THE GRADED RING OF INVARIANTS OF TERNARY QUARTICS I — GENERATORS AND RELATIONS —

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ABSTRACT. The moduli spaces of curves are known only for genus 1, genus 2 and hyperelliptic case of genus 3. In this paper, we obtained the graded ring of invariants of ternary quartics except for syzygies. I.e. we obtained homogeneous coordinate ring of moduli space of non-hyperelliptic curves of genus 3. I.e. we obtained moduli space of non-hyperelliptic curves of genus 3.

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1. INTRODUCTION

Throughout this paper, let $\mathbb{C}$ be the complex number field.

We consider two examples at first.

(1) We consider the one to one corespondance between binary quadratics and projective space of dimension 2:

$$ax^2 + 2bxy + cy^2 \longleftrightarrow (a : b : c) \in \mathbb{P}^2 = \mathrm{Proj}\mathbb{C}[a, b, c].$$

We put

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{for any } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2).$$

We compute

$$ax'^2+2bx'y'+cy'^2 = \{\alpha^2a+2\alpha\gamma b+\gamma^2c\}x^2+2\{\alpha\beta a+(\alpha\delta+\beta\gamma)b+\gamma\delta c\}xy+\{\beta^2a+2\beta\delta b+\delta^2c\}y^2.$$

We define that SL(2) operate to the ring $R = \mathbb{C}[a, b, c]$ as follows:

$$(a, b, c) \mapsto (\alpha^2a + 2\alpha\gamma b + \gamma^2c,\ \alpha\beta a + (\alpha\delta + \beta\gamma)b + \gamma\delta c,\ \beta^2a + 2\beta\delta b + \delta^2c).$$

Let S be the invariant subring of R by the above action of SL(2): $S = R^{\mathrm{SL}(2)}$. It is known that S is the polynomial ring generated by one element of degree 2 (i.e. discriminant):

$$S = \mathbb{C}[I_2], \qquad (I_2 = b^2 - ac).$$

(2) We consider the one to one corespondance between ternary cubics (eliptic curves) and projective space of dimension 9:

$$\begin{array}{l} ax^3 + 3bx^2y + 3cxy^2 + dy^3 \\ +3ex^2z + 6fxyz + 3gy^2z \\ +3hxz^2 + 3iyz^2 \\ +jz^3 \end{array} \longleftrightarrow \begin{array}{l}(a : b : c : d : e : f : g : h : i : j) \\ \in \mathbb{P}^9 = \mathrm{Proj}\mathbb{C}[a, b, c, d, e, f, g, h, i, j].\end{array}$$

We put

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{for any } A \in \mathrm{SL}(3).$$

We compute and put

$$\begin{array}{c} ax'^3 + 3bx'^2y' + 3cx'y'^2 + dy'^3 \\ +3ex'^2z' + 6fx'y'z' + 3gy'^2z' \\ +3hx'z'^2 + 3iy'z'^2 \\ +jz'^3 \end{array} = \begin{array}{c} a'x^3 + 3b'x^2y + 3c'xy^2 + d'y^3 \\ +3e'x^2z + 6f'xyz + 3g'y^2z \\ +3h'xz^2 + 3i'yz^2 \\ +j'z^3. \end{array}$$

We define that SL(3) operate to the ring $R = \mathbb{C}[a, b, c, d, e, f, g, h, i, j]$ as follows:

$$(a, b, c, d, e, f, g, h, i, j) \mapsto (a', b', c', d', e', f', g', h', i', j').$$

Let S be the invariant subring of R by the above action of SL(3): $S = R^{\mathrm{SL}(3)}$. It is known by Aronhold ([1]) that S is the polynomial ring generated by two elements of degree 4 and 6:

$$S = \mathbb{C}[I_4, I_6].$$

We note that $\mathbb{P}^1 = \mathrm{Proj}S$. It means the well known result that the moduli space of eliptic curves is the projective space of dimension 1.

In this way, invariant theory is the important theory related with moduli theory ([4]). But it is hard work. Let $S(n, r)$ denote the graded ring of invariants of homogeneous polynomials of order r in n variables after Shioda. $S(n, 2)$ $(n \geq 2)$, $S(n, 3)$ $(n = 2, 3, 4)$, $S(2, 4)$, $S(2, 5)$, $S(2, 6)$ and $S(2, 8)$ are only known. The previous two examples of (1) and (2) are $S(2, 2)$ and $S(3, 3)$ respectively. In particular, $S(2, 8)$ is computed by Shioda ([6]). It gives the moduli space of hyperelliptic curves of genus 3. In this paper, our aim is the computation of $S(3, 4)$. It gives the moduli space of non-hyperelliptic curves of genus 3.

We denote $S(3, 4)$ by S in this paper. We put ternary quartics $\varphi(x, y, z)$ as follows:

$$\varphi(x, y, z) = \begin{array}{c} ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 \\ +4fx^3z + 12gx^2yz + 12hxy^2z + 4iy^3z \\ +6jx^2z^2 + 12kxyz^2 + 6ly^2z^2 \\ +4mxz^3 + 4nyz^3 \\ +pz^4. \end{array}$$

We consider the one to one corespondance between ternary quartics and projective space of dimension 14:

$$\varphi(x, y, z) \longleftrightarrow (a : b : c : d : e : f : g : h : i : j : k : l : m : n : p)$$
$$\in \mathbb{P}^{14} = \mathrm{Proj}\mathbb{C}[a, b, c, d, e, f, g, h, i, j, k, l, m, n, p].$$

We put

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{for any } A \in \mathrm{SL}(3).$$

We compute and put

$$\varphi(x', y', z') = \begin{array}{c} a'x^4 + 4b'x^3y + 6c'x^2y^2 + 4d'xy^3 + e'y^4 \\ +4f'x^3z + 12g'x^2yz + 12h'xy^2z + 4i'y^3z \\ +6j'x^2z^2 + 12k'xyz^2 + 6l'y^2z^2 \\ +4m'xz^3 + 4n'yz^3 \\ +p'z^4. \end{array}$$

We define that SL(3) operate to the ring $R = \mathbb{C}[a, b, c, d, e, f, g, h, i, j, k, l, m, n, p]$ as follows:

$$(a, b, c, d, e, f, g, h, i, j, k, l, m, n, p) \mapsto (a', b', c', d', e', f', g', h', i', j', k', l', m', n', p').$$

S is the invariant subring of R by the above action of SL(3): $S = R^{\mathrm{SL}(3)}$.

Salmon obtained some first results about S ([5]). We will use them.

By Hochster and Roberts, S is a Cohen-Macaulay ring. Moreover, it is a Gorenstein ring in this case ([3]).

Shioda computed the generating function of S in [6]. The generating function $G(t)$ of graded ring S is defined as follows. Let S_q be the submodule of the elements of degree q in S. Let L_q be the dimension of S_q over $\mathbb{C}$. We define that

$$G(t) = \sum_{q=0}^{\infty} L_q t^q = 1 + L_1 t + L_2 t^2 + L_3 t^3 + L_4 t^4 + L_5 t^5 + \cdots\cdots .$$

It is the important function because it deside the almost structure of S.

For example, the generating functions of two examples, which we describe before, are as follows:

$$(1) \qquad \frac{1}{(1-t^2)},$$

$$(2) \qquad \frac{1}{(1-t^4)(1-t^6)}.$$

We describe Shioda's Theorem.

Theorem 1.1 (Shioda, [6], Appendix, Theorem)**.** *The generating function $G(t)$ of S is*

$$G(t) = \frac{N(t)}{(1-t^3)(1-t^6)(1-t^9)(1-t^{12})(1-t^{15})(1-t^{18})(1-t^{27})},$$

where

$$N(t) = 1 + t^9 + t^{12} + t^{15} + 2t^{18} + 3t^{21} + 2t^{24} + 3t^{27} + 4t^{30} + 3t^{33} + 4t^{36} + 4t^{39} + 3t^{42} + 4t^{45} + 3t^{48} + 2t^{51} + 3t^{54} + 2t^{57} + t^{60} + t^{63} + t^{66} + t^{75}.$$

The denominator of $G(t)$ relates to the algebraically independent generators. S has seven algebraically independent generators. Their degrees are 3, 6, 9, 12, 15, 18 and 27. We put a polynomial ring generated by these elements over $\mathbb{C}$ into P. S is a free P-module. The numerator of $G(t)$ relates to the rank. We add all the coefficients of the power of t in numerator $N(t)$ of $G(t)$, and we obtain 50:

$$\begin{aligned} &1+1+1+1+2+3+2+3+4+3+4 \\ &+4+3+4+3+2+3+2+1+1+1+1 \\ &=50. \end{aligned}$$

S has rank 50 over P.

Dixmier decided a set of algebraically independent generators ([2], Theorem 3.2).

Shioda multiplied numerator and denominator of $G(t)$ by

$$(1-t^9)(1-t^{12})(1-t^{15})(1-t^{18})(1-t^{21})^2$$

and expanded the numerator and obtained the next Conjecture.

Conjecture 1.2 (Shioda, [6], Appendix, p1046)**.**

$$G(t) = \frac{M(t)}{(1-t^3)(1-t^6)(1-t^9)^2(1-t^{12})^2(1-t^{15})^2(1-t^{18})^2(1-t^{21})^2(1-t^{27})},$$

where

$$\begin{aligned} M(t) = 1 \\ &- t^{30} - 2t^{33} - 3t^{36} - 3t^{39} - 4t^{42} - t^{45} \\ &+ 3t^{51} + 6t^{54} + 7t^{57} + 8t^{60} + 6t^{63} + 4t^{66} + 2t^{69} \\ &- 2t^{72} - 4t^{75} - 4t^{78} - 7t^{81} - 6t^{84} - 6t^{87} - 7t^{90} - 4t^{93} - 4t^{96} - 2t^{99} \\ &+ 2t^{102} + 4t^{105} + 6t^{108} + 8t^{111} + 7t^{114} + 6t^{117} + 3t^{120} \\ &- t^{126} - 4t^{129} - 3t^{132} - 3t^{135} - 2t^{138} - t^{141} \\ &+ t^{171}. \end{aligned}$$

I.e. these are as follows. We add six algebraically dependent generators to P. Their degrees are 9, 12, 15, 18, 21 and 21. S is generated by the 13 elements. S has 14 relations, 36 first syzygies, 46 second syzygies, 36 third syzygies, 14 fourth syzygies and 1 fifth (last) syzygy.

But we discovered one relation of degree 48 and one first syzygy of degree 48. Hence we must add these and two more syzygies to Conjecture 3.2.

Conjecture 1.3. In Conjecture 3.2, the numerator M(t) should be as follows.

$$\begin{aligned} M(t) = 1 \\ &- t^{30} - 2t^{33} - 3t^{36} - 3t^{39} - 4t^{42} - t^{45} - t^{48} \\ &+ t^{48} + 3t^{51} + 6t^{54} + 7t^{57} + 8t^{60} + 6t^{63} + 4t^{66} + 2t^{69} \\ &- 2t^{72} - 4t^{75} - 4t^{78} - 7t^{81} - 6t^{84} - 6t^{87} - 7t^{90} - 4t^{93} - 4t^{96} - 2t^{99} \\ &+ 2t^{102} + 4t^{105} + 6t^{108} + 8t^{111} + 7t^{114} + 6t^{117} + 3t^{120} + t^{123} \\ &- t^{123} - t^{126} - 4t^{129} - 3t^{132} - 3t^{135} - 2t^{138} - t^{141} \\ &+ t^{171}. \end{aligned}$$

I.e. we must add one relation of degree 48, one first syzygy of degree 48, one third syzygy of degree 123 and one fourth syzygy of degree 123 to Conjecture 3.2. I.e. S has 15 relations, 37 first syzygies, 46 second syzygies, 37 third syzygies, 15 fourth syzygies and 1 fifth (last) syzygy. We show the details in the next Tables.

Degree	30	33	36	39	42	45	48
The number of relations	1	2	3	3	4	1	1

Degree	48	51	54	57	60	63	66	69
The number of first syzygies	1	3	6	7	8	6	4	2

Degree	72	75	78	81	84	87	90	93	96	99
The number of second syzygies	2	4	4	7	6	6	7	4	4	2

Degree	102	105	108	111	114	117	120	123
The number of third syzygies	2	4	6	8	7	6	3	1

Degree	123	126	129	132	135	138	141
The number of fourth syzygies	1	1	4	3	3	2	1

Degree	171
The number of fifth syzygies	1

S is different from conjecture which is led by the generating function. We obtain such a result as this for the first time in mathematical history.

We give the remaining six algebraically dependent generators by using Dixmier's method. We compute the 15 relations concretely. We prove that S is generated by the 13 elements.

We expand $G(t)$ for the later Theorems:

$$G(t) = 1 + t^3 + 2t^6 + 4t^9 + 7t^{12} + 11t^{15} + 19t^{18} + 29t^{21} + 44t^{24} + 67t^{27} \\ + 98t^{30} + 139t^{33} + 199t^{36} + 275t^{39} + 375t^{42} + 509t^{45} + 678t^{48} + \cdots\cdots .$$

2. Definition of some covariants, contravariants and invariants

We denote ternary quartic by φ:

$$\varphi = \varphi(x, y, z) \qquad \text{(this is in §1).}$$

Its Hessian He is

$$\mathrm{He} = \frac{1}{1728}\begin{vmatrix} \frac{\partial^2\varphi}{\partial x^2} & \frac{\partial^2\varphi}{\partial x\partial y} & \frac{\partial^2\varphi}{\partial x\partial z} \\ \frac{\partial^2\varphi}{\partial y\partial x} & \frac{\partial^2\varphi}{\partial y^2} & \frac{\partial^2\varphi}{\partial y\partial z} \\ \frac{\partial^2\varphi}{\partial z\partial x} & \frac{\partial^2\varphi}{\partial z\partial y} & \frac{\partial^2\varphi}{\partial z^2} \end{vmatrix}.$$

Two contravariant σ and ψ exist and these symbolic expressions are as follows ([5]):

$$\sigma = (\alpha, 1, 2)^4/2,$$
$$\psi = (\alpha, 1, 2)^2(\alpha, 2, 3)^2(\alpha, 3, 1)^2/6.$$

The above notations mean as follows.
In the case of σ,

$$\sigma = (\alpha, 1, 2)^4/2 = \frac{1}{2}\begin{vmatrix} u & x[1] & x[2] \\ v & y[1] & y[2] \\ w & z[1] & z[2] \end{vmatrix}^4$$
$$= (3y[1]^2z[1]^2y[2]^2z[2]^2 - 2y[1]^3z[1]y[2]z[2]^3 - 2y[2]^3z[2]y[1]z[1]^3 \\ + y[1]^4z[2]^4/2 + y[2]^4z[1]^4/2)u^4 + \cdots\cdots .$$

We exchange $x[t]^4$, $x[t]^3y[t]$, $x[t]^2y[t]^2$, $x[t]y[t]^3$, $y[t]^4$, $x[t]^3z[t]$, $x[t]^2y[t]z[t]$, $x[t]y[t]^2z[t]$, $y[t]^3z[t]$, $x[t]^2z[t]^2$, $x[t]y[t]z[t]^2$, $y[t]^2z[t]^2$, $x[t]z[t]^3$, $y[t]z[t]^3$ and $z[t]^4$ $(t = 1,\ 2)$ for a, b, c, d, e, f, g, h, i,

j, k, l, m, n and p. We obtain σ as follows:

$$\sigma = (3l^2 - 4in + ep)u^4 + \cdots\cdots .$$

In the case of ψ,

$$\psi = (\alpha,1,2)^2(\alpha,2,3)^2(\alpha,3,1)^2/6 = \frac{1}{6}\begin{vmatrix} u & x[1] & x[2] \\ v & y[1] & y[2] \\ w & z[1] & z[2] \end{vmatrix}^2 \begin{vmatrix} u & x[2] & x[3] \\ v & y[2] & y[3] \\ w & z[2] & z[3] \end{vmatrix}^2 \begin{vmatrix} u & x[3] & x[1] \\ v & y[3] & y[1] \\ w & z[3] & z[1] \end{vmatrix}^2 .$$

We execute above exchange ($t = 1,\ 2,\ 3$), and we obtain ψ.

We define differential operator. Let A be a covariant of order q:

$$\begin{aligned} A =& A_{q,0,0}x^q + A_{q-1,1,0}x^{q-1}y + \cdots + A_{1,q-1,0}xy^{q-1} + A_{0,q,0}y^q \\ &+ A_{q-1,0,1}x^{q-1}z + \cdots\cdots\cdots\cdots\cdots\cdots + A_{0,q-1,1}y^{q-1}z \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &+ A_{1,0,q-1}xz^{q-1} + A_{0,1,q-1}yz^{q-1} \\ &+ A_{0,0,q}z^q. \end{aligned}$$

We replace A, q, x, y and z with B, r, u, v and w in the above, and obtain a contravariant B of order r. We define that

$$\begin{aligned} \mathrm{D}_A &= A_{q,0,0}\frac{\partial^q}{\partial u^q} + A_{q-1,1,0}\frac{\partial^q}{\partial u^{q-1}\partial v} + \cdots\cdots + A_{0,0,q}\frac{\partial^q}{\partial w^q}, \\ \mathrm{D}_B &= B_{r,0,0}\frac{\partial^r}{\partial x^r} + B_{r-1,1,0}\frac{\partial^r}{\partial x^{r-1}\partial y} + \cdots\cdots + B_{0,0,r}\frac{\partial^r}{\partial z^r}. \end{aligned}$$

By using this differential operator, we define eight covariants or contravariants as follows:

$$\begin{aligned} \rho &= \mathrm{D}_\varphi\psi/144 && \text{(by Salmon)}, \\ \tau &= \mathrm{D}_\rho\varphi/12 && \text{(by Salmon)}, \\ \xi &= \mathrm{D}_\sigma\mathrm{He}/72 && \text{(by Salmon)}, \\ \pi &= \mathrm{D}_\rho\mathrm{He}/2, \\ \eta &= \mathrm{D}_\xi\sigma/12, \\ \zeta &= \mathrm{D}_\tau\psi/2, \\ \chi &= \mathrm{D}_\tau\zeta/4, \\ \nu &= \mathrm{D}_\eta\pi/4. \end{aligned}$$

In Table 1, we denote these covariants and contravariants.

