Uniqueness of line orientation in ...

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Abstract

The Lie orientation of a Lie orientable partial linear space is essentially unique.

1 Introduction

Given a partial linear space $(X, \mathcal{L})$ with lines of size 3 and a cyclic orientation (xyz) on each line $\{x, y, z\}$, we can define an algebra A (over some unspecified field) with basis X and multiplication $xy = 0$ when $y = x$ or when x, y are not joined by a line, and $xy = -yx = z$ when the line $\{x, y, z\}$ has orientation (xyz).

In his thesis [3], Erik Postma considered the question when A is a Lie algebra. Checking the Jacobi identity involves only subspaces generated by three points, and hence A is a Lie algebra iff the subalgebras generated by the planes are, where a *plane* is a subspace generated by two intersecting lines. It turns out that a plane must be either (i) the partial linear space F_6 of 6 points and 4 lines obtained by deleting one point from the Fano plane, or (ii) the Fano plane F_7. (And possibility (ii) can occur over fields of characteristic 3 only.)

By Hale [1] and Hall [2] any connected partial linear space such that all planes are isomorphic to F_6 or F_7 is one of (i) $\mathrm{Sp}(V, B)$, (ii) $\mathrm{O}(V, Q)$, (iii) $\mathrm{T}(\Omega, \Omega')$, or (iv) $PG(n, 2) - PG(k, 2)$ for some $k \leq n-3$. (For definitions, see Section 2 below.)

An orientation of the lines of a partial linear space with lines of size 3 is called a *Lie orientation* if the above construction produces a Lie algebra A. The partial linear space is called *Lie orientable* when it has a Lie orientation. Postma [3] shows that each of the geometries in the Hale-Hall classification has a Lie orientation, except that in case (iv) a Lie orientation exists only in case $k = n - 3$.

Finally, one wonders whether such an orientation is essentially unique. If one maps z to $-z$ and changes the orientation of all lines incident with z, then the multiplication rules do not change. Two orientations are called equivalent when one arises from the other by such point-flips, that is, when there is a set of points X_0 such that the lines that differ in orientation for the two orientations are precisely the lines that meet X_0 in an odd number of points. Postma conjectured that any two Lie orientations of a Lie orientable partial linear space are equivalent, and we show that this is indeed the case.

2 Definitions

In the Hale-Hall classification mentioned in the introduction four types of partial linear space occurred.

(i) $\mathrm{Sp}(V,B)$ is the partial linear space found from a binary vector space V provided with a symplectic form B by taking as points the vectors outside the radical of B, and as lines the hyperbolic lines.

(ii) $\mathrm{O}(V,Q)$ is the partial linear space found from a binary vector space V provided with a quadratic form Q by taking as points the vectors where Q is nonzero and that lie outside the radical of B, the symplectic form associated to Q, and as lines the elliptic lines.

(iii) $\mathrm{T}(\Omega,\Omega')$ is the partial linear space found from two disjoint sets Ω,Ω' by taking as points the subsets A of $\Omega\cup\Omega'$ with $|A\cap\Omega|=2$, and as lines the triples $\{A,B,C\}$ of points, where $A+B+C=0$ and addition is in the binary vectorspace $2^{\Omega\cup\Omega'}$.

(iv) $PG(n,2)-PG(k,2)$ is the partial linear space with as points the points of $PG(n,2)$ outside a fixed (possibly empty) subspace $PG(k,2)$, and as lines the projective lines disjoint from that subspace.

In case (iv) we may assume that $-1\le k\le n-3$, since the case $k=n-2$ is a special case of (i), and there are no lines if $k>n-2$. Now case (iv) is the only case where Fano planes occur; in all other examples all planes are of type F_6.

3 Uniqueness of Lie orientation

Let an *even* collection of lines be one that covers each point an even number of times.

Proposition 1 *In each of the above partial linear spaces, any finite even collection of lines is the sum of even collections of size four.*

(Here 'sum' is addition in the binary vectorspace $2^{\mathcal{L}}$.)

