Integral trees homeomorphic to a double star

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Abstract

Trees with two nonadjacent vertices of degree larger than two are not integral. This settles a question by Watanabe & Schwenk (1979).

1 Introduction

The spectrum of a graph is the spectrum of its (0,1)-adjacency matrix. A graph is called *integral* when its spectrum is integral, i.e., when all eigenvalues of its adjacency matrix are integers. A tree is a connected graph without circuits. In [4] Watanabe & Schwenk studied integral trees, and characterized those integral trees that have a single vertex of degree larger than two, or two adjacent vertices of degree larger than two.

They write

> We would now like to examine the trees homeomorphic to a double star, that is, to a tree obtained by joining the centers of two stars with an edge. Unfortunately, the details are too involved to allow us to analyze trees in which the two vertices of high degree are nonadjacent.

The purpose of this note is to settle this question. The result is that there are no integral trees with precisely two vertices of degree larger than two when these two vertices are nonadjacent. This is shown using eigenvalue interlacing and Godsil's Lemma.

Lemma 1 ('Interlacing') *Let T be a graph on n vertices with eigenvalues $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n$, and let x be a vertex of T. Let $T\backslash x$ have eigenvalues $\eta_1 \geq \ldots \geq \eta_{n-1}$. Then $\theta_1 \geq \eta_1 \geq \theta_2 \geq \eta_2 \geq \ldots \geq \eta_{n-1} \geq \theta_n$.*

Lemma 2 (Godsil [2]) *Let T be a tree and θ an eigenvalue of multiplicity $m > 1$. Let P be a path in T. Then θ is eigenvalue of $T \backslash P$ with multiplicity at least $m - 1$.*

2 Theorem

Theorem 3 *Let T be a tree with integral spectrum and with precisely two vertices of degree larger than two. Then these vertices are adjacent.*

Proof: Let K_n denote the complete graph on n vertices. Let P_n denote the path of length $n-1$, with n vertices. For a graph G on n vertices, let $\theta_1(G), \ldots, \theta_n(G)$ be its eigenvalues, where $\theta_1(G) \geq \ldots \geq \theta_n(G)$.

Let u, v be the two vertices of degree larger than two in T, and assume that they are nonadjacent.

Apply interlacing twice to compare the spectra of T and $T\backslash u, v$. We find that $\theta_1(T\backslash u, v) \geq \theta_2(T\backslash u) \geq \theta_3(T)$. But $T\backslash u, v$ is a union of paths, so all eigenvalues of $T\backslash u, v$ are smaller than 2. It follows, since T is integral, that $\theta_3(T) \leq 1$.

Let w be a vertex separating u and v. Then $\theta_3(T\backslash w) \leq \theta_3(T) \leq 1$. Both components of $T\backslash w$ have at least three vertices and hence contain the path P_3 on 3 vertices which has largest eigenvalue $\sqrt{2} > 1$. So, $T\backslash w$ has precisely two eigenvalues larger than one, and the same holds for T. So the spectrum of T is $0^a \ \pm 1^b \ \pm m_1 \ \pm m_2$ where $m_2 > m_1 > 1$. (A tree is bipartite, so has symmetric spectrum.) It follows that T has at most 7 distinct eigenvalues.

As was shown by Watanabe [3], an integral tree has eigenvalue 0, so a is nonzero. If $b = 0$, then T has precisely 5 distinct eigenvalues, and by Godsil's Lemma cannot contain a path of length more than 4. In any case, T cannot contain a path of length more than 6.

2.1 Diameter 6

Suppose T contains a path P of length 6. Then $T\backslash P$ is integral with spectrum $0^{a-1} \ \pm 1^{b-1}$. That is, $T\backslash P$ consists of $a-1$ isolated points K_1 and $b-1$ copies of K_2. It follows that P passes through u and v.

Let Q be the path between u and v, including u and v. The Q has at least 3 vertices, and Q is contained in P. Let P_u and P_v be the components of $P\backslash u$ (resp. $P\backslash v$) not containing v (resp. u). Then P_u and P_v each contain at least one vertex, and together contain at most 4 vertices.