We define Dixmier's symbols ([2], 2.2).

Let P be ternary quadratic in x, y, z:

$$P = Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2.$$

Let Q be ternary quadratic in u, v, w:

$$Q = A'u^2 + 2B'uv + C'v^2 + 2D'uw + 2E'vw + F'w^2.$$

We define $J_{1,1}(P,Q)$ as

$$J_{1,1}(P,Q) = AA' + 2BB' + CC' + 2DD' + 2EE' + FF'.$$

TABLE 1. Table of covariants and contravariants

Covariants

Degree \ Order	1	2	3	4	5	6
1				φ		
2						
3						He
4						
5		τ, ξ				
6						
7				π		
8						
$\vdots$						
14		ν				

Contravariants

Degree \ Order	1	2	3	4	5	6
1						
2				σ		
3						ψ
4		ρ				
5						
6						
7		η				
8				ζ		
$\vdots$						
13		χ				

We define $J_{3,0}(P)$ and $J_{0,3}(Q)$ as discriminant of P and discriminant of Q respectively. P^* (dual of P) is well-known contravariant of P:

$$P^* = (E^2 - CF)u^2 + 2(BF - DE)uv + (D^2 - AF)v^2 + 2(CD - BE)uw + 2(AE - BD)vw + (B^2 - AC)w^2.$$

Q^* (dual of Q) is well-known covariant of Q:

$$Q^* = (E'^2 - C'F')x^2 + 2(B'F' - D'E')xy + (D'^2 - A'F')y^2 + 2(C'D' - B'E')xz + 2(A'E' - B'D')yz + (B'^2 - A'C')z^2.$$

We define $J_{2,2}(P, Q)$ as

$$J_{2,2}(P, Q) = J_{1,1}(Q^*, P^*).$$

Definition 2.1. We define 13 invariants as follows:

$I_3 = (1, 2, 3)^4/6$ (by Salmon),

$$I_6 = (\text{Hankel determinant}) = \text{Hank} = \begin{vmatrix} a & b & c & f & g & j \\ b & c & d & g & h & k \\ c & d & e & h & i & l \\ f & g & h & j & k & m \\ g & h & i & k & l & n \\ j & k & l & m & n & p \end{vmatrix} \quad \text{(by Salmon)},$$

$I_{9a} = J_{1,1}(\tau, \rho)$ (by Dixmier),

$I_{9b} = J_{1,1}(\xi, \rho)$,

$I_{12a} = J_{0,3}(\rho)$ (by Dixmier),

$I_{12b} = J_{1,1}(\tau, \eta)$,

$I_{15a} = J_{3,0}(\tau)$ (by Dixmier),

$I_{15b} = J_{3,0}(\xi)$,

$I_{18a} = J_{2,2}(\tau, \rho)$ (by Dixmier),

$I_{18b} = J_{2,2}(\xi, \rho)$,

$I_{21a} = J_{0,3}(\eta)$,

$$
\begin{aligned}
I_{21b} &= J_{1,1}(\nu,\eta),\\
I_{27} &= -8028160000I_3^9 + 428145867450000I_3^7I_6 + 1971756610875I_3^6I_{9a}\\
&\quad - 1686034281375I_3^6I_{9b} + 1162957661107704 0I_3^5I_6^2 + 33814214842500I_3^5I_{12a}\\
&\quad - 4012475231250I_3^5I_{12b} + 3655178399903994I_3^4I_6I_{9a} - 1672044447324402I_3^4I_6I_{9b}\\
&\quad + 81207445436861184 0I_3^3I_6^3 - 24494522472000I_3^4I_{15a} + 95178083994000I_3^4I_{15b}\\
&\quad + 22806490561563960I_3^3I_6I_{12a} - 4314783659551740I_3^3I_6I_{12b} + 121004804693376I_3^3I_{9a}^2\\
&\quad - 97476319111212I_3^3I_{9a}I_{9b} + 847991509162 92I_3^3I_{9b}^2 - 361984327188006240I_3^2I_6^2I_{9a}\\
&\quad - 79347650799790224 0I_3^2I_6^2I_{9b} - 1688893290246113280 0I_3I_6^4 - 3720270708000I_3^3I_{18a}\\
&\quad - 125785510578000I_3^3I_{18b} - 18457443717793920I_3^2I_6I_{15a} + 4987010422792944 0I_3^2I_6I_{15b}\\
&\quad + 2214447975692640I_3^2I_{9a}I_{12a} - 285128594624880I_3^2I_{9a}I_{12b} + 106279176263040I_3^2I_{9b}I_{12a}\\
&\quad - 120575640927600I_3^2I_{9b}I_{12b} + 309803553391769280 0I_3I_6^2I_{12a} - 913473864947966400I_3I_6^2I_{12b}\\
&\quad - 54229328492537520I_3I_6I_{9a}^2 + 48626442451830600I_3I_6I_{9a}I_{9b} + 45556110267265080I_3I_6I_{9b}^2\\
&\quad - 6380119235131084800 0I_6^3I_{9a} + 1054989415047859200 0I_6^3I_{9b} - 232827845040000I_3^2I_{21a}\\
&\quad + 35462073780000I_3^2I_{21b} - 3083647902552000I_3I_6I_{18a} - 3987662276329200 0I_3I_6I_{18b}\\
&\quad - 1482601753473120I_3I_{9a}I_{15a} + 5212832418491040I_3I_{9a}I_{15b} + 660308035393440I_3I_{9b}I_{15a}\\
&\quad - 151824734419680I_3I_{9b}I_{15b} + 2045336427648000I_3I_{12a}^2 - 1105108365052800I_3I_{12a}I_{12b}\\
&\quad + 92597146099200I_3I_{12b}^2 + 59257819371945600 0I_6^2I_{15a} + 45325102635993600 00I_6^2I_{15b}\\
&\quad + 25545522494701440 0I_6I_{9a}I_{12a} - 3781102048264800 0I_6I_{9a}I_{12b} + 9502470925476480 0I_6I_{9b}I_{12a}\\
&\quad - 67523657918724000I_6I_{9b}I_{12b} - 345178995739200I_{9a}^3 + 1208971901138400I_{9a}^2I_{9b}\\
&\quad - 518386438332000I_{9a}I_{9b}^2 + 97618348800I_{9b}^3 - 162742199578560000I_6I_{21a}\\
&\quad + 21912486324120000I_6I_{21b} + 396224647660800I_{9a}I_{18a} - 4949715628771200I_{9a}I_{18b}\\
&\quad + 203121379641600I_{9b}I_{18a} + 129328080537600I_{9b}I_{18b} - 295928456745600I_{12a}I_{15a}\\
&\quad + 1349544393187200I_{12a}I_{15b} + 167522957784000I_{12b}I_{15a} + 71111529720000I_{12b}I_{15b}\\
&\quad + 12499492320000J_{27},
\end{aligned}
$$

where $J_{27} = J_{1,1}(\nu,\chi)$.

The above notation of I_3 mean as follows:

$$
I_3 = (1,2,3)^4/6 = \frac{1}{6}\begin{vmatrix} x[1] & x[2] & x[3] \\ y[1] & y[2] & y[3] \\ z[1] & z[2] & z[3] \end{vmatrix}^4 .
$$

We execute the same exchange as making ψ, and we obtain I_3.

Remark 2.2. If we expand the invariants in Definition 2.1, then how many terms are there? We can easily expand the invariants of low degree. But we hardly expand the invariants of high degree. Fortunately, we can achieve our aim without expansion. But we can easily compute their upper limits. I.e. we compute the number of "weights zero" terms. Invariants are some linear combination with "weights zero" terms.

SL(3) has two subgroups which are isomorphic to $\mathbb{G}_m$:

$$
\begin{pmatrix} s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1/t \end{pmatrix}, \qquad (s,t \in \mathbb{C}\setminus\{0\}).
$$

Let wt_1 and wt_2 denote the weight by the former and the latter subgroup respectively. We describe the weights of coefficients of ternary quartics, i.e. weights for degree one, in the next Table.

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	p
wt_1	4	3	2	1	0	2	1	0	-1	0	-1	-2	-2	-3	-4
wt_2	0	1	2	3	4	-1	0	1	2	-2	-1	0	-3	-2	-4

By using these data, we compute the weights for high degree. For each degree, we compute the number of terms whose weight is zero by wt_1 and zero by wt_2, i.e. the number of "weights zero" terms. We describe the results in the next Table.

Degree	The number of "weights zero" terms
3	23
6	561
9	6992
12	55447
15	321927
18	1486602
21	5758887
27	58456030

Let Disc be discriminant. It is known that Disc is an invariant of degree 27. We put

$$\begin{aligned} P &= \mathbb{C}[I_3, I_6, I_{9a}, I_{12a}, I_{15a}, I_{18a}, \mathrm{Disc}], \\ P' &= \mathbb{C}[I_3, I_6, I_{9a}, I_{12a}, I_{15a}, I_{18a}, I_{27}], \\ S' &= \mathbb{C}[I_3, I_6, I_{9a}, I_{9b}, I_{12a}, I_{12b}, I_{15a}, I_{15b}, I_{18a}, I_{18b}, I_{21a}, I_{21b}, \mathrm{Disc}], \\ S'' &= \mathbb{C}[I_3, I_6, I_{9a}, I_{9b}, I_{12a}, I_{12b}, I_{15a}, I_{15b}, I_{18a}, I_{18b}, I_{21a}, I_{21b}, I_{27}]. \end{aligned}$$

It is clear that

$$S \supset S',\ S''.$$

Theorem 2.3 (Dixmier, [2], Theorem 3.2)**.** *I_3, I_6, I_{9a}, I_{12a}, I_{15a}, I_{18a} and* Disc *are algebraically independent over $\mathbb{C}$.*

Theorem 2.4. *If $S = S'$, then* $\mathrm{Disc} = I_{27}$: *$P = P'$ and $S' = S''$.*

Proof. We consider monomials of degree 27. They are made by I_3, I_6, I_{9a}, I_{9b}, I_{12a}, I_{12b}, I_{15a}, I_{15b}, I_{18a}, I_{18b}, I_{21a}, I_{21b} and J_{27}. The number of their monomials is 67: I_3^9, $I_3^7 I_6$, $\cdots$, J_{27}. We put

$$BAS_{disc} = \begin{pmatrix} I_3^9 \\ I_3^7 I_6 \\ \vdots \\ J_{27} \end{pmatrix}.$$

We can express Disc by the linear combination with BAS_{disc}. (By the proof of Theorem 3.2, there is only one relation among BAS_{30}. It is the intrinsic relation of degree 30. Then there is no relation among BAS_{disc}.) We consider

$$\alpha_1 I_3^9 + \alpha_2 I_3^7 I_6 + \cdots + \alpha_{67} J_{27},$$

where $\alpha_r \in \mathbb{C}$. Disc shuld be satisfied that the coefficients of term excluding m, n and p are 0. We prepare 66 terms in the file `ter-disc.nb`. They are a^9l^{18}, $\cdots$, $a^5e^2i^4j^4k^8l^4$. We put

$$TER_{disc} = \begin{pmatrix} a^9l^{18} \\ \vdots \\ a^5e^2i^4j^4k^8l^4 \end{pmatrix}.$$

Let $\beta_{q,r}$ ($\in \mathbb{C}$) be the coefficient of $TER_{disc}[[q]]$ in the $BAS_{disc}[[r]]$. We obtain

$$\begin{array}{ccccccc}
\alpha_1 I_3^9 & + & \cdots\cdots & + & \alpha_{67}J_{27} & & \\
|| & & & & || & & \\
\alpha_1\beta_{1,1}a^9l^{18} & + & \cdots\cdots & + & \alpha_{67}\beta_{1,67}a^9l^{18} & = & 0 \\
+ & & & & + & & \\
\vdots & & & & \vdots & & \\
+ & & & & + & & \\
\alpha_1\beta_{66,1}a^5e^2i^4j^4k^8l^4 & + & \cdots\cdots & + & \alpha_{67}\beta_{66,67}a^5e^2i^4j^4k^8l^4 & = & 0. \\
+ & & & & + & & \\
\vdots & & & & \vdots & &
\end{array}$$

I.e. we obtain

$$\begin{pmatrix} \beta_{1,1} & \cdots\cdots & \beta_{1,67} \\ \vdots & & \vdots \\ \beta_{66,1} & \cdots\cdots & \beta_{66,67} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_{67} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

We put

$$ENT_{disc} = \begin{pmatrix} \beta_{1,1} & \cdots\cdots & \beta_{1,67} \\ \vdots & & \vdots \\ \beta_{66,1} & \cdots\cdots & \beta_{66,67} \end{pmatrix}.$$

If the rank of ENT_{disc} is 66, then the null-space is just Disc. It is equal to I_{27}. □

3. Relations

Definition 3.1. We define the notation

$$W = \begin{pmatrix} W_1 \\ \vdots \\ \vdots \\ W_{50} \end{pmatrix},$$

where $(W_1, \cdots\cdots, W_{50}) = (1,\ I_{9b},\ I_{12b},\ I_{15b},\ I_{9b}^2,\ I_{18b},\ I_{9b}I_{12b},\ I_{21a},\ I_{21b},\ I_{9b}I_{15b},\ I_{12b}^2,\ I_{9b}^3,\ I_{9b}I_{18b},\ I_{12b}I_{15b},\ I_{9b}^2I_{12b},\ I_{9b}I_{21a},\ I_{9b}I_{21b},\ I_{12b}I_{18b},\ I_{9b}^2I_{15b},\ I_{9b}I_{12b}^2,\ I_{15b}I_{18b},\ I_{9b}^2I_{18b},\ I_{9b}I_{12b}I_{15b},\ I_{15b}I_{21a},\ I_{15b}I_{21b},\ I_{9b}^3I_{12b},\ I_{9b}^2I_{21a},\ I_{9b}^2I_{21b},\ I_{18b}I_{21b},\ I_{9b}I_{15b}I_{18b},\ I_{12b}^2I_{18b},\ I_{21a}I_{21b},\ I_{9b}^2I_{12b}I_{15b},\ I_{9b}I_{15b}I_{21a},\ I_{9b}I_{15b}I_{21b},\ I_{12b}I_{15b}I_{18b},\ I_{9b}^3I_{21a},\ I_{9b}^3I_{21b},\ I_{9b}I_{18b}I_{21b},\ I_{9b}^2I_{15b}I_{18b},\ I_{9b}I_{21a}I_{21b},\ I_{9b}^2I_{15b}I_{21a},\ I_{9b}^2I_{15b}I_{21b},\ I_{15b}I_{18b}I_{21b},\ I_{9b}^2I_{18b}I_{21b},\ I_{15b}I_{21a}I_{21b},\ I_{9b}^2I_{21a}I_{21b},\ I_{9b}I_{15b}I_{18b}I_{21b},\ I_{9b}I_{15b}I_{21a}I_{21b},\ I_{9b}^2I_{15b}I_{21a}I_{21b})$.

Theorem 3.2. *S'' has following 15 independent relations.*

$$\begin{aligned}
R_{30} &= -15680000I_{15b}^2 \\
&\quad + A_1W_1 + \cdots\cdots + A_{18}W_{18} \\
&\quad = 0, \\
R_{33a} &= -54432000000I_{12b}I_{21a}
\end{aligned}$$

$$
\begin{aligned}
&+ B_1W_1 + \cdots\cdots + B_{21}W_{21}\\
&= 0,\\
R_{33b} &= -756000000I_{12b}I_{21b}\\
&+ C_1W_1 + \cdots\cdots + C_{21}W_{21}\\
&= 0,\\
R_{36a} &= -11312206295040000I_{9b}^4\\
&+ D_1W_1 + \cdots\cdots + D_{25}W_{25}\\
&= 0,\\
R_{36b} &= -1119744000000I_{12b}^3\\
&+ E_1W_1 + \cdots\cdots + E_{25}W_{25}\\
&= 0,\\
R_{36c} &= -480090240000I_{18b}^2\\
&+ F_1W_1 + \cdots\cdots + F_{25}W_{25}\\
&= 0,\\
R_{39a} &= -36380363589348116889600000I_{9b}I_{12b}I_{18b}\\
&+ G_1W_1 + \cdots\cdots + G_{29}W_{29}\\
&= 0,\\
R_{39b} &= -92394574195169820672000000I_{12b}^2I_{15b}\\
&+ H_1W_1 + \cdots\cdots + H_{29}W_{29}\\
&= 0,\\
R_{39c} &= -46197287097584910336000000I_{18b}I_{21a}\\
&+ K_1W_1 + \cdots\cdots + K_{29}W_{29}\\
&= 0,\\
R_{42a} &= -34570947031217597982965760000000I_{9b}^3I_{15b}\\
&+ L_1W_1 + \cdots\cdots + L_{32}W_{32}\\
&= 0,\\
R_{42b} &= -16891453898385668290893091700736000000000I_{9b}^2I_{12b}^2\\
&+ M_1W_1 + \cdots\cdots + M_{32}W_{32}\\
&= 0,\\
R_{42c} &= -70944106373219806821750985143091200000000I_{21a}^2\\
&+ N_1W_1 + \cdots\cdots + N_{32}W_{32}\\
&= 0,\\
R_{42d} &= -13137797476522186448472404656128000000000I_{21b}^2\\
&+ P_1W_1 + \cdots\cdots + P_{32}W_{32}\\
&= 0,\\
R_{45} &= -8435059613739145813529177727518638080000000I_{9b}^3I_{18b}\\
&+ Q_1W_1 + \cdots\cdots + Q_{36}W_{36}\\
&= 0,\\
R_{48} &= -6265245515277441978431754609818785415282514229985280000000000I_{9b}^2I_{15b}^2
\end{aligned}
$$

$$+ T_1 W_1 + \cdots\cdots + T_{39} W_{39}$$
$$= 0,$$

where $A_1, A_2, \cdots\cdots\cdots\cdots, T_{38}, T_{39} \in P'$.