Corollary 2 *Up to equivalence the Lie orientation of a Lie orientable partial linear space is unique.*

Proof (of the corollary): From the classification we see that each Lie orientable partial linear space is the union of an increasing sequence of finite subspaces, and hence we may assume that the partial linear space $(X,\mathcal{L})$ is finite.

We may also assume that $(X,\mathcal{L})$ is connected. If it is reduced to a single plane then the statement is true, as one verifies directly.

Let N be the point-line incidence matrix of $(X,\mathcal{L})$. Consider two Lie orientations of $(X,\mathcal{L})$. Let x be the binary rowvector indexed by $\mathcal{L}$ with $x_L=1$ when the two orientations differ on L. We have to show that x is in the row span of N, given that the restriction of x to the set of lines in any plane is in the row span of N restricted to that same set of lines.

To this end, it suffices to show that if y is a binary rowvector orthogonal to the row space of N, then y is the sum of such vectors that are zero outside the set of lines in a plane. In other words, if we have an even collection of lines, then that collection is the sum of even collections contained in a plane.

In the plane F_6 there is only one nonempty such collection, the set of all four lines. In the plane F_7 any such collection is the set of four lines missing a some point. □

4 Triangulation

Proof (of the proposition): Induction on the dimension. Let H be a hyperplane of $(X, \mathcal{L})$ (arbitrary, or to be chosen later). Given a finite even collection $\mathcal{C}$ of lines in $(X, \mathcal{L})$, we show that $\mathcal{C}$ is the sum of an even collection in H and some even collections of size 4.

The lines in $\mathcal{L}$ not contained in H become edges of a graph G on the complement of H, and the collection $\mathcal{C}$ becomes the set of edges of a subgraph C of G that has even degree at each vertex. If C can be triangulated (i.e., is a sum of triangles in G), then we are done, since a triangle x, y, z in G is derived from lines axy, byz, czx with $a, b, c \in H$, and abc is a line, and $\{axy, byz, czx, abc\}$ is even, so that a triangle is the sum of an even 4-set and a line contained in H.

Remains to triangulate C. Induction on the size of C. Distinguish cases (i)-(iv) as in the Hale-Hall classification.

In case (iv), $PG(n,2) - PG(k,2)$, choose H a hyperplane not containing $PG(k,2)$ (for $k \geq 0$). The points outside $PG(k,2)$ fall into $2^{n-k}-1 \geq 7$ cosets, and two points are joined by an edge when they lie in distinct cosets. Given a cycle $... \sim w \sim x \sim y \sim z \sim ...$ we can pick p outside H in a coset not containing w, x, y, z and see that the cycle is the sum of the shorter cycle $... \sim w \sim p \sim z \sim ...$ and the triangles w, x, p and x, y, p and y, z, p.

In case (iii), $\mathrm{T}(\Omega, \Omega')$, first suppose $|\Omega| > 3$. Let $\alpha \in \Omega$, and let H be the hyperplane of points not containing α. The complement of H consists of the points A with $\alpha \in A$, and two points A, A' with $A \cap \Omega = \{\alpha, \beta\}$ and $A' \cap \Omega = \{\alpha, \beta'\}$ are adjacent when $\beta \neq \beta'$. Given a cycle without diagonals $... \sim w \sim x \sim y \sim z \sim ...$ we have $\beta, \gamma \in \Omega$ with $w \cap \Omega = y \cap \Omega = \{\alpha, \beta\}$ and $x \cap \Omega = z \cap \Omega = \{\alpha, \gamma\}$. Now the point $p = \{\alpha, \delta\}$ for some δ distinct from α, β, γ lies outside H and is adjacent to each of w, x, y, z.

