



Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

A.E. Brouwer

Uniqueness and nonexistence of some graphs related to M_{22}

Department of Pure Mathematics

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Uniqueness and nonexistence of some graphs related to M_{22}

A.E. Brouwer

Centre for Mathematics and Computer Science, Amsterdam

There is a unique distance regular graph with intersection array $i(7,6, 4,4; 1,1,1,6)$; it has 330 vertices, and its automorphism group $M_{22}.2$ acts distance transitively. It does not have an antipodal 2-cover, but it has a unique antipodal 3-cover, and this latter graph has automorphism group $3.M_{22}.2$ acting distance transitively. As a side result we show uniqueness of the strongly regular graph with parameters $(v,k,\lambda,\mu) = (231,30,9,3)$ under the assumption that it is a gamma space with lines of size 3.

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1. UNIQUENESS OF THE CAMERON GRAPH

There exists a strongly regular graph (sometimes called the Cameron graph) on 231 vertices with full automorphism group $M_{22}.2$ constructed by taking as vertices the unordered pairs from a 22-set and joining two pairs whenever they are disjoint and their union is contained in a block of a (fixed) Steiner system $S(3,6,22)$ on this 22-set. (For undefined terminology, see e.g. CAMERON & VAN LINT [3].) This graph becomes the collinearity graph of a partial linear space with lines of size 3 if one takes as lines the triples of pairwise disjoint pairs whose union is a block of the Steiner system. This partial linear space is a gamma space, that is, given a line L and a point x outside, then x is collinear with zero, one or all points of L . The next theorem shows that this property characterizes our graph.

THEOREM 1. *Let (X,\mathcal{L}) be a gamma space with lines of size 3 such that its collinearity graph Γ is strongly regular with parameters $(v,k,\lambda,\mu)=(231,30,9,3)$. Then Γ is isomorphic to the Cameron graph described above.*

(Here, following common practice but unlike [3], v denotes the number of vertices, k the valency, λ the number of common neighbours of two adjacent vertices and μ the number of common neighbours of two nonadjacent vertices of a graph Γ .)

PROOF. Write $\Gamma(x)$ for the set neighbours of a vertex x ; $\mu(x,y) = \Gamma(x) \cap \Gamma(y)$ for the set of common neighbours of two nonadjacent vertices x and y . The graph induced by Γ on $\mu(x,y)$ is called a μ -graph.

- (1) Each μ -graph is either a 3-coclique or a line.
(For: each $\Gamma(x)$ is a subspace of (X,\mathcal{L}) , and the intersection of subspaces is again a subspace.)
- (2) Each vertex is in ten 7-cliques and each line in two 7-cliques; each 7-clique is a subspace isomorphic to the Fano plane.
(For: maximal cliques are subspaces and have at most 11 vertices (e.g. because $\lambda=9$), and at least 7 vertices; but no STS (11) exists, so there are two maximal cliques on each line and each has 7 vertices.)

The m -clique extension of a graph is obtained by replacing each vertex x by an m -clique C_x , and joining each vertex of C_x to each vertex of C_y whenever $x \sim y$.

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P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

- (3) For each vertex x , $\Gamma(x)$ is isomorphic to the 2-clique extension of the line graph of the Petersen graph.

(For: form a graph Δ with the Eano planes on x as vertices and the lines on x as edges and (reverse) inclusion as incidence. Then Δ has 10 vertices, valency 3, no triangles and no quadrangles (otherwise Γ would have $\mu \geq 5$), so Δ is the Petersen graph.)

Note that the line graph of the Petersen graph is an antipodal 3-cover of the complete graph K_5 , so that we have a concept of antipodal concurrent lines.

- (4) Each line L is contained in a unique subspace isomorphic to the $GQ(2,2)$ generalised quadrangle (For: let $L = \{x, y, z\}$ and let M, N be the two lines on x antipodal to L , say $M = \{x, u, v\}$. Let p be a common neighbour of u and y distinct from x . Since M is antipodal to L we have that $\mu(u, y)$ is a 3-coclique, so $p \not\sim x$ and $\mu(p, x)$ is a 3-coclique, so py is antipodal to L and pu is antipodal to M . It follows that the 8 common neighbours of a point of $L \setminus \{x\}$ and a point of $M \setminus \{x\}$ lie on the 8 lines not on x antipodal to L or M , and we find two 3×3 grids having $L \cup M$ in common. But these same 8 points are also joined to $N \setminus \{x\}$ by the 4 lines not on x antipodal to N , and the 15 points and 15 lines we have found form a $GQ(2,2)$. Uniqueness follows since in a $GQ(2,2)$ all μ -graphs are 3-cocliques so that any two intersecting lines are antipodal and the whole construction was forced.)