$$A_1 = 695296000 I_3^6 I_6^2 + 49503280 I_3^5 I_6 I_{9a} + 27130880000 I_3^4 I_6^3 + 205376000 I_3^4 I_6 I_{12a} + 799578 I_3^4 I_{9a}^2 - 14793029120 I_3^3 I_6^2 I_{9a} + 1231519744000 I_3^2 I_6^4 - 379008000 I_3^3 I_6 I_{15a} - 2297480 I_3^3 I_{9a} I_{12a} + 111713792000 I_3^2 I_6^2 I_{12a} - 430762080 I_3^2 I_6 I_{9a}^2 - 3769352294400 I_3 I_6^3 I_{9a} + 15606743040000 I_6^5 + 169344000 I_3^2 I_6 I_{18a} - 9555840 I_3^2 I_{9a} I_{15a} - 267872000 I_3^2 I_{12a}^2 + 32449536000 I_3 I_6^2 I_{15a} + 9542908800 I_3 I_6 I_{9a} I_{12a} - 2142000 I_3 I_{9a}^3 + 2838011904000 I_6^3 I_{12a} - 64666828800 I_6^2 I_{9a}^2 - 1108800 I_3 I_{9a} I_{18a} + 165312000 I_3 I_{12a} I_{15a} + 40642560000 I_6^2 I_{18a} + 2773612800 I_6 I_{9a} I_{15a} - 42591744000 I_6 I_{12a}^2 + 59270400 I_{9a}^2 I_{12a} - 108864000 I_{12a} I_{18a},$$

$$A_2 = -74135640 I_3^5 I_6 - 2957943 I_3^4 I_{9a} - 50322589440 I_3^3 I_6^2 - 29731260 I_3^3 I_{12a} + 1238296080 I_3^2 I_6 I_{9a} + 695956531200 I_3 I_6^3 + 27457920 I_3^2 I_{15a} - 6539390400 I_3 I_6 I_{12a} + 19686240 I_3 I_{9a}^2 + 195203635200 I_6^2 I_{9a} - 11037600 I_3 I_{18a} - 3491510400 I_6 I_{15a} - 103420800 I_{9a} I_{12a},$$

$$A_3 = -11328000 I_3^4 I_6 + 1087620 I_3^3 I_{9a} - 42176256000 I_3^2 I_6^2 + 86160000 I_3^2 I_{12a} - 2382321600 I_3 I_6 I_{9a} - 228372480000 I_6^3 - 30240000 I_3 I_{15a} + 12807936000 I_6 I_{12a} - 10584000 I_{9a}^2 + 30240000 I_{18a},$$

$$A_4 = 1485568000 I_3^3 I_6 + 44586080 I_3^2 I_{9a} + 294676480000 I_3 I_6^2 - 753536000 I_3 I_{12a} + 4173926400 I_6 I_{9a} - 141120000 I_{15a},$$

$$A_5 = 1664577 I_3^4 + 4629174480 I_3^2 I_6 - 4698540 I_3 I_{9a} - 29373926400 I_6^2 - 253411200 I_{12a},$$

$$A_6 = -1427328000 I_3^2 I_6 - 43596000 I_3 I_{9a} - 48771072000 I_6^2 + 762048000 I_{12a},$$

$$A_7 = 3518190 I_3^3 - 353383200 I_3 I_6 - 21470400 I_{9a},$$

$$A_8 = -8709120000 I_3 I_6 - 108864000 I_{9a},$$

$$A_9 = 1209600000 I_3 I_6 + 18648000 I_{9a},$$

$$A_{10} = -115589040 I_3^2 - 12782515200 I_6,$$

$$A_{11} = -6768000 I_3^2 - 898560000 I_6,$$

$$A_{12} = -122256540 I_3,$$

$$A_{13} = 113022000 I_3,$$

$$A_{14} = 143808000 I_3,$$

$$A_{15} = 176677200,$$

$$A_{16} = 417312000,$$

$$A_{17} = -59724000,$$

$$A_{18} = -127008000,$$

$$B_1 = 31470551448000 I_3^7 I_6^2 + 2251256087640 I_3^6 I_6 I_{9a} + 1213981632640000 I_3^5 I_6^3 + 9967616148000 I_3^5 I_6 I_{12a} + 36684298014 I_3^5 I_{9a}^2 - 669819652194560 I_3^4 I_6^2 I_{9a} + 51584456476672000 I_3^3 I_6^4 - 17127621504000 I_3^4 I_6 I_{15a} - 64439496240 I_3^4 I_{9a} I_{12a}$$

$$
\begin{aligned}
&+ 5058755153536000 I_3^3 I_6^2 I_{12a} - 19787325596640 I_3^3 I_6 I_{9a}^2 - 1676613010770432 00 I_3^2 I_6^3 I_{9a} \\
&+ 643475384401920000 I_3 I_6^5 + 7702061472000 I_3^3 I_6 I_{18a} - 437122465920 I_3^3 I_{9a} I_{15a} \\
&- 115965501960000 I_3^3 I_{12a}^2 + 1396695178368000 I_3^2 I_6^2 I_{15a} + 419234774918400 I_3^2 I_6 I_{9a} I_{12a} \\
&- 98620952400 I_3^2 I_{9a}^3 + 123761283781632000 I_3 I_6^3 I_{12a} - 2825454244838400 I_3 I_6^2 I_{9a}^2 \\
&- 185468944711680000 I_6^4 I_{9a} - 43917854400 I_3^2 I_{9a} I_{18a} + 7150320576000 I_3^2 I_{12a} I_{15a} \\
&+ 1683406932480000 I_3 I_6^2 I_{18a} + 122752795334400 I_3 I_6 I_{9a} I_{15a} - 1829678098752000 I_3 I_6 I_{12a}^2 \\
&+ 2584418587200 I_3 I_{9a}^2 I_{12a} + 1386927360000000 I_6^3 I_{15a} - 510224762880000 I_6^2 I_{9a} I_{12a} \\
&+ 413538048000 I_6 I_{9a}^3 - 4751363232000 I_3 I_{12a} I_{18a} - 1253145600000 I_6 I_{9a} I_{18a} \\
&- 4025427840000 I_6 I_{12a} I_{15a} - 9398592000 I_{9a}^2 I_{15a} + 73163520000 I_{9a} I_{12a}^2, \\
B_2 &= -3332856658320 I_3^6 I_6 - 133171614009 I_3^5 I_{9a} - 2223511448094720 I_3^4 I_6^2 \\
&- 1349476774380 I_3^4 I_{12a} + 56315471642640 I_3^3 I_6 I_{9a} + 27937308583833600 I_3^2 I_6^3 \\
&+ 1230142152960 I_3^3 I_{15a} - 283189446187200 I_3^2 I_6 I_{12a} + 889240425120 I_3^2 I_{9a}^2 \\
&+ 8593690274265600 I_3 I_6^2 I_{9a} + 27206201917440000 I_6^4 - 496933768800 I_3^2 I_{18a} \\
&- 151586212579200 I_3 I_6 I_{15a} - 4441859186400 I_3 I_{9a} I_{12a} + 911136522240000 I_6^2 I_{12a} \\
&- 1982340864000 I_6 I_{9a}^2 + 4013452800000 I_6 I_{18a} + 72394560000 I_{9a} I_{15a} \\
&- 379244160000 I_{12a}^2, \\
B_3 &= -618323454000 I_3^5 I_6 + 43185990060 I_3^4 I_{9a} - 1891598616768000 I_3^3 I_6^2 \\
&+ 3744828810000 I_3^3 I_{12a} - 105431394484800 I_3^2 I_6 I_{9a} - 11864320151040000 I_3 I_6^3 \\
&- 1316800800000 I_3^2 I_{15a} + 548790307488000 I_3 I_6 I_{12a} - 465556680000 I_3 I_{9a}^2 \\
&+ 23069007360000 I_6^2 I_{9a} + 1334581920000 I_3 I_{18a} - 49520160000 I_{9a} I_{12a}, \\
B_4 &= 67704523824000 I_3^4 I_6 + 2056330043040 I_3^3 I_{9a} + 13064232231680000 I_3^2 I_6^2 \\
&- 322803912480000 I_3^2 I_{12a} + 174062768947200 I_3 I_6 I_{9a} + 134180659200000000 I_6^3 \\
&- 6387655680000 I_3 I_{15a} + 397548748800000 I_6 I_{12a} - 38638656000 I_{9a}^2 \\
&- 10584000000 I_{18a}, \\
B_5 &= 746133177751 I_3^5 + 205253934333840 I_3^3 I_6 - 221346887220 I_3^2 I_{9a} \\
&- 1113449164531200 I_3 I_6^2 - 11366179005600 I_3 I_{12a} + 2332628928000 I_6 I_{9a} \\
&- 135454032000 I_{15a}, \\
&\qquad\qquad \vdots \\
&\qquad\qquad \vdots \\
&\qquad\qquad \vdots \\
&\qquad\qquad \vdots
\end{aligned}
$$

All data are in relation.tex, relation.ps or relation.pdf.

Proof. We explain the case of degree 30. We consider monomials of degree 30. They are made by I_3, I_6, I_{9a}, I_{9b}, I_{12a}, I_{12b}, I_{15a}, I_{15b}, I_{18a}, I_{18b}, I_{21a}, I_{21b} and J_{27}. The number of their monomials is 99: I_3^{10}, $I_3^8 I_6$, $\cdots$, I_{15b}^2. We put

$$BAS_{30} = \begin{pmatrix} I_3^{10} \\ I_3^8 I_6 \\ \vdots \\ I_{15b}^2 \end{pmatrix}.$$

By generating function, we have $\dim_{\mathbb{C}} S_{30} = 98$. Then there is at least 1 relation among BAS_{30}. We put

$$\alpha_1 I_3^{10} + \alpha_2 I_3^8 I_6 + \cdots + \alpha_{99} I_{15b}^2 = 0,$$

where $\alpha_r \in \mathbb{C}$. We prepare 98 terms in the file `ter-30.nb`. They are $a^{10}e^{10}p^{10}$, $\cdots$, $a^8c^2e^2i^8l^2m^4p^4$. We put

$$TER_{30} = \begin{pmatrix} a^{10}e^{10}p^{10} \\ \vdots \\ a^8c^2e^2i^8l^2m^4p^4 \end{pmatrix}.$$

Let $\beta_{q,r}$ $(\in \mathbb{C})$ be the coefficient of $TER_{30}[[q]]$ in the $BAS_{30}[[r]]$. We obtain

$$\begin{array}{ccccccc}
\alpha_1 I_3^{10} & + & \cdots\cdots & + & \alpha_{99} I_{15b}^2 & = & 0 \\
\| & & & & \| & & \\
\alpha_1 \beta_{1,1} a^{10}e^{10}p^{10} & + & \cdots\cdots & + & \alpha_{99}\beta_{1,99} a^{10}e^{10}p^{10} & = & 0 \\
+ & & & & + & & \\
\vdots & & & & \vdots & & \\
+ & & & & + & & \\
\alpha_1 \beta_{98,1} a^8c^2e^2i^8l^2m^4p^4 & + & \cdots\cdots & + & \alpha_{99}\beta_{98,99} a^8c^2e^2i^8l^2m^4p^4 & = & 0. \\
+ & & & & + & & \\
\vdots & & & & \vdots & &
\end{array}$$

I.e. we obtain

$$\begin{pmatrix} \beta_{1,1} & \cdots\cdots & \beta_{1,99} \\ \vdots & & \vdots \\ \beta_{98,1} & \cdots\cdots & \beta_{98,99} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_{99} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

We put

$$ENT_{30} = \begin{pmatrix} \beta_{1,1} & \cdots\cdots & \beta_{1,99} \\ \vdots & & \vdots \\ \beta_{98,1} & \cdots\cdots & \beta_{98,99} \end{pmatrix}.$$

We check the rank of ENT_{30} and obtain 98. The null-space is the relation. We exchange J_{27} for I_{27} and obtain R_{30}.

The cases of other degree are similarly proved. □

4. Generators

Theorem 4.1. *$S = S' = S''$.*

$$\begin{aligned} S = \mathbb{C}[&I_3, I_6, I_{9a}, I_{9b}, I_{12a}, I_{12b}, I_{15a}, I_{15b}, I_{18a}, I_{18b}, I_{21a}, I_{21b}, I_{27}] \\ &/(R_{30}, R_{33a}, R_{33b}, R_{36a}, R_{36b}, R_{36c}, R_{39a}, R_{39b}, R_{39c}, R_{42a}, R_{42b}, R_{42c}, R_{42d}, R_{45}, R_{48}). \end{aligned}$$

Proof. By Lemma 4.4, $S' = PW_1 + \cdots\cdots + PW_{50}$. If there is no relation over P among W, then we can say $S = S'$. Lemma 4.2 shows that I_{9b} is an element of rank 50 over P. Lemma 4.5 shows that W is equivalent to $(1,\ I_{9b},\ \cdots,\ I_{9b}^{49})$ over P. Then we can say that there is no relation over P among W. Then $S = S'$. By Theorem 2.4, $S' = S''$. We are done. □

Lemma 4.2. *I_{9b} is an element of rank 50 over P.*

Proof. S has rank 50 over P. It is possible that the rank of I_{9b} is 50, 25, 10, 5 or 2. If we prove that the rank of I_{9b} is not equal or less than 25, then the rank of I_{9b} is 50. We will show that there is no relation in I_{9b} over P on the condition that the degree of I_{9b} is equal or less than 25.

We recall $P = \mathbb{C}[I_3, I_6, I_{9a}, I_{12a}, I_{15a}, I_{18a}, \mathrm{Disc}]$. If we can assume $I_3 = I_6 = I_{9a} = I_{12a} = \mathrm{Disc} = 0$, then I_{15a} and I_{18a} remain. We consider monomials of degree 225. They are made by I_{9b}, I_{15a} and I_{18a}. The number of their monomials is 42: I_{9b}^{25}, $I_{9b}^{23}I_{18a}$, $\cdots$, $I_{15a}^{3}I_{18a}^{10}$. We put

$$ABAS_{225} = \begin{pmatrix} I_{9b}^{25} \\ I_{9b}^{23}I_{18a} \\ \vdots \\ I_{15a}^{3}I_{18a}^{10} \end{pmatrix}.$$

We put

$$\alpha_1 I_{9b}^{25} + \alpha_2 I_{9b}^{23} I_{18a} + \cdots + \alpha_{42} I_{15a}^{3} I_{18a}^{10} = 0,$$

where $\alpha_r \in \mathbb{C}$.

We prepare 14 ternary quartics such that their coefficients are defined by the next Table.

No.	a	b	c	d	e	f	g	h	i	j	k	l	m	n	p
1,2,3	-2	b	$-b$	d	$b-2d+1$	-1	1	-2	5	0	-4	4	0	0	0
4,5,6	-2	b	$-b$	d	$b-2d-6$	1	4	-2	6	0	-8	8	0	0	0
7,8,9	-4	b	$-b$	d	$b-2d+1$	-1	-2	1	2	0	-5	5	0	0	0
10,11,12	-4	b	$-b$	d	$b-2d-1$	-1	3	-2	4	0	-5	5	0	0	0
13,14,15	-4	b	$-b$	d	$b-2d-1$	-2	2	-1	4	0	-8	8	0	0	0
16,17,18	-4	b	$-b$	d	$b-2d-3$	1	-1	-2	2	0	-3	3	0	0	0
19,20,21	-4	b	$-b$	d	$b-2d-4$	-1	-3	4	7	0	-8	8	0	0	0
22,23,24	-4	b	$-b$	d	$b-2d-5$	1	-1	-4	4	0	-5	5	0	0	0
25,26,27	-8	b	$-b$	d	$b-2d+1$	-2	-4	2	1	0	-7	7	0	0	0
28,29,30	-8	b	$-b$	d	$b-2d+2$	-2	-3	1	4	0	-8	8	0	0	0
31,32,33	-8	b	$-b$	d	$b-2d-3$	2	-2	-1	1	0	-3	3	0	0	0
34,35,36	-8	b	$-b$	d	$b-2d-5$	2	-2	-3	3	0	-5	5	0	0	0
37,38,39	-8	b	$-b$	d	$b-2d-7$	2	6	-3	5	0	-7	7	0	0	0
40,41,42	-8	b	$-b$	d	$b-2d-8$	1	-1	-3	3	0	-2	2	0	0	0

We have $\mathrm{Disc} = 0$ in all the cases because m, n and p are equal to 0.

We choose No. 1,2,3. We substitute their values for I_3, I_6, I_{9a}, I_{9b}, I_{12a}, I_{15a} and I_{18a}. We obtain $I_3 = I_6 = 0$. The values of b and d are not yet fixed. We compute the Gröbner basis of (I_{9a}, I_{12a}) with lexicographic monomial order and variable order (d, b). We put the result into

$GB_{1,2,3}$. We have

$$
\begin{aligned}
GB_{1,2,3} = \ (& \ 24739011624966b^8 + 27854709750497728b^7 - 83882941318430 72b^6 \\
& - 7298219973279744b^5 + 5143043055255552 0b^4 - 4390341644265062 4b^3 \\
& - 4589902764802195 2b^2 + 8508012424444720 8b - 3320981915767704 9\ , \\
& - 400304690783107722071695687768b^7 - 451315937476353021468573287055536b^6 \\
& + 685619314609745957375926882795552b^5 + 22149191227148958971428059676672 0b^4 \\
& - 506214233160671495334363075143680b^3 - 423480704005388280684148368249 28b^2 \\
& + 685407340035322901179739704849089b - 362052122350181764939156616482563 \\
& + 106898127799576366253469519375 0d\ ) .
\end{aligned}
$$

We eliminate d from I_{9a}, I_{9b}, I_{12a}, I_{15a} and I_{18a} by using the second component. The first component has 4 real solutions. We choose 3 solutions near 0 among them. We put them into b_1, b_2 and b_3 ($b_1 > b_2 > b_3$). We substitute b_1, b_2 or b_3 for I_{9a}, I_{9b}, I_{12a}, I_{15a} and I_{18a}. We obtain $I_{9a} = I_{12a} = 0$. I_{9b}, I_{15a} and I_{18a} are not equal to 0. We substitute them for $ABAS_{225}[[r]]$. We obtain $\beta_{1,r}$, $\beta_{2,r}$ or $\beta_{3,r}$. Similarly, we repeat this until No. 40,41,42. We obtain

$$
\begin{array}{ccccccc}
\alpha_1\beta_{1,1} & + & \cdots\cdots & + & \alpha_{42}\beta_{1,42} & = & 0, \\
\vdots & & & & & & \\
\vdots & & & & & & \\
\alpha_1\beta_{42,1} & + & \cdots\cdots & + & \alpha_{42}\beta_{42,42} & = & 0.
\end{array}
$$

I.e. we obtain

$$
\begin{pmatrix} \beta_{1,1} & \cdots\cdots & \beta_{1,42} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \beta_{42,1} & \cdots\cdots & \beta_{42,42} \end{pmatrix}
\begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_{42} \end{pmatrix}
=
\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.
$$

The coefficient matrix $(\beta_{q,r})$ has rank 42. We obtain $\alpha_1 = \cdots\cdots = \alpha_{42} = 0$. Then there is no non-trivial relation over $\mathbb{C}[I_{15a}, I_{18a}]$. Then this is so over P. We are done. $\square$

At the moment, we can not say $\mathrm{Disc} = I_{27}$. But we obtain the next Lemma.