Next suppose $|\Omega| = 3$. Now the collinearity graph of the partial linear space is complete tripartite. Let H be the hyperplane of points not containing $\alpha' \in \Omega'$. Given a cycle without diagonals $... \sim w \sim x \sim y \sim z \sim ...$ the points w, x, y, z must lie in two of the three parts of the tripartition, and we can pick a point p adjacent to w, x, y, z in the third part.

In case (i), $\mathrm{Sp}(V, B)$, two points x, y are joined by an edge when $B(x, y) = 1$. Put $R := \mathrm{Rad}(B)$. Then $\dim V/R$ is even. If $\dim V/R = 2$ or 4, we are in case $\mathrm{T}(\Omega, \Omega')$ with $|\Omega| = 3$ or 6, which was treated already. So assume $\dim V/R \geq 6$. Pick a point a and put $H = a^{\perp}$. Since a is in an even number of lines of $\mathcal{C}$, we can add even 4-sets to $\mathcal{C}$ in order to remove all lines that pass through a. Given a cycle $... \sim w \sim x \sim y \sim z \sim ...$ in C, we can find a point p in the complement of H adjacent to each of w, x, y, z when a does not lie in the span of w, x, y, z. Suppose a lies in the span of w, x, y, z. If $a = w + x + y + z$ then $B(x, a) = 0$, contradiction. Hence $a = w + z$ and $B(w, z) = 1$. Since we had removed all lines on a from $\mathcal{C}$, the edge wz is a diagonal in the cycle. This shows that if $x_0 \sim x_1 \sim ... \sim x_{g-1} \sim x_0$ is a shortest cycle in C, then we can make C smaller by adding triangles, unless $g = 6$ and $a = x_0 + x_3 = x_1 + x_4 = x_2 + x_5$. But now modulo even sets of size 4 the sum of the lines on the edges of this 6-cycle is zero.

In case (ii), $\mathrm{O}(V, Q)$, we have the subspace of the previous example induced by the points x with $Q(x) = 1$. In low dimensions this space is one of the examples seen already. (Let Q be the set of projective points

x with $Q(x) = 0$, and put $S = R \cap Q$ so that $Q()$ induces a nondegenerate quadratic form on V/S. We may assume that Q is nonempty. If $\dim V/S = 2$, then Q is hyperbolic and we have $PG(n,2) - PG(-1,2)$. If $\dim V/S = 3$, then we have $T(4,\Omega')$. If $\dim V/S = 4$, then either Q is hyperbolic, and we have a disconnected space (two copies of $T(3,\Omega')$), or Q is elliptic, and we have $T(5,\Omega')$. If $\dim V/S = 5$, then we have $T(6,\Omega')$. If $\dim V/S = 6$ and Q is hyperbolic, then we have $T(8,\Omega')$.)

So assume $\dim V/S \geq 6$. Again remove all lines through some point a (with $Q(a) = 1$), and take $H = a^{\perp}$. Consider a cycle $... \sim w \sim x \sim y \sim z \sim ...$ in C. If it has length 6 and opposite vertices sum to a, then as before modulo even sets of size 4 the sum of the lines on the edges of this 6-cycle is zero. Otherwise, for suitable choice of w, we have $a \neq w + z$, and then a is independent from w, x, y, z. If w, x, y, z are dependent, then they lie in a plane, and the three edges wx, xy, yz can be replaced by the single edge wz. So we may assume that a, w, x, y, z are independent. Desired: a point p with $Q(p) = 1$ and $B(p,a) = B(p,w) = B(p,x) = B(p,y) = B(p,z) = 1$. If $\dim V/S \geq 7$ then we can restrict attention to a nondegenerate subspace of dimension 7 and satisfy the conditions on $B(p,.)$ there, and then satisfy $Q(p) = 1$ by adding the nucleus to p, if necessary. So we may assume that $\dim V/S = 6$ and Q is elliptic. Now there is a unique point p that satisfies the conditions, as one sees by explicit inspection. □

References

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[3] Erik Postma, *From Lie Algebras to Geometry and Back*, Ph. D. Thesis, TUE, Eindhoven, 2007.