The genus of a plane curve

1 A formula for the genus of a nice plane curve

The genus g of a nonsingular plane curve of degree d equals $\binom{d-1}{2}$. The genus g of a plane curve of degree d with only ordinary multiple points equals

$$g = \binom{d-1}{2} - \sum_P \binom{m(P)}{2}$$

where the sum is over the multiple points P (with multiplicity $m(P)$).

For example, the genus of a quartic curve with a single double point and no other singularities equals $3 - 1 = 2$, and that is not of the form $\binom{d-1}{2}$, so there is no nonsingular plane model of this curve.

Nonsingular models can be found by blowup, but they will in general live in some higherdimensional space.

However, it is always possible, given an irreducible plane curve, to find a plane curve with only ordinary multiple points that is birationally equivalent, so that the above formula gives the genus.

2 Nice

A multiple point is *ordinary* when all tangents at the point are distinct.

The point $P = (0,0)$ is a point of multiplicity m of the curve $F(X,Y) = 0$ when the terms in F of lowest degree have degree m. It is ordinary when there are m distinct tangents, that is, when the sum of the terms of lowest degree factors as a product of m distinct factors.

For example, the curve $X^4 - XY + Y^4 = 0$ has lowest degree part XY of degree 2 with two distinct factors, so the origin is an ordinary double point.

For example, the curve $Y^2 - X^3 = 0$ has lowest degree part Y^2 of degree 2 with two equal factors, so the origin is a double point, but it is a cusp, not an ordinary double point.

3 Quadratic transformation

The map $(x,y,z) \mapsto (yz, xz, xy)$ is its own inverse: applied twice we get the point (x^2yz, y^2xz, z^2xy) which is the same projective point as (x,y,z), assuming that $xyz \neq 0$.

So, applying this map turns a curve into a birationally equivalent one, and if we are lucky or careful, into one with better properties.

Example (failure) For example, the curve $Y^2 = X^3$, homogeneous $Y^2Z = X^3$, becomes $(XZ)^2(XY) = (YZ)^3$, i.e., $X^3 = Y^2Z$, no improvement.

Example (failure) And if we turn the axes a little by sending (X, Y) to $(X+Y, X-Y)$ so that the equation becomes $(X-Y)^2 = (X+Y)^3$, then the above quadratic transformation turns this into $(YZ-XZ)^2XY = (YZ+XZ)^3$, i.e., $XY(Y-X)^2 = (Y+X)^3Z$ which is even worse: the origin is a point with multiplicity 3 with 3 coinciding tangents.

Example (success) But if we send (X, Y, Z) to $(X+Y, X-Y, X+Z)$, then $(X-Y)^2(X+Z) = (X+Y)^3$ is transformed into $(YZ-XZ)^2(YZ+XY) = (YZ+XZ)^3$, i.e. $(Y-X)^2Y = Z(X^2+2XY+5Y^2)$, and the origin is a double point with two distinct tangents $X = (-1 \pm 2i)Y$, and the genus equals $1-1=0$.

(One can see directly that the genus is 0: put $Y = XT$, then the curve becomes $T^2 = X$, nonsingular of degree 2.)

Instead of trying, one can follow an algorithm that always works. In order to improve the situation one fixes coordinate axes in such a way that the origin is a nonordinary multiple point of multiplicity m, and otherwise the axes are in general position: the axes are not tangent to the curve, and the line $Z = 0$ meets the curve in d distinct points that have $xy \neq 0$, and the lines $X = 0$ and $Y = 0$ meet the curve in $d-m$ distinct points other than P. In such a situation, a quadratic transformation improves things, and after finitely many steps we arrive at a nice plane curve: the only singularities are ordinary multiple points.

4 The measure of improvement

For an irreducible plane curve with multiple points P of multiplicity $m(P)$ (where these multiple points need not be ordinary), put $g^* = \binom{d-1}{2} - \sum_P \binom{m(P)}{2}$. Then $g^* \geq 0$ and when not all P are ordinary multiple points, a quadratic transformation can be chosen that either diminishes g^*, or leaves g^* invariant and diminishes the number of nonordinary multiple points, Thus, after finitely many steps all multiple points will be ordinary. (And then $g^* = g$.)