Lemma 4.3. *If* $\mathrm{Disc} = 0$, *then* $I_{27} = 0$: $P = P'$ *and* $S' = S''$.

Proof. We refer to the section 4 in Dixmier's paper ([2], 4). If $\mathrm{Disc} = 0$, then there are three cases:

(1) the point $(0 : 0 : 1)$ is a multiple point of order ≥ 3,
(2) the point $(0 : 0 : 1)$ is a double point and the two tangent lines at $(0 : 0 : 1)$ are equal and the double tangent is the line $x = 0$,
(3) the point $(0 : 0 : 1)$ is a double point and the two tangent lines at $(0 : 0 : 1)$ are distinct and these tangent lines are $x = 0$ and $y = 0$.

The condition of each case is as follows:

(1) $j = k = l = m = n = p = 0$,
(2) $k = l = m = n = p = 0$,
(3) $j = l = m = n = p = 0$.

I_{27} is 0 on each condition. $\square$

Lemma 4.4. $S' = PW_1 + \cdots\cdots + PW_{50}$.

Proof. $S' \supset PW_1 + \cdots\cdots + PW_{50}$ is clear. We will prove $S' \subset PW_1 + \cdots\cdots + PW_{50}$. It is sufficient that we prove

$$IW_1 \in PW_1 + \cdots\cdots + PW_{50},$$
$$\vdots$$
$$\vdots$$
$$IW_{50} \in PW_1 + \cdots\cdots + PW_{50},$$

where $I = I_{9b}, I_{12b}, I_{15b}, I_{18b}, I_{21a}, I_{21b}$. (We note that $W_1 = 1$, $W_2 = I_{9b}$, $W_3 = I_{12b}$, $W_4 = I_{15b}$, $W_6 = I_{18b}$, $W_8 = I_{21a}$ and $W_9 = I_{21b}$.)

We recall that I_3, I_6, I_{9a}, I_{12a}, I_{15a}, I_{18a} and Disc are algebraically independent over $\mathbb{C}$. We may assume Disc $= 0$. By Lemma 4.3, we obtain $I_{27} = 0$. I.e. we obtain $P = P'$ and $S' = S''$. Moreover, we may assume $I_3 = I_6 = I_{9a} = I_{18a} = 0$ and $I_{12a} = I_{15a} = 1$. Theoretically, we do not need them. We need them to compute quickly.

We obtain

$$I_{15b}^2 = U_{1,1}W_1 + \cdots\cdots + U_{1,50}W_{50}, \tag{1}$$
$$I_{12b}I_{21a} = U_{2,1}W_1 + \cdots\cdots + U_{2,50}W_{50}, \tag{2}$$
$$I_{12b}I_{21b} = U_{3,1}W_1 + \cdots\cdots + U_{3,50}W_{50}, \tag{3}$$
$$I_{9b}^4 = U_{4,1}W_1 + \cdots\cdots + U_{4,50}W_{50}, \tag{4}$$
$$I_{12b}^3 = U_{5,1}W_1 + \cdots\cdots + U_{5,50}W_{50}, \tag{5}$$
$$I_{18b}^2 = U_{6,1}W_1 + \cdots\cdots + U_{6,50}W_{50}, \tag{6}$$
$$I_{9b}I_{12b}I_{18b} = U_{7,1}W_1 + \cdots\cdots + U_{7,50}W_{50}, \tag{7}$$
$$I_{12b}^2I_{15b} = U_{8,1}W_1 + \cdots\cdots + U_{8,50}W_{50}, \tag{8}$$
$$I_{18b}I_{21a} = U_{9,1}W_1 + \cdots\cdots + U_{9,50}W_{50}, \tag{9}$$
$$I_{9b}^3I_{15b} = U_{10,1}W_1 + \cdots\cdots + U_{10,50}W_{50}, \tag{10}$$
$$I_{9b}^2I_{12b}^2 = U_{11,1}W_1 + \cdots\cdots + U_{11,50}W_{50}, \tag{11}$$
$$I_{21a}^2 = U_{12,1}W_1 + \cdots\cdots + U_{12,50}W_{50}, \tag{12}$$
$$I_{21b}^2 = U_{13,1}W_1 + \cdots\cdots + U_{13,50}W_{50}, \tag{13}$$
$$I_{9b}^3I_{18b} = U_{14,1}W_1 + \cdots\cdots + U_{14,50}W_{50}, \tag{14}$$
$$I_{9b}^2I_{15b}^2 = U_{15,1}W_1 + \cdots\cdots + U_{15,50}W_{50} \tag{15}$$

from the 15 independent relations in Theorem 3.2.

We make $(1) \times I_{9b}$. If the terms of left hand side of relations $(1) \sim (15)$ exist in it, then we apply $(1) \sim (15)$ to it and eliminate the terms from it and denote it by $(*16)$. We solve $(*16)$ for $I_{9b}I_{15b}^2$ and obtain

$$I_{9b}I_{15b}^2 = U_{16,1}W_1 + \cdots\cdots + U_{16,50}W_{50}. \tag{16}$$

We make $(2) \times I_{9b}$, $(3) \times I_{9b}$ and $(1) \times I_{12b}$ respectively. If the terms of left hand side of relations $(1) \sim (16)$ exist in these, then we apply $(1) \sim (16)$ to these and eliminate the terms from these and denote these by $(*17)$, $(*18)$ and $(*19)$ respectively. We solve $(*17)$, $(*18)$ and $(*19)$ for $I_{9b}I_{12b}I_{21a}$, $I_{9b}I_{12b}I_{21b}$ and $I_{12b}I_{15b}^2$ and obtain

$$I_{9b}I_{12b}I_{21a} = U_{17,1}W_1 + \cdots\cdots + U_{17,50}W_{50}, \tag{17}$$
$$I_{9b}I_{12b}I_{21b} = U_{18,1}W_1 + \cdots\cdots + U_{18,50}W_{50}, \tag{18}$$
$$I_{12b}I_{15b}^2 = U_{19,1}W_1 + \cdots\cdots + U_{19,50}W_{50}. \tag{19}$$

We repeat this operation until the degree becomes 96. We show the method in the next Table.

Degree	How to make	Left hand side	Equation number
39	$(1) \times I_{9b}$	$I_{9b}I_{15b}^2$	(16)
42	$(2) \times I_{9b}$	$I_{9b}I_{12b}I_{21a}$	(17)
	$(3) \times I_{9b}$	$I_{9b}I_{12b}I_{21b}$	(18)
	$(1) \times I_{12b}$	$I_{12b}I_{15b}^2$	(19)
45	$(5) \times I_{9b}$	$I_{9b}I_{12b}^3$	(20)
	$(6) \times I_{9b}$	$I_{9b}I_{18b}^2$	(21)
	$(2) \times I_{12b}$	$I_{12b}^2I_{21a}$	(22)
	$(3) \times I_{12b}$	$I_{12b}^2I_{21b}$	(23)
	$(1) \times I_{15b}$	I_{15b}^3	(24)
48	$(4) \times I_{12b}$	$I_{9b}^4I_{12b}$	(25)
	$(7) \times I_{9b}$	$I_{9b}^2I_{12b}I_{18b}$	(26)
	$(8) \times I_{9b}$	$I_{9b}I_{12b}^2I_{15b}$	(27)
	$(9) \times I_{9b}$	$I_{9b}I_{18b}I_{21a}$	(28)
	$(2) \times I_{15b}$	$I_{12b}I_{15b}I_{21a}$	(29)
	$(3) \times I_{15b}$	$I_{12b}I_{15b}I_{21b}$	(30)
	$(6) \times I_{12b}$	$I_{12b}I_{18b}^2$	(31)
	$(1) \times I_{18b}$	$I_{15b}^2I_{18b}$	(32)
51	$(11) \times I_{9b}$	$I_{9b}^3I_{12b}^2$	(33)
	$(17) \times I_{9b}$	$I_{9b}^2I_{12b}I_{21a}$	(34)
	$(18) \times I_{9b}$	$I_{9b}^2I_{12b}I_{21b}$	(35)
	$(7) \times I_{12b}$	$I_{9b}I_{12b}^2I_{18b}$	(36)
	$(19) \times I_{9b}$	$I_{9b}I_{12b}I_{15b}^2$	(37)
	$(12) \times I_{9b}$	$I_{9b}I_{21a}^2$	(38)
	$(13) \times I_{9b}$	$I_{9b}I_{21b}^2$	(39)
	$(9) \times I_{12b}$	$I_{12b}I_{18b}I_{21a}$	(40)
	$(3) \times I_{18b}$	$I_{12b}I_{18b}I_{21b}$	(41)
	$(1) \times I_{21a}$	$I_{15b}^2I_{21a}$	(42)
	$(1) \times I_{21b}$	$I_{15b}^2I_{21b}$	(43)
	$(6) \times I_{15b}$	$I_{15b}I_{18b}^2$	(44)
54	$(10) \times I_{12b}$	$I_{9b}^3I_{12b}I_{15b}$	(45)
	$(21) \times I_{9b}$	$I_{9b}^2I_{18b}^2$	(46)
	$(22) \times I_{9b}$	$I_{9b}I_{12b}^2I_{21a}$	(47)
	$(23) \times I_{9b}$	$I_{9b}I_{12b}^2I_{21b}$	(48)
	$(7) \times I_{15b}$	$I_{9b}I_{12b}I_{15b}I_{18b}$	(49)
	$(5) \times I_{18b}$	$I_{12b}^3I_{18b}$	(50)
	$(3) \times I_{21a}$	$I_{12b}I_{21a}I_{21b}$	(51)
	$(9) \times I_{15b}$	$I_{15b}I_{18b}I_{21a}$	(52)

Degree	How to make	Left hand side	Equation number
57	(4) $\times I_{21a}$	$I_{9b}^4 I_{21a}$	(53)
	(4) $\times I_{21b}$	$I_{9b}^4 I_{21b}$	(54)
	(26) $\times I_{9b}$	$I_{9b}^3 I_{12b} I_{18b}$	(55)
	(27) $\times I_{9b}$	$I_{9b}^2 I_{12b}^2 I_{15b}$	(56)
	(28) $\times I_{9b}$	$I_{9b}^2 I_{18b} I_{21a}$	(57)
	(29) $\times I_{9b}$	$I_{9b} I_{12b} I_{15b} I_{21a}$	(58)
	(30) $\times I_{9b}$	$I_{9b} I_{12b} I_{15b} I_{21b}$	(59)
	(32) $\times I_{9b}$	$I_{9b} I_{15b}^2 I_{18b}$	(60)
	(8) $\times I_{18b}$	$I_{12b}^2 I_{15b} I_{18b}$	(61)
	(24) $\times I_{12b}$	$I_{12b} I_{15b}^3$	(62)
	(12) $\times I_{15b}$	$I_{15b} I_{21a}^2$	(63)
	(13) $\times I_{15b}$	$I_{15b} I_{21b}^2$	(64)
	(6) $\times I_{21b}$	$I_{18b}^2 I_{21b}$	(65)
60	(34) $\times I_{9b}$	$I_{9b}^3 I_{12b} I_{21a}$	(66)
	(35) $\times I_{9b}$	$I_{9b}^3 I_{12b} I_{21b}$	(67)
	(14) $\times I_{15b}$	$I_{9b}^3 I_{15b} I_{18b}$	(68)
	(37) $\times I_{9b}$	$I_{9b}^2 I_{12b} I_{15b}^2$	(69)
	(38) $\times I_{9b}$	$I_{9b}^2 I_{21a}^2$	(70)
	(39) $\times I_{9b}$	$I_{9b}^2 I_{21b}^2$	(71)
	(41) $\times I_{9b}$	$I_{9b} I_{12b} I_{18b} I_{21b}$	(72)
	(42) $\times I_{9b}$	$I_{9b} I_{15b}^2 I_{21a}$	(73)
	(43) $\times I_{9b}$	$I_{9b} I_{15b}^2 I_{21b}$	(74)
	(44) $\times I_{9b}$	$I_{9b} I_{15b} I_{18b}^2$	(75)
	(31) $\times I_{12b}$	$I_{12b}^2 I_{18b}^2$	(76)
	(32) $\times I_{12b}$	$I_{12b} I_{15b}^2 I_{18b}$	(77)
	(24) $\times I_{15b}$	I_{15b}^4	(78)
	(9) $\times I_{21b}$	$I_{18b} I_{21a} I_{21b}$	(79)
	(13) $\times I_{18b}$	$I_{18b} I_{21b}^2$	(80)
63	(10) $\times I_{21a}$	$I_{9b}^3 I_{15b} I_{21a}$	(81)
	(10) $\times I_{21b}$	$I_{9b}^3 I_{15b} I_{21b}$	(82)
	(49) $\times I_{9b}$	$I_{9b}^2 I_{12b} I_{15b} I_{18b}$	(83)
	(51) $\times I_{9b}$	$I_{9b} I_{12b} I_{21a} I_{21b}$	(84)
	(52) $\times I_{9b}$	$I_{9b} I_{15b} I_{18b} I_{21a}$	(85)
	(40) $\times I_{12b}$	$I_{12b}^2 I_{18b} I_{21a}$	(86)
	(41) $\times I_{12b}$	$I_{12b}^2 I_{18b} I_{21b}$	(87)
	(44) $\times I_{12b}$	$I_{12b} I_{15b} I_{18b}^2$	(88)
	(12) $\times I_{21b}$	$I_{21a}^2 I_{21b}$	(89)
	(13) $\times I_{21a}$	$I_{21a} I_{21b}^2$	(90)
66	(57) $\times I_{9b}$	$I_{9b}^3 I_{18b} I_{21a}$	(91)
	(14) $\times I_{21b}$	$I_{9b}^3 I_{18b} I_{21b}$	(92)
	(58) $\times I_{9b}$	$I_{9b}^2 I_{12b} I_{15b} I_{21a}$	(93)
	(59) $\times I_{9b}$	$I_{9b}^2 I_{12b} I_{15b} I_{21b}$	(94)
	(60) $\times I_{9b}$	$I_{9b}^2 I_{15b}^2 I_{18b}$	(95)
	(63) $\times I_{9b}$	$I_{9b} I_{15b} I_{21a}^2$	(96)
	(64) $\times I_{9b}$	$I_{9b} I_{15b} I_{21b}^2$	(97)
	(65) $\times I_{9b}$	$I_{9b} I_{18b}^2 I_{21b}$	(98)
	(52) $\times I_{12b}$	$I_{12b} I_{15b} I_{18b} I_{21a}$	(99)
	(41) $\times I_{15b}$	$I_{12b} I_{15b} I_{18b} I_{21b}$	(100)

Degree	How to make	Left hand side	Equation number
69	$(70)\times I_{9b}$	$I_{9b}^3 I_{21a}^2$	(101)
	$(32)\times I_{21b}$	$I_{9b}^3 I_{21a} I_{21b}$	(102)
	$(71)\times I_{9b}$	$I_{9b}^3 I_{21b}^2$	(103)
	$(72)\times I_{9b}$	$I_{9b}^2 I_{12b} I_{18b} I_{21b}$	(104)
	$(73)\times I_{9b}$	$I_{9b}^2 I_{15b}^2 I_{21a}$	(105)
	$(74)\times I_{9b}$	$I_{9b}^2 I_{15b}^2 I_{21b}$	(106)
	$(75)\times I_{9b}$	$I_{9b}^2 I_{15b} I_{18b}^2$	(107)
	$(79)\times I_{9b}$	$I_{9b} I_{18b} I_{21a} I_{21b}$	(108)
	$(80)\times I_{9b}$	$I_{9b} I_{18b} I_{21b}^2$	(109)
	$(51)\times I_{15b}$	$I_{12b} I_{15b} I_{21a} I_{21b}$	(110)
	$(43)\times I_{18b}$	$I_{15b}^2 I_{18b} I_{21b}$	(111)
72	$(84)\times I_{9b}$	$I_{9b}^2 I_{12b} I_{21a} I_{21b}$	(112)
	$(85)\times I_{9b}$	$I_{9b}^2 I_{15b} I_{18b} I_{21a}$	(113)
	$(62)\times I_{15b}$	$I_{9b}^2 I_{15b} I_{18b} I_{21b}$	(114)
	$(78)\times I_{12b}$	$I_{12b} I_{15b}^4$	(115)
	$(89)\times I_{9b}$	$I_{9b} I_{21a}^2 I_{21b}$	(116)
	$(90)\times I_{9b}$	$I_{9b} I_{21a} I_{21b}^2$	(117)
	$(43)\times I_{21a}$	$I_{15b}^2 I_{21a} I_{21b}$	(118)
	$(65)\times I_{15b}$	$I_{15b} I_{18b}^2 I_{21b}$	(119)
75	$(96)\times I_{9b}$	$I_{9b}^2 I_{15b} I_{21a}^2$	(120)
	$(97)\times I_{9b}$	$I_{9b}^2 I_{15b} I_{21b}^2$	(121)
	$(98)\times I_{9b}$	$I_{9b}^2 I_{18b}^2 I_{21b}$	(122)
	$(100)\times I_{9b}$	$I_{9b} I_{12b} I_{15b} I_{18b} I_{21b}$	(123)
	$(79)\times I_{15b}$	$I_{15b} I_{18b} I_{21a} I_{21b}$	(124)
	$(80)\times I_{15b}$	$I_{15b} I_{18b} I_{21b}^2$	(125)
78	$(108)\times I_{9b}$	$I_{9b}^2 I_{18b} I_{21a} I_{21b}$	(126)
	$(109)\times I_{9b}$	$I_{9b}^2 I_{18b} I_{21b}^2$	(127)
	$(110)\times I_{9b}$	$I_{9b} I_{12b} I_{15b} I_{21a} I_{21b}$	(128)
	$(111)\times I_{9b}$	$I_{9b} I_{15b}^2 I_{18b} I_{21b}$	(129)
	$(89)\times I_{15b}$	$I_{15b} I_{21a}^2 I_{21b}$	(130)
	$(90)\times I_{15b}$	$I_{15b} I_{21a} I_{21b}^2$	(131)
81	$(116)\times I_{9b}$	$I_{9b}^2 I_{21a}^2 I_{21b}$	(132)
	$(117)\times I_{9b}$	$I_{9b}^2 I_{21a} I_{21b}^2$	(133)
	$(118)\times I_{9b}$	$I_{9b} I_{15b}^2 I_{21a} I_{21b}$	(134)
	$(119)\times I_{9b}$	$I_{9b} I_{15b} I_{18b}^2 I_{21b}$	(135)
84	$(102)\times I_{15b}$	$I_{9b}^3 I_{15b} I_{21a} I_{21b}$	(136)
	$(124)\times I_{9b}$	$I_{9b} I_{15b} I_{18b} I_{21a} I_{21b}$	(137)
	$(125)\times I_{9b}$	$I_{9b} I_{15b} I_{18b} I_{21b}^2$	(138)
87	$(128)\times I_{9b}$	$I_{9b}^2 I_{12b} I_{15b} I_{21a} I_{21b}$	(139)
	$(130)\times I_{9b}$	$I_{9b} I_{15b} I_{21a}^2 I_{21b}$	(140)
	$(131)\times I_{9b}$	$I_{9b} I_{15b} I_{21a} I_{21b}^2$	(141)
90	$(134)\times I_{9b}$	$I_{9b}^2 I_{15b}^2 I_{21a} I_{21b}$	(142)
93	$(137)\times I_{9b}$	$I_{9b}^2 I_{15b} I_{18b} I_{21a} I_{21b}$	(143)
96	$(140)\times I_{9b}$	$I_{9b}^2 I_{15b} I_{21a}^2 I_{21b}$	(144)
	$(141)\times I_{9b}$	$I_{9b}^2 I_{15b} I_{21a} I_{21b}^2$	(145)

We obtain the multiplication table.

