

# 1 Dimension

Intuitively we all know what the dimension of something is supposed to be. A point is 0-dimensional, a line 1-dimensional, a plane 2-dimensional, the space we live in 3-dimensional. The dimension gives the number of free parameters, the number of coordinates needed to specify a point.

Also a curved line has dimension 1, and a curved surface like that of a sphere in 3-space has dimension 2.

This intuitive notion has been made precise in many branches of mathematics, and we have dimension concepts in linear algebra, in ring theory, in topology, in convexity theory, in algebraic geometry, etc.

Sometimes several of these concepts apply to the same object, and there is no guarantee at all that the same value is found for each definition. It is a game in topology to construct topological spaces that have dimensions 0, 1, 2 for three different concepts of dimension.

A dimension does not have to be an integer. In cases where one has a concept of area or volume or content or so, one may say that an object has dimension  $d$  if, when it is enlarged by a factor  $c$ , the volume increases by a factor  $c^d$ . Now non-integral ('fractal') dimensions occur.

# 2 Dimension in linear algebra

The dimension of a vector space  $V$  over a field  $k$  is the cardinality of a basis. One can show that all bases have the same cardinality.

Note that the dimension depends on the field  $k$ . The field of complex numbers has dimension 1 over itself but dimension 2 over the field of real numbers.

# 3 Dimension in topology

There are many dimension concepts in topology. Below the two most important ones,  $\dim X$  and  $\text{Ind } X$ . For metric spaces  $X$  one has  $\dim X = \text{Ind } X$ , and one has monotonicity: if  $X \subseteq Y$ , then  $\dim X \leq \dim Y$ . For non-metrizable spaces this is no longer true.

## 3.1 Covering dimension

A collection  $\mathcal{U}$  of subsets in a topological space  $X$  is called an *open cover* of  $X$  when each  $U \in \mathcal{U}$  is open, and the union of  $\mathcal{U}$  is  $X$ .

An open cover  $\mathcal{V}$  of  $X$  is called a *refinement* of the open cover  $\mathcal{U}$  if for each  $V \in \mathcal{V}$  there is a  $U \in \mathcal{U}$  such that  $V \subseteq U$ .

Let  $\mathcal{U}$  be a collection of sets in a topological space  $X$ . The *order* of  $\mathcal{U}$  at a point  $p \in X$  is by definition the number of elements of  $\mathcal{U}$  that contain  $p$ . The topological space  $X$  is said to have *covering dimension* (or Lebesgue dimension) at most  $d$ , if for every finite open cover of  $X$  there is a refinement that has order at most  $d + 1$  at each point. It is said to have covering dimension  $d$  if the dimension is at most  $d$  but not at most  $d - 1$ . This dimension is noted  $\dim X$ . The empty space has covering dimension  $-1$ .

### 3.2 Strong inductive dimension

Let the *boundary* of a subset  $A$  of a topological space  $X$  be  $\partial A = \bar{A} \setminus A^\circ$  the difference between its closure and its interior. This is a closed set.

Strong inductive dimension  $\text{Ind } X$  of a topological space  $X$  is defined inductively by  $\text{Ind } \emptyset = -1$ , and if for any two disjoint closed sets  $C$  and  $D$  of  $X$  there exists an open set  $U$  containing  $C$  and disjoint from  $D$  with  $\text{Ind } \partial U \leq d - 1$  then  $\text{Ind } X \leq d$ . Again,  $\text{Ind } X = d$  if  $\text{Ind } X \leq d$  but not  $\text{Ind } X \leq d - 1$ .

## 4 Dimension in ring theory

Again, there are many inequivalent definitions.

The *Krull dimension of a ring*  $R$  is the largest number  $d$  such that there exists a strictly increasing chain of prime ideals  $P_0 \subset P_1 \subset \cdots \subset P_d$  in  $R$ . (Note that  $R$  is not a prime ideal.)

## 5 Dimension in algebraic geometry

(Assume that our ground field  $k$  is algebraically closed.) We give a handful of equivalent definitions of dimension.

### 5.1 Chains of irreducible closed subvarieties

An algebraic variety  $X$  is said to have dimension  $d$  when  $d$  is the largest integer such that there is a strictly increasing chain  $X_0 \subset X_1 \subset \cdots \subset X_d$  of closed irreducible subsets of  $X$ . (Note that the empty set is not irreducible.)

Because of the Nullstellensatz, there is a 1-1 correspondence between algebraic sets  $Y$  in  $\mathbf{A}^m = k^m$  and radical ideals  $I$  in the polynomial ring  $k[x_1, \dots, x_m]$ .

