: we know that $P$ is an $m$-fold point. All other intersections of $C$ with the triangle must be simple, and $C$ must not meet the other two corners.)

It is clear that given $P$ we can change coordinates in such a way that the triangle is in excellent position (since the field is infinite).

Now $C^\phi$ has the following multiple points:

(i) Outside the triangle, $C$ and $C^\phi$ are isomorphic, and the same multiplicities occur.

(ii) $P, P', P''$ are ordinary multiple points on $C^\phi$ with multiplicities $n, n-m, n-m$, respectively. (This follows from excellent position.)

(iii) $C^\phi$ has no points on $X = 0$ or $Y = 0$ other than $P, P', P''$. (The sum of the intersection multiplicities equals the multiplicity of $P'$ or $P''$ on $C$, that is, 0.)

Have we made progress? One arbitrary singularity was turned into three ordinary singularities and a number of points on the line $Z = 0$, that may be bad themselves. Let us introduce a parameter to measure progress.

Let the irreducible plane projective curve $C$ have degree $n$ and multiple points of multiplicity $m_i = m_{P_i}(C)$. Put

$$g^*(C) = (n-1)(n-2)/2 - \sum m_i(m_i-1)/2.$$

Then $g^* \geq 0$.

We prove this below. But let us assume $g^* \geq 0$ for the moment, and compute. We have $2g^*(C^\phi) = (2n-m-1)(2n-m-2) - \sum m_i(m_i-1) + m(m-1) - n(n-1) - 2(n-m)(n-m-1) - \sigma = 2g^*(C) - \sigma$, where $\sigma = \sum m_i(m_i-1)$ summed over the points of $C^\phi$ on $Z = 0$. Thus, either $g^*$ decreases, or it stays the same, but then $\sigma = 0$, and hence all points of $C^\phi$ on $Z = 0$ are simple points. This shows that after finitely many steps all multiple points will be ordinary multiple points.

Remains to show that $g^* \geq 0$. This will follow from Bezout.

First apply Bezout to $F$ and its derivative $F' = (\partial/\partial Z)F$. If $F$ has multiple points $P_i$ with multiplicities $m_i$, then these same points have multiplicity at least $m_i - 1$ for $F'$, and Bezout tells us that $n(n-1) \geq \sum m_i(m_i-1)$. (Needed: $F$ and $F'$ do not have a common factor, that is, $F$ does not have a squared factor.)

Next improve this inequality by picking a different curve to intersect $F$ with. Take a polynomial $G$, homogeneous of degree $n-1$ (there are $n(n+1)/2$ coefficients to choose), and require for each $i$ that $P_i$ has multiplicity at least $m_i - 1$ on $G$ (this imposes $m_i(m_i-1)/2$ linear conditions). We know that $n(n-1)/2 \geq \sum m_i(m_i-1)/2$, so this leaves at least $n$ degrees of freedom, and we can require $G$ to pass through $t := n(n+1)/2 - 1 - \sum m_i(m_i-1)/2$ further points of $F$, and still have a nonzero solution. Now Bezout tells us that $n(n-1) \geq \sum m_i(m_i-1) + t$, i.e., $g^* \geq 0$. (Needed: $F$ and $G$ do not have a common factor. Since $G$ is unknown of degree $n-1$ we need that $F$ is irreducible.)