Proposition 1 *We have $g^* \geq 0$ for any irreducible plane curve.*

Proof: First of all, let F be the equation of the curve, of degree d, and let F_X be the derivative of F w.r.t. X. Then F_X is the equation of a curve of degree $d-1$ and since F is irreducible, F and F_X do not have common components, so by Bezout they have at most $d(d-1)$ common points (counting multiplicities). But each point of multiplicity m on $V(F)$ has multiplicity at least $m-1$ on $V(F_X)$, so we see at least $\sum m(P)(m(P)-1)$ common points, and $d(d-1) \geq \sum m(P)(m(P)-1)$. Put $r := \frac{1}{2}(d-1)(d+2) - \sum \frac{1}{2}m(m-1)$ so that $r \geq 0$.

The curves of degree d live in a projective space of dimension $\frac{1}{2}d(d+3)$. (The coordinates are the coefficients of the equation, up to a constant.) Passing through a point is one linear condition on the coordinates. Having multiplicity m at a point gives $\frac{1}{2}m(m+1)$ linear conditions (e.g. at the origin: all coefficients of terms X^iY^j with $i+j < m$ must be zero). Let G be the equation of a curve of degree $d-1$ that has the multiple points P of $V(F)$ with multiplicity at least $m(P)-1$, and has r more points in common with $V(F)$. There is such a curve G, because $\frac{1}{2}(d-1)(d+2) - \sum \frac{1}{2}(m-1)m - r \geq 0$. Now apply Bezout to F and G and find $d(d-1) \geq \sum m(m-1) + r$, i.e., $\frac{1}{2}(d-1)(d-2) \geq \sum \frac{1}{2}m(m-1)$. $\square$

5 Properties of the quadratic transformation

The map $\phi : (x, y, z) \mapsto (yz, xz, xy)$ induces an isomorphism from U onto itself, where $U = \mathbf{P}^2 \setminus V(XYZ)$ is the set of points with three nonzero coordinates.

Let $P = (0, 0, 1)$, $P' = (0, 1, 0)$ and $P'' = (1, 0, 0)$. Then ϕ maps the line $Z = 0$ to the point P, and hence ϕ^{-1} blows up the point P to the line $Z = 0$, i.e., acts as a blowup, but without introducing a new coordinate.

Lemma 2 *Let $F(X, Y, Z)$ be homogeneous polynomial of degree n without factors X, Y or Z. Let the points P, P', P'' have multiplicities m, m', m'' on the curve $F = 0$. Define*

$$\overline{F}(X, Y, Z) = \frac{F(YZ, XZ, XY)}{X^{m''} Y^{m'} Z^{m}}.$$

(i) $\overline{F}$ is a homogeneous polynomial of degree $2n - m - m' - m''$ and $\overline{\overline{F}} = F$.
(ii) The points P, P', P'' have multiplicities $n - m' - m'', n - m - m'', n - m - m'$ on the curve $\overline{F} = 0$.
(iii) If F is irreducible then $\overline{F}$ is irreducible.

Proof:

(i) If P has multiplicity m on the curve $F = 0$, then the terms of lowest degree in X, Y (that is, of highest degree in Z) have degree m in X, Y (and $n-m$ in Z). So, $F(X, Y, Z) = F_m(X, Y)Z^{n-m} + ... + F_{n-m}(X, Y)$ with $F_i(X, Y)$ homogeneous of degree i, and $F(YZ, XZ, XY) = F_m(YZ, XZ)(XY)^{n-m} + ... + F_{n-m}(YZ, XZ)$ has a factor Z^m that is divided out in $\overline{F}$. Since $\overline{F}$ can be described as $F(YZ, XZ, XY)$ with all factors X, Y, Z removed, and since $\phi(\phi(F)) = (XYZ)^n F$, doing this twice gets us back to F, since F has no factors X, Y, Z.

(ii) We have $\overline{F}(X, Y, Z) = \sum_{i=0}^{n-m} F_{m+i}(Y, X) X^{n-m-m''-i} Y^{n-m-m'-i} Z^i$ with highest degree in Z in the part $F_n(Y, X) X^{-m''} Y^{-m'} Z^{n-m}$.

(iii) Any factorization of one of F, $\overline{F}$ immediately gives a corresponding factorization of the other. □

Let $C = V(F)$ and $\overline{C} = V(\overline{F})$ be the curves defined by F and $\overline{F}$, and assume that P is a point of multiplicity m on the curve C of degree n.