The set  $Y$  will be irreducible iff the corresponding ideal  $I(Y)$  is prime. (And now  $k[x_1, \dots, x_m]/I(Y)$  is a domain that is finitely generated as  $k$ -algebra.)

Thus, a chain  $X_0 \subset X_1 \subset \cdots \subset X_d$  of closed irreducible subsets of  $X$  corresponds to a chain  $I(X_0) \supset I(X_1) \supset \cdots \supset I(X_d)$  of prime ideals in  $k[X]$ , and the dimension of  $X$  equals the Krull dimension of  $k[X]$ .

The empty set has dimension  $-1$ . Finite sets have dimension 0, and conversely, any set of dimension 0 is finite.

### 5.2 Transcendence degree

An irreducible algebraic variety  $X$  is said to have dimension  $d$  when  $d$  is the transcendence degree of its function field  $k(X)$  over  $k$ .

(If a field  $L$  is an extension field of the field  $K$ , then  $L$  has transcendence degree  $n$  over  $K$  if there is a set of  $n$  elements  $x_1, \dots, x_n$  in  $L$  that are algebraically independent over  $K$ : no nonzero polynomial  $f(x_1, \dots, x_n)$  with coefficients in  $K$  vanishes, while  $L$  is an algebraic extension of  $K(x_1, \dots, x_n)$ .)

This concept of dimension coincides with the previous one.

### 5.3 Intersection with linear subspaces

Each time one imposes an additional restriction, the dimension of solutions goes down by one. Now that we know what sets have dimension zero (the finite sets) or  $-1$  (the empty set), we can count how many additional restrictions are needed before we get at a finite set or at the empty set.

An irreducible variety  $X$  in a projective space  $\mathbf{P}^n$  has dimension  $k$  iff the general  $(n - k)$ -space hits  $X$  in a finite set (but larger spaces do not).

An irreducible variety  $X$  in a projective space  $\mathbf{P}^n$  has dimension  $k$  iff the general  $(n - k - 1)$ -space hits  $X$  in the empty set (but larger spaces do not).

(Here the conditions we impose are linear conditions:  $X$  is hit by a series of hyperplanes. The adjective "general" is needed because hitting  $X$  with a hyperplane that contains it does not lower the dimension.)

### 5.4 Finite-to-one dominant maps

Let  $X$  be an irreducible variety, and  $Y$  a variety, and  $\phi : X \rightarrow Y$  a rational map.

(What is that? Roughly speaking, a map of which all coordinate functions are rational functions. But the denominators that occur may be zero at some places, and that means that at some places the value of  $\phi$  may be undefined. Also, different ways of writing the same rational functions may yield zeros at different places - e.g. if  $X$  is the circle defined by  $x^2 + y^2 = z^2$  then  $(y + z)/x = x/(z - y)$ . Thus, a more precise way of defining "rational map" is as equivalence class of regular maps  $\psi : U \rightarrow Y$  where  $U$  is open (and therefore dense) in  $X$ , where such maps are equivalent when they agree on the intersection of their domains.)

The *image* of  $\phi$  is by definition  $\pi_2(\text{Graph}(\phi))$  where  $\text{Graph}(\phi)$  is the subvariety of  $X \times Y$  that is the closure of the subvariety of  $U \times Y$  defined by  $y = \psi(u)$ . (This is independent of the choice of  $\psi$ .)

The map  $\phi$  is called *dominant* if  $Y$  is the image of  $\phi$ . Intuitively this is like "surjective".

An irreducible variety  $X$  has dimension  $k$  iff it has a finite-to-one dominant map to  $\mathbf{P}^k$ .

("Finite-to-one" means that all fibers  $\phi^{-1}(y)$  are finite.)

This definition can be compared with the last one from the previous subsection: if  $X$  is a subvariety of  $\mathbf{P}^n$ , and there is a  $(n - k - 1)$ -space  $S$  that hits  $X$  in the empty set while larger spaces all hit  $X$ , then we can map  $X$  to the  $\mathbf{P}^k$  that is the space of all subspaces of  $\mathbf{P}^n$  containing  $S$ , by mapping each point  $x \in X$  to the subspace spanned by  $x$  and  $S$ . This map is finite-to-one since  $(n - k)$ -spaces hit  $X$  in a finite set, and dominant since none of these  $(n - k)$ -spaces is disjoint from  $X$ .

This definition can also be compared with the one by transcendence degree:

**Theorem 5.1** *The dominant map  $\phi : X \rightarrow Y$  has finite general fibers  $\phi^{-1}(y)$  iff  $k(X)$  is a finite extension of  $k(Y)$  via  $\phi^*$ . In characteristic 0, the general fiber has size  $[k(X) : k(Y)]$ .*