HOW TO CHOMP FORESTS, AND SOME OTHER GRAPHS

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Abstract

Two persons play the following game: starting with a finite undirected graph G, they take turns in removing either an edge from G, or a vertex together with all incident edges. He who faces the awkward task of removing something from the empty graph, loses the game. In general a winning strategy for this game seems to be unknown; this note treats two special cases: we completely solve the case where G is a forest, and we show how certain involutions can be employed to reduce the graph under investigation. Interesting consequences are: first, that the starting player has a winning strategy for any non-empty tree; and second, that he has a winning strategy for the complete graph on n vertices if and only if n is not a multiple of 3.

1. Introduction

The game of *graph chomp* is played on a finite simple undirected graph, i.e., a graph $G = (V, E)$ for which the vertex set V is finite and the edge set E is a set of unordered pairs of vertices. Two players, A (who starts) and B, take turns alternatingly, and in each turn, either an edge is removed, or a vertex together with all incident edges. Note that, even when all edges incident with a given vertex have been removed, the vertex itself stays in the graph until it is explicitly deleted in a later turn. The name 'graph chomp' finds its origin in the similarity of this game to the game of Chomp [2]; see also [3], where graph chomp is mentioned as a special case of a chomp-like game on a simplicial complex.

An element k of $V \cup E$ is simply called an element of G, and the graph obtained by removing k from G is denoted by $G - k$. It is clear that for any graph G, either A has a winning strategy for graph chomp on G, or B has such a strategy. In the latter case, we call G *lost*; in the former case, there exist elements k of G such that $G - k$ is lost. Such elements are called *winning moves*, and G is called *won*.

Proposition 1. *Every non-empty path is won.*

Proof. The element k in the middle of the path P is a winning move. Indeed, $P - k$ has an involution τ fixing no elements, and A wins the game by answering $\tau(l)$ to every move l of B. □

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Both the statement of this proposition and its proof are subject to considerable generalisation: Theorem 2 below implies that all non-empty trees are won, and Proposition 3 shows that certain involutions that do fix elements can nevertheless be used to simplify the question of whether a given graph is won or lost. As an application of this proposition, we answer that question for all complete graphs in Corollary 4.

2. Grundy numbers

When playing graph chomp, our heroes follow a path in the directed graph Γ having as vertex set the set $\mathcal{G}$ of isomorphism classes of finite undirected graphs, and arrows $G \to H$ if $H = G - k$ for some element k of G. Clearly $\to$ is acyclic, and all paths in Γ are finite, so that we can inductively define an $\mathbb{N}$-valued function g on $\mathcal{G}$ as follows: $g(\emptyset) = 0$, and for all non-empty $G \in \mathcal{G}$:

$$g(G) = \min(\mathbb{N} \setminus \{g(H) \mid G \to H\}).$$

The number $g(G)$ is called the *Grundy*-number of G, and it is easy to see that $g(G) = 0$ if and only if G is lost. Grundy numbers are also called nim-numbers, and much more can be said about them [1].

3. Grundy numbers of forests

The subset of $\mathcal{G}$ consisting of all forests (disjoint unions of trees) is closed under $\to$, so that one can compute Grundy numbers of forests without having to compute Grundy numbers of graphs other than forests. The following theorem states the result.

Theorem 2. *Let F be a forest. Then $g(F) \leq 3$, and the following table shows how the exact value is determined by the parity of the number c of connected components of F and the parity of its number v of vertices.*

$g(F)$	*c even*	*c odd*
v even	*0*	*2*
v odd	*3*	*1*

Proof. First observe

(1) that removing an edge from F changes the parity of c and not that of v; and
(2) that removing a vertex of degree d from F changes the parity of v, and adds $d - 1$ to c.

We proceed by induction. The table is evidently correct for the empty forest; now suppose that $F \neq \emptyset$ and that the table predicts the Grundy number correctly for all F' with $F \to F'$. As in the statement of the theorem, denote by c and v the numbers of connected components and of vertices of F, respectively; and let g_0 be the value of $g(F)$ predicted by the table above. The above observations show that any move changes the parity of at least one of the numbers c and v, so that $g(F') \neq g_0$ for any F' with $F \to F'$. This shows that $g(F) \leq g_0$, and it suffices to prove that for any natural number $g_1 < g_0$, there exists an F' with $F \to F'$ and $g(F') = g_1$.

