Coherent configurations

A. E. Brouwer

Abstract

Definition and a few examples.

0.1 Relations

A *coherent configuration* is a finite set X (of *points*) together with a collection $\mathcal{R} = \{R_i \mid i \in I\}$ of nonempty binary relations on X, satisfying the following four conditions:

(i) $\mathcal{R}$ is a partition of $X \times X$, that is, any ordered pair of points is in a unique relation R_i.

(ii) There is a subset H of the index set I such that $\{R_h \mid h \in H\}$ is a partition of the diagonal $\{(x,x) \mid x \in X\}$.

(iii) For each R_i, its converse $\{(y,x) \mid (x,y) \in R_i\}$ is also one of the relations in $\mathcal{R}$, say, $R_{i'}$.

(iv) For $i,j,k \in I$ and $(x,y) \in R_k$, the number of $z \in X$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$ is a constant p^k_{ij} that does not depend on the choice of x,y.

Coherent configurations were introduced by Higman in order to 'do group theory without groups', see example (ii) below.

The number $|I|$ of relations is called the *rank* of the coherent configuration.

From (ii) we get a partition of X into sets X_h ($h \in H$) called *fibers*, defined by $R_h = \{(x,x) \mid x \in X_h\}$ for $h \in H$. It follows from (iv) that for any $i \in I$ we have $R_i \subseteq X_s \times X_t$ for certain fibers X_s, X_t. Consequently, any subset H_0 of H determines a sub-cc with point set $\bigcup_{h \in H_0} X_h$.

0.2 Matrices

Let A_i be the *adjacency matrix* of R_i, defined by $(A_i)_{xy} = 1$ if $(x,y) \in R_i$ and $(A_i)_{xy} = 0$ otherwise. In terms of the A_i the above definition becomes: The A_i ($i \in I$) are nonzero 0-1 matrices with rows and columns indexed by X such that

(i) $\sum_{i \in I} A_i = J$, where J is the all-1 matrix.

(ii) $\sum_{h \in H} A_h = I$, where I is the identity matrix.

(iii) $(A_i)^\top = A_{i'}$ for $i \in I$.

(iv) $A_i A_j = \sum_k p^k_{ij} A_k$.

0.3 The adjacency algebra

By (iv) above, the matrices A_i form the basis for an $|I|$-dimensional algebra $\mathcal{A}$ (over an arbitrary field F) called the (F-)*adjacency algebra*. The algebra $\mathcal{A}$ is closed for both matrix multiplication and Hadamard (entrywise) multiplication.

If $F = \mathbb{C}$, the algebra $\mathcal{A}$ is closed for taking the conjugate transpose $M^* = \overline{M}^\top$ of a matrix M, and $\mathcal{A}$ is a C*-algebra.

By (iii), the real or complex adjacency algebra $\mathcal{A}$ is semisimple. (Its radical contains together with a matrix M also $M^\top M$ or $M^* M$, but these matrices can be diagonalized and are nilpotent so are zero. Hence rad $\mathcal{A} = 0$.) It follows that $\mathcal{A}$ is the direct sum of simple two-sided ideals: $\mathcal{A} = \sum_k \mathcal{I}_k$ where the $\mathcal{I}_k$ annihilate each other. Take $F = \mathbb{C}$, then each $\mathcal{I}_k$ is isomorphic to a matrix algebra $M_{n_k}(\mathbb{C})$ of matrices of order n_k over $\mathbb{C}$. We see that $\dim \mathcal{A} = \sum_k n_k^2$.

Since $M_n(\mathbb{C})$ has a basis of matrices e_{ij} (with a 1 entry at the (i,j)-position and 0 elsewhere) that multiply according to $e_{ij}e_{kl} = e_{il}$ if $j = k$, and $e_{ij}e_{kl} = 0$ otherwise, we can find matrices E_h in $\mathcal{A}$, n_k^2 in each ideal $\mathcal{I}_k$, where the E_h in $\mathcal{I}_k \cong M_{n_k}(\mathbb{C})$ multiply like the e_{ij} (taken in some order). Let $E_{h'} = E_h^*$.

Now $\mathcal{A}$ has two bases, namely that of the A_i and that of the E_j, and we can express each basis in terms of the other. Define constants P_{ij} and Q_{ij} by $A_i = \sum P_{ji} E_j$ and $E_j = \frac{1}{|X|} \sum Q_{ij} A_i$. (The order of indices, and the factor $\frac{1}{|X|}$ are traditional.)

Consider the bilinear form on $\mathcal{A}$ given by $(M,N) = \operatorname{tr} M^* N$. This form is nondegenerate, and the A_i are mutually orthogonal, and the E_j are, too. It follows that if M is an arbitrary matrix of order $v = |X|$, the projection πM of M on $\mathcal{A}$ is given by $\pi M = \sum_i \frac{(A_i, M)}{(A_i, A_i)} A_i = \sum_j \frac{(E_j, M)}{(E_j, E_j)} E_j$. For $M = yx^*$ of rank 1, we find

$$\sum_i \frac{x^* A_{i'} y}{(A_i, A_i)} A_i = \sum_j \frac{x^* E_{j'} y}{(E_j, E_j)} E_j$$

for all $x, y \in V$. (This is a form of Roos' identity.)

For $M = xx^*$, which is symmetric and positive semidefinite, the projection πM is also symmetric and positive semidefinite, and we find Hobart's result that $\sum_i \frac{1}{(A_i, A_i)} (x^* A_i x) A_i$ is psd. (This is a form of Delsarte's LP bound.)