	I_{9b}	I_{12b}	I_{15b}	I_{18b}	I_{21a}	I_{21b}
W_1	W_2	W_3	W_4	W_6	W_8	W_9
W_2	W_5	W_7	W_{10}	W_{13}	W_{16}	W_{17}
W_3	W_7	W_{11}	W_{14}	W_{18}	(2)	(3)
W_4	W_{10}	W_{14}	(1)	W_{21}	W_{24}	W_{25}
W_5	W_{12}	W_{15}	W_{19}	W_{22}	W_{27}	W_{28}
W_6	W_{13}	W_{18}	W_{21}	(6)	(9)	W_{29}
W_7	W_{15}	W_{20}	W_{23}	(7)	(17)	(18)
W_8	W_{16}	(2)	W_{24}	(9)	(12)	W_{32}
W_9	W_{17}	(3)	W_{25}	W_{29}	W_{32}	(13)
W_{10}	W_{19}	W_{23}	(16)	W_{30}	W_{34}	W_{35}
W_{11}	W_{20}	(5)	(8)	W_{31}	(22)	(23)
W_{12}	(4)	W_{26}	(10)	(14)	W_{37}	W_{38}
W_{13}	W_{22}	(7)	W_{30}	(21)	(28)	W_{39}
W_{14}	W_{23}	(8)	(19)	W_{36}	(29)	(30)
W_{15}	W_{26}	(11)	W_{33}	(26)	(34)	(35)
W_{16}	W_{27}	(17)	W_{34}	(28)	(38)	W_{41}
W_{17}	W_{28}	(18)	W_{35}	W_{39}	W_{41}	(39)
W_{18}	(7)	W_{31}	W_{36}	(31)	(40)	(41)
W_{19}	(10)	W_{33}	(15)	W_{40}	W_{42}	W_{43}
W_{20}	(11)	(20)	(27)	(36)	(47)	(48)
W_{21}	W_{30}	W_{36}	(32)	(44)	(52)	W_{44}
W_{22}	(14)	(26)	W_{40}	(46)	(57)	W_{45}
W_{23}	W_{33}	(27)	(37)	(49)	(58)	(59)
W_{24}	W_{34}	(29)	(42)	(52)	(63)	W_{46}
W_{25}	W_{35}	(30)	(43)	W_{44}	W_{46}	(64)
W_{26}	(25)	(33)	(45)	(55)	(66)	(67)
W_{27}	W_{37}	(34)	W_{42}	(57)	(70)	W_{47}
W_{28}	W_{38}	(35)	W_{43}	W_{45}	W_{47}	(71)
W_{29}	W_{39}	(41)	W_{44}	(65)	(79)	(80)
W_{30}	W_{40}	(49)	(60)	(75)	(85)	W_{48}
W_{31}	(36)	(50)	(61)	(76)	(86)	(87)
W_{32}	W_{41}	(51)	W_{46}	(79)	(89)	(90)
W_{33}	(45)	(56)	(69)	(83)	(93)	(94)
W_{34}	W_{42}	(58)	(73)	(85)	(96)	W_{49}
W_{35}	W_{43}	(59)	(74)	W_{48}	W_{49}	(97)
W_{36}	(49)	(61)	(77)	(88)	(99)	(100)
W_{37}	(53)	(66)	(81)	(91)	(101)	(102)
W_{38}	(54)	(67)	(82)	(92)	(102)	(103)
W_{39}	W_{45}	(72)	W_{48}	(98)	(108)	(109)
W_{40}	(68)	(83)	(95)	(107)	(113)	(114)
W_{41}	W_{47}	(84)	W_{49}	(108)	(116)	(117)
W_{42}	(81)	(93)	(105)	(113)	(120)	W_{50}
W_{43}	(82)	(94)	(106)	(114)	W_{50}	(121)
W_{44}	W_{48}	(100)	(111)	(119)	(124)	(125)
W_{45}	(92)	(104)	(114)	(122)	(126)	(127)
W_{46}	W_{49}	(110)	(118)	(124)	(130)	(131)
W_{47}	(102)	(112)	W_{50}	(126)	(132)	(133)
W_{48}	(114)	(123)	(129)	(135)	(137)	(138)
W_{49}	W_{50}	(128)	(134)	(137)	(140)	(141)
W_{50}	(136)	(139)	(142)	(143)	(144)	(145)

We are done. □

Lemma 4.5. *W is equivalent to $(1,\ I_{9b},\ \cdots,\ I_{9b}^{49})$ over P.*

Proof. We show the existence of matrix $U_{I_{9b}}$ which is rank 50 over P and satisfy

$$\begin{pmatrix} 1 \\ I_{9b} \\ \vdots \\ I_{9b}^{49} \end{pmatrix} = U_{I_{9b}} \begin{pmatrix} W_1 \\ W_2 \\ \vdots \\ W_{50} \end{pmatrix}.$$

We may assume the same condition as Lemma 4.4.

We prepare the relation (4):

$$I_{9b}^4 = U_{4,1}W_1 + \cdots\cdots + U_{4,50}W_{50}.$$

We make (4) $\times$ I_{9b}. We use the multiplication table in the proof of Lemma 4.4. We express it by using W. We obtain

$$I_{9b}^5 = U_{146,1}W_1 + \cdots\cdots + U_{146,50}W_{50}.$$

We make (above relation) $\times$ I_{9b}. We use the multiplication table in the proof of Lemma 4.4. We express it by using W. We obtain

$$I_{9b}^6 = U_{147,1}W_1 + \cdots\cdots + U_{147,50}W_{50}.$$

We repeat this operation until the degree of I_{9b} becomes 49:

$$\begin{aligned} I_{9b}^7 &= U_{148,1}W_1 + \cdots\cdots + U_{148,50}W_{50}, \\ I_{9b}^8 &= U_{149,1}W_1 + \cdots\cdots + U_{149,50}W_{50}, \\ &\vdots \\ &\vdots \\ I_{9b}^{48} &= U_{189,1}W_1 + \cdots\cdots + U_{189,50}W_{50}, \\ I_{9b}^{49} &= U_{190,1}W_1 + \cdots\cdots + U_{190,50}W_{50}. \end{aligned}$$

We obtain

$$U_{I_{9b}} = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & & 1 & & & \\ & & & & & 1 & & \\ U_{4,1} & U_{4,2} & \cdots & U_{4,5} & \cdots\cdots & U_{4,12} & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & U_{4,50} \\ U_{146,1} & U_{146,2} & \cdots & U_{146,5} & \cdots\cdots & U_{146,12} & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & U_{146,50} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ U_{190,1} & U_{190,2} & \cdots & U_{190,5} & \cdots\cdots & U_{190,12} & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & U_{190,50} \end{pmatrix}.$$

The rank of $U_{I_{9b}}$ is practically 50. □

Remark 4.6. We can prove Lemma 4.5 on the condition that we use I_{12b}, I_{15b}, I_{18b}, I_{21a} or I_{21b} in place of I_{9b}. I.e. we can show the existence of matrix U_I which is rank 50 over P and satisfy

$$\begin{pmatrix} 1 \\ I \\ \vdots \\ I^{49} \end{pmatrix} = U_I \begin{pmatrix} W_1 \\ W_2 \\ \vdots \\ W_{50} \end{pmatrix},$$

where $I = I_{12b}, I_{15b}, I_{18b}, I_{21a}, I_{21b}$. Then we can say that I_{12b}, I_{15b}, I_{18b}, I_{21a} and I_{21b} are also elements of rank 50 over P.

5. Appendix 1 - First syzygy of degree 48

The first syzygies have the form

$$\theta_1 R_{30} + \theta_2 R_{33a} + \cdots + \theta_{15} R_{48} = 0,$$

where θ_i are the polynomials expression of I_3, I_6, I_{9a}, I_{9b}, I_{12a}, I_{12b}, I_{15a}, I_{15b}, I_{18a}, I_{18b}, I_{21a}, I_{21b} and I_{27}. We express it by

$$(\theta_1,\ \theta_2,\ \cdots,\ \theta_{15}).$$

We obtain the first syzygy of degree 48 in the file `v48.nb`. Let V_{48} denote it.

Proposition 5.1. *The V_{48} is the first syzygy of degree 48.*

Proof. We put

$$R = (R_{30},\ R_{33a},\ \cdots,\ R_{48}).$$

We compute the inner product of V_{48} and R. We obtain 0. □

6. Appendix 2 - Computer files

Almost all the computation in this paper can not be computed by the human power. Then we use computer. We made some computer files. We put them on a directory named `S-3-4`. Please get the directory `S-3-4` by download.

In §6.1, we describe the expression of covariants and contravariants in computer programing. In §6.2, we describe each file. In §6.3, we describe the proofs in this paper by using their files.

6.1. **Expression of covariants and contravariants.** In computer programing, we express covariants and contravariants by list by using the next order. Let A be a covariant of order q:

$$\begin{aligned} A =& A_{q,0,0}x^q + A_{q-1,1,0}x^{q-1}y + \cdots + A_{1,q-1,0}xy^{q-1} + A_{0,q,0}y^q \\ &+ A_{q-1,0,1}x^{q-1}z + \cdots\cdots\cdots + A_{0,q-1,1}y^{q-1}z \\ &+ \cdots\cdots\cdots \\ &+ A_{1,0,q-1}xz^{q-1} + A_{0,1,q-1}yz^{q-1} \\ &+ A_{0,0,q}z^q. \end{aligned}$$

We express A by

$$\begin{gathered} \{\ A_{q,0,0},\ A_{q-1,1,0},\ \cdots\cdots\cdots,\ A_{1,q-1,0},\ A_{0,q,0}, \\ A_{q-1,0,1},\ \cdots\cdots\cdots,\ A_{0,q-1,1}, \\ \cdots\cdots\cdots, \\ A_{1,0,q-1},\ A_{0,1,q-1}, \\ A_{0,0,q}\ \}. \end{gathered}$$

If we replace A, q, x, y and z with B, r, u, v and w in the above, then we obtain a contravariant B of order r and its expression.

Let A be a co(contra)variant of order 4:

$$A = \{\ A[[1]],\ A[[2]],\ \cdots,\ A[[15]]\ \}.$$

Let B be a contra(co)variant of order 6:

$$B = \{\ B[[1]],\ B[[2]],\ \cdots,\ B[[28]]\ \}.$$

In our saving form, the differential operator $\mathrm{D}_A B$ is equivalent to as follows:

$$\begin{aligned}
\mathrm{D}_A B = \{\ & 360A[[1]]B[[1]] + 60A[[2]]B[[2]] + \cdots + 24A[[15]]B[[23]], \\
& 120A[[1]]B[[2]] + 48A[[2]]B[[3]] + \cdots + 24A[[15]]B[[24]], \\
& \quad \cdots\cdots\cdots, \\
& 24A[[1]]B[[14]] + 6A[[2]]B[[15]] + \cdots + 360A[[15]]B[[28]]\ \} \\
= \{\ & t0[[1,1]]A[[1]]B[[t2[[1,1]]]] + t0[[1,2]]A[[2]]B[[t2[[1,2]]]] + \\
& \qquad \cdots + t0[[1,15]]A[[15]]B[[t2[[1,15]]]], \\
& t0[[2,1]]A[[1]]B[[t2[[2,1]]]] + t0[[2,2]]A[[2]]B[[t2[[2,2]]]] + \\
& \qquad \cdots + t0[[2,15]]A[[15]]B[[t2[[2,15]]]], \\
& \quad \cdots\cdots\cdots, \\
& t0[[6,1]]A[[1]]B[[t2[[6,1]]]] + t0[[6,2]]A[[2]]B[[t2[[6,2]]]] + \\
& \qquad \cdots + t0[[6,15]]A[[15]]B[[t2[[6,15]]]]\ \},
\end{aligned}$$

where $t0$ and $t2$ are matrixes in the file of d-4-6.nb. The file name d-4-6.nb means that a co(contra)variant of order 4 operates on a contra(co)variant of order 6. Similarly, we made the files of d-2-4.nb and d-2-6.nb more.

6.2. **Description of files.** We describe all the file names on the directory `S-3-4` as follows.

`abas-225.nb`	`con-psi.nb`	`j11.nb`	`relation.pdf`
`bas-30.nb`	`con-rho.nb`	`j3.nb`	`relation.ps`
`bas-33.nb`	`con-sigma.nb`	`jexc27.nb`	`relation.tex`
`bas-36.nb`	`cov-he.nb`	`make-he.nb`	`sec-4-lemma-i9b.nb`
`bas-39.nb`	`cov-tau.nb`	`make-i3.nb`	`sec-4-lemma-i12b.nb`
`bas-42.nb`	`cov-varphi.nb`	`make-psi.nb`	`sec-4-lemma-i15b.nb`
`bas-45.nb`	`cov-xi.nb`	`make-sigma.nb`	`sec-4-lemma-i18b.nb`
`bas-48.nb`	`d-2-4.nb`	`quarticsdata.nb`	`sec-4-lemma-i21a.nb`
`bas-disc.nb`	`d-2-6.nb`	`r30.nb`	`sec-4-lemma-i21b.nb`
`coe-30.nb`	`d-4-6.nb`	`r33a.nb`	`sumdisc.nb`
`coe-33a.nb`	`disc.nb`	`r33b.nb`	`ter-30.nb`
`coe-33b.nb`	`dual.nb`	`r36a.nb`	`ter-33.nb`
`coe-36a.nb`	`entry-30.nb`	`r36b.nb`	`ter-36.nb`
`coe-36b.nb`	`entry-33.nb`	`r36c.nb`	`ter-39.nb`
`coe-36c.nb`	`entry-36.nb`	`r39a.nb`	`ter-42.nb`
`coe-39a.nb`	`entry-39.nb`	`r39b.nb`	`ter-45.nb`
`coe-39b.nb`	`entry-42.nb`	`r39c.nb`	`ter-48.nb`
`coe-39c.nb`	`entry-45.nb`	`r42a.nb`	`ter-disc.nb`
`coe-42a.nb`	`entry-48.nb`	`r42b.nb`	`unexistence.nb`
`coe-42b.nb`	`entry-disc.nb`	`r42c.nb`	`unexistsolulist.nb`
`coe-42c.nb`	`entry.c`	`r42d.nb`	`v48.nb`
`coe-42d.nb`	`entry.nb`	`r45.nb`	
`coe-45.nb`	`inv-i3.nb`	`r48.nb`	
`coe-48.nb`	`inv-i6.nb`	`rank.nb`	
`coe-disc.nb`	`inv-sec-4.nb`	`relalist.nb`	

The files are classified as follows.

`*.c`	:	Source program of C,
`*.nb`	:	Mathematica file,
`*.pdf`	:	pdf-file of TeX,
`*.ps`	:	ps-file of TeX,
`*.tex`	:	Source file of TeX.

We describe each file. Moreover, if it is a program file, then we add its list and how to use it. And moreover, if the program file uses other files, then we also add their file names. In this way, some file uses other files. Then please compute on the directory `S-3-4`. We use the computer which has the specification of 3GHz, 1GB.

6.2.1. `abas-225.nb`.

$$\texttt{abas-225.nb} = ABAS_{225}.$$

6.2.2. `bas-*.nb`.

$$\begin{aligned} \texttt{bas-30.nb} &= BAS_{30}, \\ \texttt{bas-33.nb} &= BAS_{33}, \\ \texttt{bas-36.nb} &= BAS_{36}, \\ \texttt{bas-39.nb} &= BAS_{39}, \\ \texttt{bas-42.nb} &= BAS_{42}, \\ \texttt{bas-45.nb} &= BAS_{45}, \\ \texttt{bas-48.nb} &= BAS_{48}, \\ \texttt{bas-disc.nb} &= BAS_{disc}. \end{aligned}$$

6.2.3. `coe-*.nb`.