Choose coordinates in such a way that the lines $X = 0$ and $Y = 0$ are not tangent to C, and intersect C in $n - m$ points other than $P = (0, 0, 1)$, and let $Z = 0$ intersect this curve in n distinct points different from P', P''.

Lemma 3 *Under these conditions, the curve $\overline{C}$ has the following multiple points:*

(a) Inside U the curves C and $\overline{C}$ are isomorphic, and the same multiplicities occur.

(b) The points P, P', P'' are ordinary multiple points of $\overline{C}$.

(c) Apart from P, P', P'', the curve $\overline{C}$ does not meet the lines $X = 0$ or $Y = 0$, and has total intersection multiplicity m with the line $Z = 0$.

Proof: Easy. □

Corollary 4 *We have $g^*(\overline{C}) = g^*(C) - \sum_R \binom{\overline{m}(R)}{2}$ where $\overline{m}(R)$ is the multiplicity of the point R on $\overline{C} \cap V(Z)$.*

Proof: Since by assumption P' and P'' are not on C (so that $m' = m'' = 0$ and P, P', P'' have multiplicities $n, n-m, n-m$ on $\overline{C}$) we find $g^*(C) = \binom{n-1}{2} - \binom{m}{2} - \sum_Q \binom{m(Q)}{2}$, where the sum is over the multiple points Q on the curve C distinct from P, and $g^*(\overline{C}) = \binom{2n-m-1}{2} - \binom{n}{2} - 2\binom{n-m}{2} - \sum_R \binom{\overline{m}(R)}{2} = \binom{n-1}{2} - \binom{m}{2} - \sum_R \binom{\overline{m}(R)}{2}$ where the sum is over the multiple points R on the curve $\overline{C}$ distinct from P, P', P'', so that $g^*(C) - g^*(\overline{C}) = \sum_R - \sum_Q$. But by choice of the coordinate axes there is no contribution to $\sum_Q$ from points on $V(XYZ)$. $\square$

This corollary says that g^* decreases, unless $\overline{m}(R) = 1$ for each R on $\overline{C} \cap V(Z)$, in which case it stays the same but the number of nonordinary multiple points has decreased.

6 Computer algebra packages

Several computer algebra packages are able to compute the genus of a plane curve.

6.1 PAFF

PAFF is a package developed by Gaétan Haché. It runs on top of axiom.

See `http://www-rocq.inria.fr/codes/Gaetan.Hache/PAFF.html`.

Example Compute the genus of the quadrifolium $(x^2 + y^2)^3 = 4x^2y^2$.

```
% cd PAFF
% axiom
(1) -> )lib )dir ./spad
(1) -> )lib )dir ./spad

(1) -> K:=PF 101
   (1)  PrimeField 101
(2) -> R:=DMP([X,Y,Z],K)
   (2)  DistributedMultivariatePolynomial([X,Y,Z],PrimeField 101)
(3) -> P:=PAFF(K,[X,Y,Z],BLQT)
   (3)  PackageForAlgebraicFunctionField(PrimeField 101,[X,Y,Z],
                                        BlowUpWithQuadTrans)
(4) -> C:R:= (X**2+Y**2)**3-4*X**2*Y**2*Z**2
          6     4 2     2 4      2 2 2    6
   (4)  X  + 3X Y  + 3X Y  + 97X Y Z  + Y
(5) -> setCurve(C)$P
(6) -> genus()$P
   (6)  0
(7) -> singularPoints()$P
                   1         1        1
   (7)  [(91:1:0) ,(10:1:0) ,(0:0:1) ]
(8) -> )quit
%
```

We see that the genus over $\mathbf{F}_{101}$ is zero, and get the singular points. This package was developed to work with algebraic geometry codes.

6.2 Singular

Singular is a package developed at the University of Kaiserslautern.

See `http://www.singular.uni-kl.de/`.

Let us again compute the genus of $(x^2 + y^2)^3 = 4x^2y^2$.

```
% Singular
> LIB "normal.lib";
> ring r=0,(x,y),dp;
> ideal i=(x^2+y^2)^3-4*x^2*y^2;
> genus(i);
0
> Auf Wiedersehen.
%
```

The ring declaration asks for $\mathbf{Q}[x, y]$. The `dp` parameter in the ring declaration asks for the use of degree reverse lexicographical ordering—a good default. The program was exited by typing Ctrl-D.