One has $(A_i, A_i) = k_i |X_g|$ when $R_i \subseteq X_g \times X_h$ and A_i has row sums k_i $(= p_{ii'}^g)$. One has $(E_i, E_i) = m_i$ when the standard module $V = \mathbb{C}^X$ contains m_i copies of the $\mathcal{I}_k$ corresponding to E_i. Computing (A_i, E_j) in two ways, we find $(A_i, E_j) = \frac{1}{|X|} Q_{ij} (A_i, A_i) = \frac{1}{|X|} Q_{ij} k_i |X_g|$ and $(A_i, E_j) = \overline{P_{ji}}\, (E_j, E_j) = \overline{P_{ji}}\, m_j$.

0.4 Association schemes

A coherent configuration (cc) is called *homogeneous* if I_0 consists of a single element, which then is written 0.

A cc is called *commutative* when $A_i A_j = A_j A_i$ for all $i, j \in I$ (or, equivalently, when $p_{ij}^k = p_{ji}^k$ for all $i, j, k \in I$).

A cc is called *symmetric* when $A_i^\top = A_i$ for all $i \in I$ (or, equivalently, when $i' = i$ for all $i \in I$).

Every symmetric cc is commutative. Every nonempty commutative cc is homogeneous. Every homogeneous cc of rank at most 5 is commutative. A nonempty symmetric cc is called a (symmetric) *association scheme.* (For most authors association schemes are by definition symmetric, but Delsarte used the term more generally for homogeneous commutative coherent configurations, so one often adds 'symmetric' to avoid ambiguity.) For association schemes the adjacency algebra is also known as the *Bose-Mesner algebra.*

0.5 Examples

(i) The empty coherent configuration has $X = \emptyset$. It has $\mathcal{R} = \emptyset$ and is symmetric but not homogeneous, and not an association scheme.

(ii) Let G be a permutation group acting on the set X. Then X together with the partition $\mathcal{R}$ of $X \times X$ into orbits of G (acting on $X \times X$ via $g(x, y) = (gx, gy)$) is a coherent configuration. A coherent configuration obtained in this way is called *Schurian*. This coherent configuration is homogeneous when G has precisely 1 orbit, i.e., when X is nonempty and G is transitive on X.

(iii) Let $\mathcal{S}$ be an arbitrary collection of relations on X. Then there is a unique coarsest $\mathcal{R}$ such that $(X, \mathcal{R})$ is a coherent configuration and all elements of $\mathcal{S}$ are unions of relations in $\mathcal{R}$. (Proof: Consider the algebra generated by the matrices I, J and the 0-1 adjacency matrices for the elements of $\mathcal{S}$ under taking ordinary and Hadamard products. It has a basis of minimal idempotents for Hadamard multiplication, and this yields $\mathcal{R}$.) In particular, a graph Γ (directed or not) determines a coherent configuration $\mathrm{cc}(\Gamma)$.

(iiia) The discrete coherent configuration on a set X is the unique coherent configuration in which all singletons are fibers. It has rank $|X|^2$ and is the finest coherent configuration on X.

(iiib) The indiscrete coherent configuration on a set X is the coherent configuration determined by the complete graph on X. It is the coarsest coherent configuration on X. If $|X| > 1$ it has rank 2.

(iv) Up to isomorphism, there are four coherent configurations on 3 points. The indiscrete one is $\mathrm{cc}(K_3)$ and is an association scheme of rank 2. The coherent configuration determined by the cyclically directed triangle is commutative and homogeneous and has rank 3. The path P_3 of length 2 determines $\mathrm{cc}(P_3)$ of rank 5. The discrete cc has rank 9.

Let us give very explicit details.

a) The indiscrete scheme $\mathrm{cc}(K_3)$ has adjacency algebra $\mathcal{A} = \langle I, J - I \rangle$ with idempotents $\frac{1}{3}J$ and $I - \frac{1}{3}J$ and $P = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$.

b) The cyclically directed triangle has adjacency algebra $\mathcal{A} = \langle I, A, A^2 \rangle$, where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ with idempotents $\frac{1}{3}(I + \rho A + \rho^2 A^2)$ for $\rho = 1,\ \omega,\ \omega^2$, where ω is a primitive cube root of unity, and $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \overline{\omega} & \omega \\ 1 & \omega & \overline{\omega} \end{bmatrix}$.

c) The scheme $\mathrm{cc}(P_3)$ has adjacency algebra

$$\mathcal{A} = \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle.$$

Here $\mathcal{A}$ is the sum of the three minimal left ideals of dimensions 2, 2, 1 generated by the idempotents $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$. The sum of the first two is a two-sided ideal of dimension 4 isomorphic to $M_2(\mathbb{C})$, the third is isomorphic to $M_1(\mathbb{C})$. The basis of the E_j can be taken as

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \right\}.$$

Now $P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$. The isomorphism of $\mathcal{A}$ with $M_2(\mathbb{C}) \oplus M_1(\mathbb{C})$ sends the adjacency matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ of the path P_3 to $\sqrt{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}$, so that A has eigenvalues $\sqrt{2}, 0, -\sqrt{2}$.

d) The discrete cc on 3 points has adjacency algebra $\mathcal{A} = M_3(\mathbb{C})$, and both the 9 matrices A_i and the 9 matrices E_j are the 9 matrices e_{ij} with a single 1.