`coe-*.nb` are the lists which satisfy

coe-30.nb	inner product	bas-30.nb	=	r'30,
coe-33a.nb	inner product	bas-33.nb	=	r'33a,
coe-33b.nb	inner product	bas-33.nb	=	r'33b,
coe-36a.nb	inner product	bas-36.nb	=	r'36a,
coe-36b.nb	inner product	bas-36.nb	=	r'36b,
coe-36c.nb	inner product	bas-36.nb	=	r'36c,
coe-39a.nb	inner product	bas-39.nb	=	r'39a,
coe-39b.nb	inner product	bas-39.nb	=	r'39b,
coe-39c.nb	inner product	bas-39.nb	=	r'39c,
coe-42a.nb	inner product	bas-42.nb	=	r'42a,
coe-42b.nb	inner product	bas-42.nb	=	r'42b,
coe-42c.nb	inner product	bas-42.nb	=	r'42c,
coe-42d.nb	inner product	bas-42.nb	=	r'42d,
coe-45.nb	inner product	bas-45.nb	=	r'45,
coe-48.nb	inner product	bas-48.nb	=	r'48,
coe-disc.nb	inner product	bas-disc.nb	=	disc.nb,

where `r'*` are following relations. We apply the inverse of `jexc27.nb` to `r*.nb`. `r'*` denote them.

6.2.4. `con-*.nb` and `cov-*.nb`.

con-psi.nb	=	ψ,
con-rho.nb	=	ρ,
con-sigma.nb	=	σ,
cov-he.nb	=	He,
cov-tau.nb	=	τ,
cov-varphi.nb	=	φ,
cov-xi.nb	=	ξ.

6.2.5. `d-2-4.nb`.

Let A be a co(contra)variant of order 2 and B be a contra(co)variant of order 4.

$$\texttt{d-2-4.nb} = \mathrm{D}_A B.$$

List

```
t0={{ 12,  3,  2,  3,  1,  2},
    {  6,  4,  6,  2,  2,  2},
    {  2,  3, 12,  1,  3,  2},
    {  6,  2,  2,  4,  2,  6},
    {  2,  2,  6,  2,  4,  6},
    {  2,  1,  2,  3,  3, 12}}
t2={{  1,  2,  3,  6,  7, 10},
    {  2,  3,  4,  7,  8, 11},
    {  3,  4,  5,  8,  9, 12},
    {  6,  7,  8, 10, 11, 13},
    {  7,  8,  9, 11, 12, 14},
```

```
    { 10, 11, 12, 13, 14, 15}}
cc3={  0,  0,  0,  0,  0,  0}
For[q=1,q<=6,q++,
  For[r=1,r<=6,r++,
    cc3[[q]]+=t0[[q,r]]*cc1[[r]]*cc2[[t2[[q,r]]]]/t
  ]
]
```

How to use
Similar to `d-4-6.nb`.

6.2.6. `d-2-6.nb`.

Let A be a co(contra)variant of order 2 and B be a contra(co)variant of order 6.

$$\texttt{d-2-6.nb} \quad = \quad \mathrm{D}_A B.$$

List

```
t0={{ 30,  5,  2,  5,  1,  2},
    { 20,  8,  6,  4,  2,  2},
    { 12,  9, 12,  3,  3,  2},
    {  6,  8, 20,  2,  4,  2},
    {  2,  5, 30,  1,  5,  2},
    { 20,  4,  2,  8,  2,  6},
    { 12,  6,  6,  6,  4,  6},
    {  6,  6, 12,  4,  6,  6},
    {  2,  4, 20,  2,  8,  6},
    { 12,  3,  2,  9,  3, 12},
    {  6,  4,  6,  6,  6, 12},
    {  2,  3, 12,  3,  9, 12},
    {  6,  2,  2,  8,  4, 20},
    {  2,  2,  6,  4,  8, 20},
    {  2,  1,  2,  5,  5, 30}}
t2={{  1,  2,  3,  8,  9, 14},
    {  2,  3,  4,  9, 10, 15},
    {  3,  4,  5, 10, 11, 16},
    {  4,  5,  6, 11, 12, 17},
    {  5,  6,  7, 12, 13, 18},
    {  8,  9, 10, 14, 15, 19},
    {  9, 10, 11, 15, 16, 20},
    { 10, 11, 12, 16, 17, 21},
    { 11, 12, 13, 17, 18, 22},
    { 14, 15, 16, 19, 20, 23},
    { 15, 16, 17, 20, 21, 24},
    { 16, 17, 18, 21, 22, 25},
    { 19, 20, 21, 23, 24, 26},
    { 20, 21, 22, 24, 25, 27},
    { 23, 24, 25, 26, 27, 28}}
cc3={  0,  0,  0,  0,  0,  0,  0,  0,  0,  0,  0,  0,  0,  0,  0}
For[q=1,q<=15,q++,
```

```
  For[r=1,r<=6,r++,
    cc3[[q]]+=t0[[q,r]]*cc1[[r]]*cc2[[t2[[q,r]]]]/t
  ]
]
```

How to use
Similar to `d-4-6.nb`.

6.2.7. `d-4-6.nb`.

Let A be a co(contra)variant of order 4 and B be a contra(co)variant of order 6.

$$\texttt{d-4-6.nb} \quad = \quad \mathrm{D}_A B.$$

List

```
t0={{360, 60, 24, 18, 24, 60, 12,  6,  6, 24,  6,  4, 18,  6, 24},
    {120, 48, 36, 48,120, 24, 12, 12, 24, 12,  8, 12, 12, 12, 24},
    { 24, 18, 24, 60,360,  6,  6, 12, 60,  4,  6, 24,  6, 18, 24},
    {120, 24, 12, 12, 24, 48, 12,  8, 12, 36, 12, 12, 48, 24,120},
    { 24, 12, 12, 24,120, 12,  8, 12, 48, 12, 12, 36, 24, 48,120},
    { 24,  6,  4,  6, 24, 18,  6,  6, 18, 24, 12, 24, 60, 60,360}}
t2={{  1,  2,  3,  4,  5,  8,  9, 10, 11, 14, 15, 16, 19, 20, 23},
    {  2,  3,  4,  5,  6,  9, 10, 11, 12, 15, 16, 17, 20, 21, 24},
    {  3,  4,  5,  6,  7, 10, 11, 12, 13, 16, 17, 18, 21, 22, 25},
    {  8,  9, 10, 11, 12, 14, 15, 16, 17, 19, 20, 21, 23, 24, 26},
    {  9, 10, 11, 12, 13, 15, 16, 17, 18, 20, 21, 22, 24, 25, 27},
    { 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28}}
cc3={  0,  0,  0,  0,  0,  0}
For[q=1,q<=6,q++,
  For[r=1,r<=15,r++,
    cc3[[q]]+=t0[[q,r]]*cc1[[r]]*cc2[[t2[[q,r]]]]/t
  ]
]
```

How to use
For example $\rho = \mathrm{D}_\varphi \psi$. The order of φ is 4. The order of ψ is 6.

```
In[1]:=cc1=Get["cov-varphi.nb"];
In[2]:=cc2=Get["con-psi.nb"];
In[3]:=t=144;        (divisor)
In[4]:=<<d-4-6.nb
In[5]:=cc3=Expand[cc3]
```

Out[5]={-5i^{2}j^2 +20hijk -12h^{2}k^2 -8gik^2 -8ejk^2 + ······

```
(In[6]:=cc3>>con-rho.nb)
```

6.2.8. `disc.nb`.

$$\texttt{disc.nb} \quad = \quad I_{27}.$$

6.2.9. `dual.nb`.

Let A be a co(contra)variant. `dual.nb` compute the dual A^* of A.

List

```
t0={{4,-1},{1,-2},{4,-1},{1,-2},{1,-2},{4,-1}}
t1={{5,3},{2,4},{4,1},{3,2},{1,2},{2,1}}
t2={{5,6},{6,5},{4,6},{4,5},{5,4},{2,3}}
cc3={0,0,0,0,0,0}
For[q=1,q<=6,q++,
  For[r=1,r<=2,r++,
    cc3[[q]]+=cc1[[t1[[q,r]]]]*cc1[[t2[[q,r]]]]/t0[[q,r]]
  ]
]
```

How to use

For example ρ^*.

```
In[1]:=cc1=Get["con-rho.nb"];
In[2]:=<<dual.nb
In[3]:=cc3=Expand[cc3]
Out[3]={ ······
(In[4]:=cc3>>cov-rhodual.nb)
```

6.2.10. `entry-*.nb`.

$$
\begin{aligned}
\texttt{entry-30.nb} &= ENT_{30},\\
\texttt{entry-33.nb} &= ENT_{33},\\
\texttt{entry-36.nb} &= ENT_{36},\\
\texttt{entry-39.nb} &= ENT_{39},\\
\texttt{entry-42.nb} &= ENT_{42},\\
\texttt{entry-45.nb} &= ENT_{45},\\
\texttt{entry-48.nb} &= ENT_{48},\\
\texttt{entry-disc.nb} &= ENT_{disc}.
\end{aligned}
$$

The size of their matrixes is written in the next Table.

File name	Size
`entry-30.nb`	98×99
`entry-33.nb`	139×142
`entry-36.nb`	199×206
`entry-39.nb`	275×289
`entry-42.nb`	375×403
`entry-45.nb`	509×557
`entry-48.nb`	678×761
`entry-disc.nb`	66×67

6.2.11. `entry.c`.

`entry.c` compute `entry-*.nb` (*=30, 33, 36, 39, 42, 45, 48, `disc`). This is a C program.

List
We omit the list, because it is long. Please see the file directly. We wrote a brief outline of it in itself.

How to use
We need a C compiler for 32-bit. It must be able to use quadruple precision real number. We compile `entry.c` and make the execute file. We execute it.
For example the computation of `entry-30.nb`.

```
Input ter-*.nb   ter-30.nb
Input bas-*.nb   bas-30.nb
Input save file name   entry-30.nb
```

(Caution! If the file of `entry-30.nb` already exists, then it is overwritten.)

```
1
2
⋮
98
```

Other cases are similarly computed. The estimated computation time is written in the next Table.

File name	Estimated computation time
`entry-30.nb`	30 seconds
`entry-33.nb`	3 minutes
`entry-36.nb`	15 minutes
`entry-39.nb`	90 minutes
`entry-42.nb`	8 hours
`entry-45.nb`	46 hours
`entry-48.nb`	10 days
`entry-disc.nb`	10 seconds

Other necessary files except for input files
`con-psi.nb`, `con-rho.nb`, `con-sigma.nb`, `cov-he.nb`, `cov-tau.nb`, `cov-varphi.nb`, `cov-xi.nb`, `inv-i3.nb` and `inv-i6.nb`.

6.2.12. `entry.nb`.

`entry.nb` compute `entry-*.nb` (`*=30, 33, 36, 39, 42, 45, 48, disc`). This is a Mathematica program.

List

```
varphi=Get["cov-varphi.nb"]
sigma=Get["con-sigma.nb"]
he=Get["cov-he.nb"]
psi=Get["con-psi.nb"]
cc1=varphi
cc2=psi
t=144
Get["d-4-6.nb"]
```

```
rho=Expand[cc3]
cc1=rho
cc2=varphi
t=12
Get["d-2-4.nb"]
tau=Expand[cc3]
cc1=sigma
cc2=he
t=72
Get["d-4-6.nb"]
xi=Expand[cc3]
cc1=rho
cc2=he
t=2
Get["d-2-6.nb"]
pi=cc3
cc1=xi
cc2=sigma
t=12
Get["d-2-4.nb"]
eta=cc3
cc1=tau
cc2=psi
t=2
Get["d-2-6.nb"]
zeta=cc3
cc1=tau
cc2=zeta
t=4
Get["d-2-4.nb"]
chi=cc3
cc1=eta
cc2=pi
t=4
Get["d-2-4.nb"]
nu=cc3
cc1=rho
Get["dual.nb"]
rhodual=cc3
cc1=tau
Get["dual.nb"]
taudual=cc3
cc1=xi
Get["dual.nb"]
xidual=cc3
I3=Get["inv-i3.nb"]
I6=Get["inv-i6.nb"]
cc1=tau
cc2=rho
Get["j11.nb"]
```

```
I9a=cc3
cc1=xi
cc2=rho
Get["j11.nb"]
I9b=cc3
cc1=rho
Get["j3.nb"]
I12a=cc3
cc1=tau
cc2=eta
Get["j11.nb"]
I12b=cc3
cc1=tau
Get["j3.nb"]
I15a=cc3
cc1=xi
Get["j3.nb"]
I15b=cc3
cc1=rhodual
cc2=taudual
Get["j11.nb"]
I18a=cc3
cc1=rhodual
cc2=xidual
Get["j11.nb"]
I18b=cc3
cc1=eta
Get["j3.nb"]
I21a=cc3
cc1=nu
cc2=eta
Get["j11.nb"]
I21b=cc3
cc1=nu
cc2=chi
Get["j11.nb"]
J27=cc3
rdim=Length[ter]
cdim=Length[bas]
s=Table[0,{im,rdim},{jm,cdim}]
For[im=1,im<=rdim,im++,{
  For[jm=1,jm<=cdim,jm++,
    s[[im,jm]]=Coefficient[bas[[jm]],ter[[im]]]
  ],
  Print[im]
}]
```

How to use
The case of `entry-30.nb`.

```
In[1]:=ter=Get["ter-30.nb"];
```

```
In[2]:=bas=Get["bas-30.nb"];
In[3]:=<<entry.nb
1
2
⋮
98
(In[4]:=s>>entry-30.nb)
```

Other cases are similarly computed. The estimated computation time is written in the next Table.

File name	Estimated computation time
entry-30.nb	320 minutes
entry-33.nb	54 hours
entry-36.nb	
entry-39.nb	
entry-42.nb	
entry-45.nb	
entry-48.nb	
entry-disc.nb	30 hours

Other necessary files except for input files

con-psi.nb, con-sigma.nb, cov-he.nb, cov-varphi.nb, d-2-4.nb, d-2-6.nb, d-4-6.nb, dual.nb, inv-i3.nb, inv-i6.nb, j11.nb and j3.nb.

6.2.13. inv-*.nb.

$$\begin{aligned}\texttt{inv-i3.nb} &= I_3,\\ \texttt{inv-i6.nb} &= I_6.\end{aligned}$$

6.2.14. inv-sec-4.nb.

inv-sec-4.nb is the 13-dimensional list

$$\{I_3, I_6, I_{9a}, I_{9b}, I_{12a}, 0, I_{15a}, 0, I_{18a}, 0, 0, 0, 0\}$$

satisfying $j = m = n = p = 0$ (Lemma 4.2).

6.2.15. j11.nb.

$$\texttt{j11.nb} = J_{1,1}.$$

List

```
t0={1,2,1,2,2,1}
cc3=0
For[q=1,q<=6,q++,
  cc3+=cc1[[q]]*cc2[[q]]/t0[[q]]
]
```

How to use

For example $I_{9a} = J_{1,1}(\tau, \rho)$.

```
In[1]:=cc1=Get["cov-tau.nb"];
In[2]:=cc2=Get["con-rho.nb"];
In[3]:=<<j11.nb
In[4]:=cc3=Expand[cc3]
Out[4]={ ······
(In[5]:=cc3>>inv-i9a.nb)
```

6.2.16. `j3.nb`.

$$\texttt{j3.nb} \quad = \quad J_{3,0} \text{ or } J_{0,3}.$$

List

```
t0={1,4,-4,-4,-4}
t1={1,2,1,2,3}
t2={3,4,5,2,4}
t3={6,5,5,6,4}
cc3=0
For[q=1,q<=5,q++,
  cc3+=cc1[[t1[[q]]]]*cc1[[t2[[q]]]]*cc1[[t3[[q]]]]/t0[[q]]
]
```

How to use

For example $I_{12a} = J_{0,3}(\rho)$.

```
In[1]:=cc1=Get["con-rho.nb"];
In[2]:=<<j3.nb
In[3]:=cc3=Expand[cc3]
Out[3]={ ······
(In[4]:=cc3>>inv-i12a.nb)
```

6.2.17. `jexc27.nb`.

`jexc27.nb` is the rule of the exchange of J_{27} for I_{27}.

How to make this file is as follows.

```
In[1]:=disc=Get["disc.nb"];
In[2]:=Solve[I27==disc,J27]
Out[2]={{J27 -> (295928456745600I12aI15a - ······
```

We obtain the rule of `jexc27.nb`.

6.2.18. `make-he.nb`.

`make-he.nb` compute `cov-he.nb`.

List

```
F=a*x^4+4*b*x^3*y+6*c*x^2*y^2+4*d*x*y^3+e*y^4+
  4*f*x^3*z+12*g*x^2*y*z+12*h*x*y^2*z+4*i*y^3*z+
```

```
     6*j*x^2*z^2+12*k*x*y*z^2+6*l*y^2*z^2+
            4*m*x*z^3+4*n*y*z^3+
                   p*z^4
A={{D[F,x,x],D[F,x,y],D[F,x,z]},
   {D[F,y,x],D[F,y,y],D[F,y,z]},
   {D[F,z,x],D[F,z,y],D[F,z,z]}}
he=Expand[Det[A]/1728]
var={x^6,x^5*y,x^4*y^2,x^3*y^3,x^2*y^4,x*y^5,y^6,
     x^5*z,x^4*y*z,x^3*y^2*z,x^2*y^3*z,x*y^4*z,y^5*z,
     x^4*z^2,x^3*y*z^2,x^2*y^2*z^2,x*y^3*z^2,y^4*z^2,
         x^3*z^3,x^2*y*z^3,x*y^2*z^3,y^3*z^3,
              x^2*z^4,x*y*z^4,y^2*z^4,
                   x*z^5,y*z^5,
                       z^6}
s=Coefficient[he,var]
```

How to use

In[1]:=<<make-he.nb
Out[1]={-(cf^2) +2bfg -ag^2 -b^{2}j +acj, ······
(In[2]:=s>>cov-he.nb)

6.2.19. `make-i3.nb`.

`make-i3.nb` compute `inv-i3.nb`.