2.1.1 Pending P_3

Suppose first that P_u has at least three vertices. Then it has precisely three vertices, and all components of $T\backslash v$ not containing u are single points. If some component P'_u of $T\backslash u$ has two vertices, then let P' be the path obtained from P by replacing P_u by P'_u. Now $T\backslash P'$ has spectrum $0^a \ \pm 1^{b-2} \ \pm\sqrt{2}$, violating Godsil's Lemma. It follows that $b = 1$. Writing down an eigenvector of T for the eigenvalue 1 yields a contradiction.

2.1.2 Pending P_2 on both sides

Suppose both P_u and P_v have 2 vertices. If $T\backslash u$ (or $T\backslash v$) has a single-point component S, then let P' be the path obtained from P by replacing P_u (or P_v) by S. Now $T\backslash P'$ has spectrum $0^{a-2} \ \pm 1^b$ contradicting Godsil's Lemma. It follows that $a = 1$. But all integral trees with eigenvalue 0 of multiplicity 1 were determined in Brouwer [1], and these have only one vertex of degree more than two.

2.1.3 Pending P_2 on one side

Suppose P_u has 2 vertices and P_v is a single point. Then the path Q has 4 vertices. If $T\backslash u$ has a single-point component S we find the same contradiction as before. Let P_u and P'_u be two components of $T\backslash u$ with two vertices each, and let P'' be the path on 5 vertices induced by $P_u \cup \{u\} \cup P'_u$. Then $T\backslash P''$ has eigenvalue 1 of multiplicity $b-2$, contradicting Godsil's Lemma.

2.1.4 Pending K_1 on both sides

Writing down an eigenvector of T for the eigenvalue 1 yields a contradiction except for the case of the tree $\tilde{D}_8$ on 9 vertices, with spectrum 0^3 ± 1 $\pm\sqrt{3}$ ± 2, that has a nonintegral eigenvalue.

Since all subcases fail, the tree T cannot have diameter 6.

2.2 Diameter 5

With diameter only 5, there is no room to have pending P_2's on both sides.

2.2.1 Pending P_2 on one side

Now the path Q has 3 vertices. Let $T\backslash u$ have components $pK_1 + qK_2$ not on v, and let $T\backslash v$ have components rK_1 not on u. With P a path of length 5, $T\backslash P$ has spectrum 0^{p+r-1} $\pm 1^{q-1}$. If $p \neq 0$, then let P' be a path of length 4 on u and v. Then $T\backslash P'$ has spectrum 0^{p+r-2} $\pm 1^q$. It follows that $a = p + r - 1$ and $b = q$. Writing down an eigenvector for 1 we see that it must vanish at u and then also on the component of $T\backslash u$ containing v, since that is a star that does not have eigenvalue 1, and the multiplicity of the eigenvalue 1 is $q - 1$, contradiction.

2.2.2 Pending K_1 on both sides

Now the path Q has 4 vertices, and T has a single eigenvalue 1. Writing down an eigenvector for 1 yields a contradiction.

Since both subcases fail, the tree T cannot have diameter 5.

2.3 Diameter 4

Now the path Q has 3 vertices. Let $T\backslash u$ have components pK_1 not on v and let $T\backslash v$ have components rK_1 not on u. Writing down the equation for a nonzero eigenvalue λ yields $(\lambda^2 - p - 1)(\lambda^2 - r - 1) = 1$. If all solutions λ are integers, then $\lambda^2 - p - 1 = \lambda^2 - r - 1 = \pm 1$, so that $p = r$, and the four roots are $\pm\sqrt{p}$, $\pm\sqrt{p+2}$. But these cannot all be integers. □

References

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[4] Mamoru Watanabe & Allen J. Schwenk, Integral starlike trees, J. Austral. Math. Soc. (Ser A) **28** (1979) 120–128.