(0) If c and v are both even, then the above already shows that $g(F) = 0$.
(1) If c and v are both odd, then F has a vertex of even degree (as does any graph with an odd number of vertices). Removing such a vertex turns F into a forest with Grundy number 0. This proves $g(F) = 1$.
(2) If c is odd and v is even, then F has both edges and vertices of degree 1. Removing an edge from F yields a forest with Grundy number 0, while removing a vertex of degree 1 results in a forest with Grundy number 1. We find $g(F) = 2$.
(3) If c is even and v is odd, then F contains edges, vertices of degree 1 and vertices of even degree, which, when removed, yield forests with Grundy numbers $1, 0$ and 2, respectively. We conclude that that $g(F) = 3$.

□

4. Involutions in graph chomp

If τ is an involution of a graph G, then one may try to win graph chomp on G by answering $\tau(k)$ to every move k of the opponent. This strategy, however, cannot be pursued when the element k is fixed under τ. One remedy is to require that τ does not fix any elements, but of course this severely limits the applicability of this strategy. The following proposition presents a more subtle version of this approach.

Proposition 3. *Let $G = (V, E)$ be a finite simple graph, and let τ be an automorphism of G satisfying $\tau^2 = 1$. Assume that $G^\tau := (V^\tau, E^\tau)$, where V^τ and E^τ denote the sets of fixed vertices and of fixed edges, respectively, is a graph; that is, the vertices of any fixed edge are fixed, rather than interchanged by τ. Then G is won if and only if G^τ is won.*

Proof. To argue by induction, we assume that the statement is true for all subgraphs of G. First suppose that G^τ is won, and let $k \in G^\tau$ be a winning move. Then $G^\tau - k$ is lost, and τ restricts to an involution on $G - k$ having the property that

$$(G - k)^\tau = G^\tau - k.$$

Now by the induction hypothesis $G - k$ is lost, hence G is won.

Second, assume that G^τ is lost, and let k be any element of G; then either $k \in G^\tau$, or not. In the first case, player B has a winning answer $l \in G^\tau$ to k (in graph chomp on G^τ), and τ restricts to an involution on $G - k - l$ satisfying

$$(G - k - l)^\tau = G^\tau - k - l.$$

Hence, $G - k - l$ is lost by the induction hypothesis, so that $G - k$ is won. In the second case, that is: $k \notin G^\tau$, player B can answer $\tau(k)$ to k, and once more, τ restricts to an involution on $G - k - \tau(k)$ satisfying

$$(G - k - \tau(k))^\tau = G^\tau.$$

We conclude that $G - k - \tau(k)$ is lost, hence $G - k$ is won, as it was in the first case. This shows that G is lost. □

Corollary 4. *The complete graph K_n on n vertices is lost if and only if n is a multiple of* 3.

Proof. We proceed by induction on n. Assume that the statement is valid for all n smaller than n_0, and consider K_{n_0}. Removing a vertex yields K_{n_0-1}, so if $n_0 \equiv 1 \mod 3$, then the latter graph is lost, and K_{n_0} is won. If, on the other hand, $n_0 \equiv 0$ or $2 \mod 3$, then it is not wise to remove a vertex, and K_{n_0} is won if and only if K_{n_0} minus one edge, say $\{1, 2\}$, is lost. Now the permutation $\tau = (1, 2)$ is an involution of $K_{n_0} - \{1, 2\}$ such that

$$(K_{n_0} - \{1, 2\})^\tau \text{ is a graph } \cong K_{n_0-2}.$$

Application of Proposition 3 shows that $K_{n_0} - \{1, 2\}$ is lost if and only if K_{n_0-2} is lost, which is the case if and only if $n_0 - 2$ is a multiple of 3. This shows that K_{n_0} is lost if and only if n_0 is a multiple of 3, as claimed. □

References

[1] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. *Winning ways for your mathematical plays.* Vol. 1: *Games in general.* Vol. 2: *Games in particular.* Academic Press. London, 1982.

[2] Andries Brouwer. *The game of Chomp.* `http://www.win.tue.nl/~aeb/games/chomp.html`

[3] *Eindejaarsprijsvraag 2002.* `http://www.win.tue.nl/~aeb/contest2002/trapafbreken.html`