List

```
A={{x[1],x[2],x[3]},
   {y[1],y[2],y[3]},
   {z[1],z[2],z[3]}}
i3=Expand[Det[A]^4/6]
exc={x[t_]^4 -> a, x[t_]^3*y[t_] -> b, x[t_]^2*y[t_]^2 -> c,
                                  x[t_]*y[t_]^3 -> d, y[t_]^4 -> e,
     x[t_]^3*z[t_] -> f, x[t_]^2*y[t_]*z[t_] -> g,
                      x[t_]*y[t_]^2*z[t_] -> h, y[t_]^3*z[t_] -> i,
     x[t_]^2*z[t_]^2 -> j, x[t_]*y[t_]*z[t_]^2 -> k,
                                              y[t_]^2*z[t_]^2 -> l,
                   x[t_]*z[t_]^3 -> m, y[t_]*z[t_]^3 -> n,
                           z[t_]^4 -> p}
s=i3//.exc
```

How to use

Similar to `make-he.nb`.

6.2.20. `make-psi.nb`.

`make-psi.nb` compute `con-psi.nb`.

List

```
A12={{u,x[1],x[2]},
     {v,y[1],y[2]},
     {w,z[1],z[2]}}
A23={{u,x[2],x[3]},
     {v,y[2],y[3]},
     {w,z[2],z[3]}}
A31={{u,x[3],x[1]},
     {v,y[3],y[1]},
     {w,z[3],z[1]}}
psi=Expand[Det[A12]^2*Det[A23]^2*Det[A31]^2/6]
exc={x[t_]^4 -> a, x[t_]^3*y[t_] -> b, x[t_]^2*y[t_]^2 -> c,
                                  x[t_]*y[t_]^3 -> d, y[t_]^4 -> e,
     x[t_]^3*z[t_] -> f, x[t_]^2*y[t_]*z[t_] -> g,
                      x[t_]*y[t_]^2*z[t_] -> h, y[t_]^3*z[t_] -> i,
     x[t_]^2*z[t_]^2 -> j, x[t_]*y[t_]*z[t_]^2 -> k,
                                              y[t_]^2*z[t_]^2 -> l,
                   x[t_]*z[t_]^3 -> m, y[t_]*z[t_]^3 -> n,
                            z[t_]^4 -> p}
psi=psi//.exc
var={u^6,u^5*v,u^4*v^2,u^3*v^3,u^2*v^4,u*v^5,v^6,
     u^5*w,u^4*v*w,u^3*v^2*w,u^2*v^3*w,u*v^4*w,v^5*w,
     u^4*w^2,u^3*v*w^2,u^2*v^2*w^2,u*v^3*w^2,v^4*w^2,
         u^3*w^3,u^2*v*w^3,u*v^2*w^3,v^3*w^3,
              u^2*w^4,u*v*w^4,v^2*w^4,
                   u*w^5,v*w^5,
                       w^6}
s=Coefficient[psi,var]
```

How to use
Similar to `make-he.nb`.

6.2.21. `make-sigma.nb`.

`make-sigma.nb` compute `con-sigma.nb`.

List

```
A12={{u,x[1],x[2]},
     {v,y[1],y[2]},
     {w,z[1],z[2]}}
sigma=Expand[Det[A12]^4/2]
exc={x[t_]^4 -> a, x[t_]^3*y[t_] -> b, x[t_]^2*y[t_]^2 -> c,
                                  x[t_]*y[t_]^3 -> d, y[t_]^4 -> e,
     x[t_]^3*z[t_] -> f, x[t_]^2*y[t_]*z[t_] -> g,
                      x[t_]*y[t_]^2*z[t_] -> h, y[t_]^3*z[t_] -> i,
     x[t_]^2*z[t_]^2 -> j, x[t_]*y[t_]*z[t_]^2 -> k,
                                              y[t_]^2*z[t_]^2 -> l,
                   x[t_]*z[t_]^3 -> m, y[t_]*z[t_]^3 -> n,
                            z[t_]^4 -> p}
sigma=sigma//.exc
```

```
var={u^4,u^3*v,u^2*v^2,u*v^3,v^4,
     u^3*w,u^2*v*w,u*v^2*w,v^3*w,
       u^2*w^2,u*v*w^2,v^2*w^2,
             u*w^3,v*w^3,
                 w^4}
s=Coefficient[sigma,var]
```

How to use
Similar to `make-he.nb`.

6.2.22. `quarticsdata.nb`.

`quarticsdata.nb` is the 14-dimensional list such that its i-th component is the 15-dimensional list of values of coefficient of ternary quartic (Lemma 4.2).

6.2.23. `r*.nb`.

`r30.nb`	=	R_{30},
`r33a.nb`	=	R_{33a},
`r33b.nb`	=	R_{33b},
`r36a.nb`	=	R_{36a},
`r36b.nb`	=	R_{36b},
`r36c.nb`	=	R_{36c},
`r39a.nb`	=	R_{39a},
`r39b.nb`	=	R_{39b},
`r39c.nb`	=	R_{39c},
`r42a.nb`	=	R_{42a},
`r42b.nb`	=	R_{42b},
`r42c.nb`	=	R_{42c},
`r42d.nb`	=	R_{42d},
`r45.nb`	=	R_{45},
`r48.nb`	=	R_{48}.

6.2.24. `rank.nb`.

`rank.nb` compute the rank of `entry-*.nb` (*=30, 33, 36, 39, 42, 45, 48).

List

```
rdim=Length[A]
For[k=1,k<=rdim,k++,{
  m=A[[k,k]],
  For[j=k,j<=rdim,j++,A[[k,j]]/=m],
  For[i=k+1,i<=rdim,i++,{
    m=A[[i,k]],
    If[m!=0,For[j=k,j<=rdim,j++,A[[i,j]]-=A[[k,j]]*m]]
  }],
  Print[k]
}]
```

How to use

The case of `entry-30.nb`.

```
In[1]:=A=Get["entry-30.nb"];
In[2]:=<<rank.nb
1
2
⋮
98
```

Other cases are similarly computed. The estimated computation time is written in the next Table.

File name	Estimated computation time
`entry-30.nb`	10 seconds
`entry-33.nb`	20 seconds
`entry-36.nb`	80 seconds
`entry-39.nb`	4 minutes
`entry-42.nb`	15 minutes
`entry-45.nb`	150 minutes
`entry-48.nb`	260 minutes

6.2.25. `relalist.nb`.

`relalist.nb` make a list `s` of all the relations in Theorem 3.2:

$$\texttt{s} = \{\texttt{r30.nb}, \texttt{r33a.nb}, \cdots, \texttt{r48.nb}\}.$$

List

```
s={0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}
fn={"r30.nb", "r33a.nb", "r33b.nb", "r36a.nb", "r36b.nb",
   "r36c.nb", "r39a.nb", "r39b.nb", "r39c.nb", "r42a.nb",
   "r42b.nb", "r42c.nb", "r42d.nb", "r45.nb", "r48.nb"}
For[i=1,i<=15,i++,s[[i]]=Get[fn[[i]]]]
```

How to use

```
In[1]:=<<relalist.nb
```

6.2.26. `relation.*`.

`relation.tex` is a source file of TeX. `relation.ps` is the PostScript File. `relation.pdf` is the Portable Document Format File. In these files, we describe the coefficient of W_i in the relation in Theorem 3.2.

6.2.27. `sec-4-lemma-*.nb`.

The computation in Lemma 4.4 and Lemma 4.5 is computed by `sec-4-lemma-i9b.nb`.
The computation in (Lemma 4.4 and) Remark 4.6 is computed by `sec-4-lemma-i12b.nb`, `sec-4-lemma-i15b.nb`, `sec-4-lemma-i18b.nb`, `sec-4-lemma-i21a.nb` and `sec-4-lemma-i21b.nb`.

We describe the only case of `sec-4-lemma-i9b.nb`. We omit other cases, because they are similar to `sec-4-lemma-i9b.nb`.

We assume the conditions of $I_3 = I_6 = I_{9a} = I_{18a} = I_{27} = 0$ and $I_{12a} = I_{15a} = 1$. See the list below. q is 240-dimensional list. $q[[j]]$ $(1 \leq j \leq 190)$ is the left hand side of (1) $\sim$ (190). $q[[j]]$ $(191 \leq j \leq 240)$ is $W_1 \sim W_{50}$. A is 190×240 matrix. $A[[i, j]]$ $(1 \leq i \leq 15,\ 1 \leq j \leq 240)$ is the coefficient of $q[[j]]$ in the relation of (1) $\sim$ (15) and the remaining entries are 0 at first (299∼314-th line).

The n $(16 \leq n \leq 190)$-th relation is maked from $s[[n]]$-th relation multiplied by $r[[n]]$. We execute this work from $n = 16$ to $n = 190$. But we do not execute it at a time. We divide $q[[n]]$ $(16 \leq n \leq 145,\ 146 \leq n \leq 190)$ according to their degree. There are 65 kinds. We express its information by list nn. The list nn has 65 rows. The first 20 rows correspond to Lemma 4.4. The remaining 45 rows correspond to Lemma 4.5 ($I_{9b}^5 \sim I_{9b}^{49}$). The 315∼347-th line correspond to the computation of Lemma 4.4 and Lemma 4.5 ($I_{9b}^5 \sim I_{9b}^{49}$). The 348∼355-th line correspond to the computation of Lemma 4.5 ($U_{I_{9b}}$).

List

We show the only list of `sec-4-lemma-i9b.nb`. We omit other lists, because they are similar to `sec-4-lemma-i9b.nb`. Please see each file directly.

```
nn={{ 16, 16},{ 17, 19},{ 20, 24},{ 25, 32},{ 33, 44},
    { 45, 52},{ 53, 65},{ 66, 80},{ 81, 90},{ 91,100},
    {101,111},{112,119},{120,125},{126,131},{132,135},
    {136,138},{139,141},{142,142},{143,143},{144,145},
    {146,146},{147,147},{148,148},{149,149},{150,150},
    {151,151},{152,152},{153,153},{154,154},{155,155},
    {156,156},{157,157},{158,158},{159,159},{160,160},
    {161,161},{162,162},{163,163},{164,164},{165,165},
    {166,166},{167,167},{168,168},{169,169},{170,170},
    {171,171},{172,172},{173,173},{174,174},{175,175},
    {176,176},{177,177},{178,178},{179,179},{180,180},
    {181,181},{182,182},{183,183},{184,184},{185,185},
    {186,186},{187,187},{188,188},{189,189},{190,190}}
s={    0,    0,    0,    0,    0,    0,    0,    0,    0,    0,
       0,    0,    0,    0,    0,    1,    2,    3,    1,    5,
       6,    2,    3,    1,    4,    7,    8,    9,    2,    3,
       6,    1,   11,   17,   18,    7,   19,   12,   13,    9,
       3,    1,    1,    6,   10,   21,   22,   23,    7,    5,
       3,    9,    4,    4,   26,   27,   28,   29,   30,   32,
       8,   24,   12,   13,    6,   34,   35,   14,   37,   38,
      39,   41,   42,   43,   44,   31,   32,   24,    9,   13,
      10,   10,   49,   51,   52,   40,   41,   44,   12,   13,
      57,   14,   58,   59,   60,   63,   64,   65,   52,   41,
      70,   32,   71,   72,   73,   74,   75,   79,   80,   51,
      43,   84,   85,   62,   78,   89,   90,   43,   65,   96,
      97,   98,  100,   79,   80,  108,  109,  110,  111,   89,
      90,  116,  117,  118,  119,  102,  124,  125,  128,  130,
     131,  134,  137,  140,  141,    4,  146,  147,  148,  149,
     150,  151,  152,  153,  154,  155,  156,  157,  158,  159,
     160,  161,  162,  163,  164,  165,  166,  167,  168,  169,
     170,  171,  172,  173,  174,  175,  176,  177,  178,  179,
```

```
     180,  181,  182,  183,  184,  185,  186,  187,  188,  189}
r={    0,    0,    0,    0,    0,    0,    0,    0,    0,    0,
       0,    0,    0,    0,    0, I9b,  I9b,  I9b, I12b,  I9b,
     I9b, I12b, I12b, I15b, I12b,  I9b,  I9b,  I9b, I15b, I15b,
    I12b, I18b,  I9b,  I9b,  I9b, I12b,  I9b,  I9b,  I9b, I12b,
    I18b, I21a, I21b, I15b, I12b,  I9b,  I9b,  I9b, I15b, I18b,
    I21a, I15b, I21a, I21b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,
    I18b, I12b, I15b, I15b, I21b,  I9b,  I9b, I15b,  I9b,  I9b,
     I9b,  I9b,  I9b,  I9b,  I9b, I12b, I12b, I15b, I21b, I18b,
    I21a, I21b,  I9b,  I9b,  I9b, I12b, I12b, I12b, I21b, I21a,
     I9b, I21b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b, I12b, I15b,
     I9b, I21b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b, I15b,
    I18b,  I9b,  I9b, I15b, I12b,  I9b,  I9b, I21a, I15b,  I9b,
     I9b,  I9b,  I9b, I15b, I15b,  I9b,  I9b,  I9b,  I9b, I15b,
    I15b,  I9b,  I9b,  I9b,  I9b, I15b,  I9b,  I9b,  I9b,  I9b,
     I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,
     I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,
     I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,
     I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,
     I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b,  I9b}
q={I15b^2, I12b*I21a, I12b*I21b, I9b^4, I12b^3, I18b^2,
     I9b*I12b*I18b, I12b^2*I15b, I18b*I21a,
     I9b^3*I15b, I9b^2*I12b^2, I21a^2, I21b^2, I9b^3*I18b, I9b^2*I15b^2,
   I9b*I15b^2,
   I9b*I12b*I21a, I9b*I12b*I21b, I12b*I15b^2,
   I9b*I12b^3, I9b*I18b^2, I12b^2*I21a, I12b^2*I21b, I15b^3,
   I9b^4*I12b, I9b^2*I12b*I18b, I9b*I12b^2*I15b, I9b*I18b*I21a,
     I12b*I15b*I21a, I12b*I15b*I21b, I12b*I18b^2, I15b^2*I18b,
   I9b^3*I12b^2, I9b^2*I12b*I21a, I9b^2*I12b*I21b,
     I9b*I12b^2*I18b, I9b*I12b*I15b^2,
     I9b*I21a^2, I9b*I21b^2, I12b*I18b*I21a, I12b*I18b*I21b,
     I15b^2*I21a, I15b^2*I21b, I15b*I18b^2,
   I9b^3*I12b*I15b, I9b^2*I18b^2, I9b*I12b^2*I21a, I9b*I12b^2*I21b,
     I9b*I12b*I15b*I18b, I12b^3*I18b, I12b*I21a*I21b, I15b*I18b*I21a,
   I9b^4*I21a, I9b^4*I21b, I9b^3*I12b*I18b, I9b^2*I12b^2*I15b,
     I9b^2*I18b*I21a, I9b*I12b*I15b*I21a, I9b*I12b*I15b*I21b,
     I9b*I15b^2*I18b, I12b^2*I15b*I18b, I12b*I15b^3,
     I15b*I21a^2, I15b*I21b^2, I18b^2*I21b,
   I9b^3*I12b*I21a, I9b^3*I12b*I21b, I9b^3*I15b*I18b,
     I9b^2*I12b*I15b^2, I9b^2*I21a^2, I9b^2*I21b^2, I9b*I12b*I18b*I21b,
     I9b*I15b^2*I21a, I9b*I15b^2*I21b, I9b*I15b*I18b^2, I12b^2*I18b^2,
     I12b*I15b^2*I18b, I15b^4, I18b*I21a*I21b, I18b*I21b^2,
   I9b^3*I15b*I21a, I9b^3*I15b*I21b, I9b^2*I12b*I15b*I18b,
     I9b*I12b*I21a*I21b, I9b*I15b*I18b*I21a,
     I12b^2*I18b*I21a, I12b^2*I18b*I21b,
     I12b*I15b*I18b^2, I21a^2*I21b, I21a*I21b^2,
   I9b^3*I18b*I21a, I9b^3*I18b*I21b,
     I9b^2*I12b*I15b*I21a, I9b^2*I12b*I15b*I21b,
     I9b^2*I15b^2*I18b, I9b*I15b*I21a^2, I9b*I15b*I21b^2,
     I9b*I18b^2*I21b, I12b*I15b*I18b*I21a, I12b*I15b*I18b*I21b,
```

```
   I9b^3*I21a^2, I9b^3*I21a*I21b, I9b^3*I21b^2,
     I9b^2*I12b*I18b*I21b, I9b^2*I15b^2*I21a, I9b^2*I15b^2*I21b,
     I9b^2*I15b*I18b^2, I9b*I18b*I21a*I21b, I9b*I18b*I21b^2,
     I12b*I15b*I21a*I21b, I15b^2*I18b*I21b,
   I9b^2*I12b*I21a*I21b, I9b^2*I15b*I18b*I21a, I9b^2*I15b*I18b*I21b,
     I12b*I15b^4, I9b*I21a^2*I21b, I9b*I21a*I21b^2,
     I15b^2*I21a*I21b, I15b*I18b^2*I21b,
   I9b^2*I15b*I21a^2, I9b^2*I15b*I21b^2, I9b^2*I18b^2*I21b,
     I9b*I12b*I15b*I18b*I21b, I15b*I18b*I21a*I21b, I15b*I18b*I21b^2,
   I9b^2*I18b*I21a*I21b, I9b^2*I18b*I21b^2, I9b*I12b*I15b*I21a*I21b,
     I9b*I15b^2*I18b*I21b, I15b*I21a^2*I21b, I15b*I21a*I21b^2,
   I9b^2*I21a^2*I21b, I9b^2*I21a*I21b^2,
     I9b*I15b^2*I21a*I21b, I9b*I15b*I18b^2*I21b,
   I9b^3*I15b*I21a*I21b, I9b*I15b*I18b*I21a*I21b, I9b*I15b*I18b*I21b^2,
   I9b^2*I12b*I15b*I21a*I21b, I9b*I15b*I21a^2*I21b, I9b*I15b*I21a*I21b^2,
   I9b^2*I15b^2*I21a*I21b,
   I9b^2*I15b*I18b*I21a*I21b,
   I9b^2*I15b*I21a^2*I21b, I9b^2*I15b*I21a*I21b^2,
   I9b^5,
     I9b^6,  I9b^7,  I9b^8,  I9b^9,  I9b^10,
     I9b^11, I9b^12, I9b^13, I9b^14, I9b^15,
     I9b^16, I9b^17, I9b^18, I9b^19, I9b^20,
     I9b^21, I9b^22, I9b^23, I9b^24, I9b^25,
     I9b^26, I9b^27, I9b^28, I9b^29, I9b^30,
     I9b^31, I9b^32, I9b^33, I9b^34, I9b^35,
     I9b^36, I9b^37, I9b^38, I9b^39, I9b^40,
     I9b^41, I9b^42, I9b^43, I9b^44, I9b^45,
     I9b^46, I9b^47, I9b^48, I9b^49,
   1, I9b, I12b, I15b, I9b^2, I18b, I9b*I12b, I21a, I21b, I9b*I15b, I12b^2,
     I9b^3, I9b*I18b, I12b*I15b, I9b^2*I12b, I9b*I21a, I9b*I21b, I12b*I18b,
     I9b^2*I15b, I9b*I12b^2, I15b*I18b,
     I9b^2*I18b, I9b*I12b*I15b, I15b*I21a, I15b*I21b,
     I9b^3*I12b, I9b^2*I21a, I9b^2*I21b, I18b*I21b,
     I9b*I15b*I18b, I12b^2*I18b, I21a*I21b,
     I9b^2*I12b*I15b, I9b*I15b*I21a, I9b*I15b*I21b, I12b*I15b*I18b,
     I9b^3*I21a, I9b^3*I21b, I9b*I18b*I21b, I9b^2*I15b*I18b, I9b*I21a*I21b,
     I9b^2*I15b*I21a, I9b^2*I15b*I21b, I15b*I18b*I21b,
     I9b^2*I18b*I21b, I15b*I21a*I21b, I9b^2*I21a*I21b,
     I9b*I15b*I18b*I21b, I9b*I15b*I21a*I21b, I9b^2*I15b*I21a*I21b}
nh=65
nu=145
nr=190
nc=240
fn={"r30.nb", "r33a.nb", "r33b.nb", "r36a.nb", "r36b.nb",
   "r36c.nb", "r39a.nb", "r39b.nb", "r39c.nb", "r42a.nb",
   "r42b.nb", "r42c.nb", "r42d.nb", "r45.nb", "r48.nb"}
A=Table[0,{i,nr},{j,nc}]
For[i=1,i<=15,i++,{
  rel=Get[fn[[i]]],
  rel=rel//.{I3->0,I6->0,I9a->0,I12a->1,I15a->1,I18a->0,I27->0},
```

```
  n=Length[rel],
  For[k=1,k<=n,k++,{
    co=rel[[k]],
    j=1,
    While[IntegerQ[co/q[[j]]]==False,j++],
    A[[i,j]]+=co/q[[j]]
  }]
}]
For[k=1,k<=15,k++,{
  m=A[[k,k]],
  For[j=k,j<=nc,j++,A[[k,j]]/=m]
}]
For[h=1,h<=nh,h++,{
  For[n=nn[[h,1]],n<=nn[[h,2]],n++,
    For[i=s[[n]],i<=nc,i++,{
      If[A[[s[[n]],i]]==0,Continue[]],
      t=q[[i]]*r[[n]],
      For[j=1,j<=nc,j++,
        If[t==q[[j]],{A[[n,j]]=A[[s[[n]],i]],Break[]}]
      ]
    }]
  ],
  For[k=1,k<=nn[[h,1]]-1,k++,
    For[i=nn[[h,1]],i<=nn[[h,2]],i++,{
      If[A[[i,k]]==0,Continue[]],
      m=A[[i,k]],
      For[j=k,j<=nc,j++,A[[i,j]]=Together[A[[i,j]]-A[[k,j]]*m]]
    }]
  ],
  For[k=nn[[h,1]],k<=nn[[h,2]],k++,{
    m=A[[k,k]],
    For[j=k,j<=nc,j++,A[[k,j]]/=m],
    For[i=k+1,i<=nn[[h,2]],i++,{
      m=A[[i,k]],
      For[j=k,j<=nc,j++,A[[i,j]]=Together[A[[i,j]]-A[[k,j]]*m]]
    }]
  }],
  For[k=nn[[h,2]],k>=nn[[h,1]],k--,
    For[i=k-1,i>=nn[[h,1]],i--,{
      m=A[[i,k]],
      For[j=k,j<=nc,j++,A[[i,j]]=Together[A[[i,j]]-A[[k,j]]*m]]
    }]
  ],
  Print[h]
}]
B=Table[0,{i,50},{j,50}]
B[[1,1]]=1
B[[2,2]]=1
B[[3,5]]=1
B[[4,12]]=1
```

```
For[j=1,j<=50,j++,B[[5,j]]=-A[[4,nr+j]]]
For[i=6,i<=50,i++,For[j=1,j<=50,j++,B[[i,j]]=-A[[nu+i-5,nr+j]]]]
Print[NullSpace[B]]
```

How to use

```
In[1]:=<<sec-4-lemma-i9b.nb
1
2
⋮
65
{}
```

Other cases are similar to `sec-4-lemma-i9b.nb`. The estimated computation time is $2 \sim 4$ minutes.

Other necessary files
`r30.nb`, `r33a.nb`, `r33b.nb`, `r36a.nb`, `r36b.nb`, `r36c.nb`, `r39a.nb`, `r39b.nb`, `r39c.nb`, `r42a.nb`, `r42b.nb`, `r42c.nb`, `r42d.nb`, `r45.nb` and `r48.nb`.

6.2.28. `sumdisc.nb`.

`sumdisc.nb` compute I_{27} on given condition.

List
We omit the list, because it is similar to `entry.nb`. Please see the file directly.

How to use
For example $j = k = l = m = n = p = 0$.

```
In[1]:=exc={j->0,k->0,l->0,m->0,n->0,p->0};
In[2]:=<<sumdisc.nb
0
```

Other necessary files
`bas-disc.nb`, `coe-disc.nb`, `con-psi.nb`, `con-sigma.nb`, `cov-he.nb`, `cov-varphi.nb`, `d-2-4.nb`, `d-2-6.nb`, `d-4-6.nb`, `dual.nb`, `inv-i3.nb`, `inv-i6.nb`, `j11.nb` and `j3.nb`.

6.2.29. `ter-*.nb`.

`ter-30.nb`	=	TER_{30},
`ter-33.nb`	=	TER_{33},
`ter-36.nb`	=	TER_{36},
`ter-39.nb`	=	TER_{39},
`ter-42.nb`	=	TER_{42},
`ter-45.nb`	=	TER_{45},
`ter-48.nb`	=	TER_{48},
`ter-disc.nb`	=	TER_{disc}.

6.2.30. `unexistence.nb`.

The computation in Lemma 4.2 is computed by `unexistence.nb`.

In practical computation, it is hard to compute exact values of b_i. We select ϵ_1 and ϵ_2 ($\epsilon_1, \epsilon_2 \geq 0$) and make a pair $\{b_i - \epsilon_1, b_i + \epsilon_2\}$. We define that a pair $\{x_1, x_2\}$ always satisfies $x_1 \leq x_2$. We define four fundamental rules of arithmetic between two pairs $\{x_1, x_2\}$ ($x_1 > 0$ or $x_2 < 0$) and $\{y_1, y_2\}$ ($y_1 > 0$ or $y_2 < 0$):

$$\begin{aligned}
\{x_1, x_2\} + \{y_1, y_2\} &= \{x_1 + y_1, x_2 + y_2\}, \\
\{x_1, x_2\} - \{y_1, y_2\} &= \{x_1 - y_2, x_2 - y_1\}, \\
\{x_1, x_2\}\{y_1, y_2\} &= \begin{cases} \{x_{min}y_{min}, x_{max}y_{max}\} & (x_1y_1 > 0) \\ \{-x_{max}y_{max}, -x_{min}y_{min}\} & (x_1y_1 < 0), \end{cases} \\
\{x_1, x_2\}/\{y_1, y_2\} &= \begin{cases} \{x_{min}/y_{max}, x_{max}/y_{min}\} & (x_1y_1 > 0) \\ \{-x_{max}/y_{min}, -x_{min}/y_{max}\} & (x_1y_1 < 0), \end{cases}
\end{aligned}$$

where

$$\begin{aligned}
x_{min} &= \min(|x_1|, |x_2|), \\
x_{max} &= \max(|x_1|, |x_2|), \\
y_{min} &= \min(|y_1|, |y_2|), \\
y_{max} &= \max(|y_1|, |y_2|).
\end{aligned}$$

We execute computation by using these rules. If the result $\{x_1, x_2\}$ satisfies $x_1x_2 > 0$, then it is not equal to 0. On the other hand, if it satisfies $x_1x_2 \leq 0$, then it may be equal to 0. The later case does not occur in our computation.

List

```
add[in1_,in2_]:=
  Module[{outmin,outmax},{
    outmin=in1[[1]]+in2[[1]],
    outmax=in1[[2]]+in2[[2]],
    If[outmin*outmax<=0,Print["This may be 0 by add."]]};
    Return[{outmin,outmax}]
  ]
sub[in1_,in2_]:=
  Module[{outmin,outmax},{
    outmin=in1[[1]]-in2[[2]],
    outmax=in1[[2]]-in2[[1]],
    If[outmin*outmax<=0,Print["This may be 0 by sub."]]};
    Return[{outmin,outmax}]
  ]
multi[in1_,in2_]:=
  Module[{min1,max1,min2,max2,outmin,outmax},{
    If[in1[[1]]>0,{min1=in1[[1]],max1=in1[[2]]},
                  {min1=-in1[[2]],max1=-in1[[1]]}],
    If[in2[[1]]>0,{min2=in2[[1]],max2=in2[[2]]},
                  {min2=-in2[[2]],max2=-in2[[1]]}],
    If[in1[[1]]*in2[[1]]>0,{outmin=min1*min2,outmax=max1*max2},
                           {outmin=-max1*max2,outmax=-min1*min2}]};
    Return[{outmin,outmax}]
```

```
  ]
div[in1_,in2_]:=
  Module[{min1,max1,min2,max2,outmin,outmax},{
    If[in1[[1]]>0,{min1=in1[[1]],max1=in1[[2]]},
                  {min1=-in1[[2]],max1=-in1[[1]]}],
    If[in2[[1]]>0,{min2=in2[[1]],max2=in2[[2]]},
                  {min2=-in2[[2]],max2=-in2[[1]]}],
    If[in1[[1]]*in2[[1]]>0,{outmin=min1/max2,outmax=max1/min2},
                           {outmin=-max1/min2,outmax=-min1/max2}]};
    Return[{outmin,outmax}]
  ]
int[in_]:=
  Module[{outmin,outmax},{
    outmin=Floor[in[[1]]],
    outmax=Ceiling[in[[2]]],
    If[outmin*outmax==0,Print["This output becomes 0 by int."]]};
    Return[{outmin,outmax}]
  ]
approx[poly_,in_]:=
  Module[{val,mid,outmin,outmax},{
    outmin=in[[1]],
    outmax=in[[2]],
    val=1,
    While[val>err,{
      mid=(outmin+outmax)/2,
      val=poly//.b->mid,
      If[val>0,outmax=mid,{outmin=mid,val=-val}]
    }]};
    Return[{outmin,outmax}]
  ]
ge=Get["inv-sec-4.nb"]
gs=Get["quarticsdata.nb"]
solulist=Get["unexistsolulist.nb"]
bas=Get["abas-225.nb"]
gesym={I3,I6,I9a,I9b,I12a,I12b,I15a,I15b,I18a,I18b,I21a,I21b,I27}
nc=Length[bas]
col=Table[0,{nc}]
beta=Table[0,{ii,nc},{jj,nc},{kk,2}]
bpow=Table[0,{ii,100},{jj,2}]
ge1=Table[0,{13}]
err=1/10^200
nr=0
For[ii=1,ii<=Length[gs],ii++,{
  For[kk=1,kk<=13,kk++,ge1[[kk]]=Expand[ge[[kk]]//.{a->gs[[ii,1]],
    c->gs[[ii,3]],e->gs[[ii,5]],f->gs[[ii,6]],g->gs[[ii,7]],
    h->gs[[ii,8]],i->gs[[ii,9]],k->gs[[ii,11]],l->gs[[ii,12]]}]],
  gb=GroebnerBasis[{ge1[[3]],ge1[[5]]},{d,b},MonomialOrder->Lexicographic],
  solud=Solve[gb[[2]]==0,d],
  For[kk=1,kk<=13,kk++,ge1[[kk]]=Expand[ge1[[kk]]//.solud[[1,1]]]],
  gf=gb[[1]],
```

```
  For[kk=1,kk<=13,kk++,ge1[[kk]]=PolynomialRemainder[ge1[[kk]],gf,b]],
  If[ge1[[4]]==0,Print["I9b is equal to 0."]],
  If[ge1[[7]]==0,Print["I15a is equal to 0."]],
  If[ge1[[9]]==0,Print["I18a is equal to 0."]],
  For[jj=1,jj<=nc,jj++,{
    gg=1,
    For[kk=1,kk<=13,kk++,
      For[ll=1,ll<=Exponent[bas[[jj]],gesym[[kk]]],ll++,{
        gg=Expand[gg*ge1[[kk]]],
        gg=PolynomialRemainder[gg,gf,b]
      }]
    ],
    If[gg==0,Print["This component is 0."]],
    col[[jj]]=gg
  }],
  collist=Table[CoefficientList[col[[jj]],b],{jj,nc}],
  For[iii=1,iii<=Length[solulist[[ii]]],iii++,{
    inmin=solulist[[ii,iii]],
    inmax=inmin+1,
    If[(gf//.b->inmin)<0,sign=1,sign=-1],
    bpow[[1]]=approx[gf*sign,{inmin,inmax}],
    For[kk=2,kk<Exponent[gf,b],kk++,
      bpow[[kk]]=multi[bpow[[kk-1]],bpow[[1]]]
    ],
    nr++,
    For[jj=1,jj<=nc,jj++,{
      val1={collist[[jj,1]],collist[[jj,1]]},
      For[kk=2,kk<=Length[collist[[jj]]],kk++,
        If[collist[[jj,kk]]!=0,{
          val2=multi[bpow[[kk-1]],{collist[[jj,kk]],collist[[jj,kk]]}],
          val1=add[val1,val2]
        }]
      ],
      beta[[nr,jj]]=int[val1]
    }],
    Print[nr,"   ",ii,"   ",ge1[[1]],"   ",ge1[[2]],
                    "   ",ge1[[3]],"   ",ge1[[5]]]
  }]
}]
For[kk=1,kk<=nc,kk++,{
  For[ii=kk+1,ii<=nr,ii++,{
    val1=div[beta[[ii,kk]],beta[[kk,kk]]],
    beta[[ii,kk]]={0,0},
    For[jj=kk+1,jj<=nc,jj++,{
      val2=multi[beta[[kk,jj]],val1],
      val2=sub[beta[[ii,jj]],val2],
      beta[[ii,jj]]=int[val2]
    }]
  }],
  For[jj=kk,jj<=nc,jj++,beta[[kk,jj]]={0,0}],
```

```
  Print[kk]
}]
```

How to use

```
In[1]:=<<unexistence.nb
1   1   0   0   0   0
2   1   0   0   0   0
3   1   0   0   0   0
4   2   0   0   0   0
5   2   0   0   0   0
6   2   0   0   0   0
⋮
40   14   0   0   0   0
41   14   0   0   0   0
42   14   0   0   0   0
1
2
3
4
5
6
⋮
40
41
42
```

The estimated computation time is 3 minutes.

Other necessary files

`abas-225.nb`, `inv-sec-4.nb`, `quarticsdata.nb` and `unexistsolulist.nb`.

6.2.31. `unexistsolulist.nb`.

`unexistsolulist.nb` is the 14-dimensional list such that its i-th component is the 3-dimensional list of greatest integers less than or equal to solutions (Lemma 4.2):

$$\begin{aligned}&\{\{[b_1],[b_2],[b_3]\},\\ &\ \{[b_4],[b_5],[b_6]\},\\ &\ \vdots\\ &\ \{[b_{37}],[b_{38}],[b_{39}]\},\\ &\ \{[b_{40}],[b_{41}],[b_{42}]\}\},\end{aligned}$$

where [ ] is the Gauss symbol.

6.2.32. `v48.nb`.

$$\texttt{v48.nb} = V_{48}.$$

6.3. Description of proofs.

6.3.1. *Theorem* 2.4.

We compute `entry-disc.nb` by using `entry.nb` or `entry.c`.

```
In[1]:=<<entry-disc.nb;
In[2]:=NullSpace[%1]
Out[2]={-8028160000,
⋮
```

%2 is equal to `coe-disc.nb`. The inner product of `coe-disc.nb` and `bas-disc.nb` is `disc.nb`.

6.3.2. *Theorem* 3.2.

We describe the case of degree 30.
We compute `entry-30.nb` by using `entry.nb` or `entry.c`.
We compute the rank of `entry-30.nb` by using `rank.nb`.

```
In[1]:=<<entry-30.nb;
In[2]:=<<coe-30.nb;
In[3]:=%1.%2
Out[3]={0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
⋮
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
In[4]:=<<bas-30.nb;
In[5]:=rel=%2.%4;
In[6]:=exc=Get["jexc27.nb"];
In[7]:=rel=rel//.exc;
In[8]:=rel=Expand[rel]
Out[8]=-141120000I15aI15b
⋮
(In[9]:=rel>>r30.nb)
```

The cases of other degree are similarly computed.

6.3.3. *Lemma* 4.2.

We execute `unexistence.nb`.

6.3.4. *Lemma* 4.3.

It is sufficient that we compute I_{27} on each condition of $k = l = m = n = p = 0$ and $j = l = m = n = p = 0$.

```
In[1]:=exc={k->0,l->0,m->0,n->0,p->0};
In[2]:=<<sumdisc.nb
0
```

```
In[3]:=exc={j->0,l->0,m->0,n->0,p->0};
In[4]:=<<sumdisc.nb
0
```

6.3.5. *Lemma* 4.4 *and Lemma* 4.5.

We execute `sec-4-lemma-i9b.nb`.

6.3.6. (*Lemma* 4.4 *and*) *Remark* 4.6.

We execute `sec-4-lemma-i12b.nb`, `sec-4-lemma-i15b.nb`, `sec-4-lemma-i18b.nb`, `sec-4-lemma-i21a.nb` and `sec-4-lemma-i21b.nb`.

6.3.7. *Proposition* 5.1.

```
In[1]:=<<relalist.nb
In[2]:=<<v48.nb;
In[3]:=Expand[s.%2]
Out[3]=0
```

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