Let us call a $GQ(2,2)$ subgraph (subgeometry) a *quad*.

- (5) There are 77 quads, 5 on each vertex, 1 on each line, and any two have at most one vertex in common. Two nonadjacent vertices x, y are in a quad if and only if $\mu(x, y)$ is a 3-coclique. Quads are geodetically closed.

(For: if $x \not\sim y$ and $\mu(x, y)$ is a 3-coclique and p is a common neighbour of x and y then the lines px and py are antipodes, and y is in the unique quad containing px .)

We shall write $Q(L)$ and $Q(x, y)$ for the unique quad on the line L or on the nonadjacent vertices x, y (this notation implying that $\mu(x, y)$ is a 3-coclique).

- (6) Let Q be a quad and $x \notin Q$. Then $\Gamma(x) \cap Q$ is either empty or a line. If we write $\Gamma_i(Q) = \{y \mid d(y, Q) = i\}$ then $|\Gamma_0(Q)| = 15$, $|\Gamma_1(Q)| = 120$, $|\Gamma_2(Q)| = 96$.

(For: let L be a line on x meeting Q in y , then L is in a Fano plane together with one of the three lines on y in Q .)

- (7) If Q, Q' are two quads, and $Q \cap Q' = \{z\}$ then the 8 nonneighbours of z in Q' are in $\Gamma_2(Q)$. There are 60 quads meeting Q in a single point, 5 on each point of $\Gamma_2(Q)$, so there are 16 quads disjoint from Q and these are entirely contained within $\Gamma_1(Q)$.

(For: let $x \in Q', y \in Q, x \sim y, x \not\sim z$ then $\Gamma(x) \cap Q$ is a line L on y . This line does not contain z , so z has a neighbour on it and we may assume $z \sim y$. But now $x \sim y \sim z$ and Q' is geodetically closed, so $y \in Q'$, contradiction.)

- (8) There are no three pairwise disjoint quads.

(For: suppose Q_1, Q_2, Q_3 are pairwise disjoint, and define $\gamma_{ij} : Q_i \rightarrow Q_j^*$ by $\gamma_{ij}(x) = \Gamma(x) \cap Q_j$, where Q^* denotes the generalized quadrangle dual to Q .

Then $\psi = \gamma_{32}^{-1} \circ \gamma_{12}$ is an isomorphism from Q_1 onto Q_3 . If $x \in Q_1$ and $x \not\sim \psi(x)$ then $\mu(x, \psi(x))$ is the line $\gamma_{12}(x)$, but $\psi(x)$ also has a neighbour on the line $\gamma_{13}(x)$, contradiction. Thus $x \sim \psi(x)$ for each $x \in Q_1$, and $\gamma_{13} \circ \psi^{-1}$ is a polarity of Q_3 where all points are absolute. But $GQ(2,2)$ has no such polarity, contradiction.)

- (9) For a graph Δ with the quads as vertices, two quads being adjacent whenever they are disjoint. Then Δ is the unique strongly regular graph with parameters $(v, k, \lambda, \mu) = (77, 16, 0, 4)$ and is isomorphic with the graph that has the blocks of $S(3, 6, 22)$ as vertices and pairs of disjoint blocks as edges.

(For: we have seen v, k, λ and $\mu = 4$ is easily checked. Now the result follows from BROUWER [2].)

Now we might continue describing Γ in terms of Δ , exploiting detailed knowledge of Δ . Instead I'll choose another way, showing the rank 4 structure of Γ .

- (10) Γ carries a 3-class association scheme with $(x,y) \in R_0$ iff $x=y$,
 $(x,y) \in R_1$, iff $x \sim y$, $(x,y) \in R_2$ iff $x \not\sim y$ and $\mu(x,y)$ is a line
 $(x,y) \in R_3$ iff $x \not\sim y$ and $\mu(x,y)$ is a 3-coclique.
The parameters are $(p_{0j}^i) = I$,

$$(p_{1j}^i)_{ij} = \begin{pmatrix} 0 & 30 & 0 & 0 \\ 1 & 9 & 16 & 4 \\ 0 & 3 & 21 & 6 \\ 0 & 3 & 24 & 3 \end{pmatrix}, \quad (p_{2j}^i)_{ij} = \begin{pmatrix} 0 & 0 & 160 & 0 \\ 0 & 16 & 112 & 32 \\ 1 & 21 & 108 & 30 \\ 0 & 24 & 120 & 16 \end{pmatrix}, \quad (p_{3j}^i)_{ij} = \begin{pmatrix} 0 & 0 & 0 & 40 \\ 0 & 4 & 32 & 4 \\ 0 & 6 & 30 & 4 \\ 1 & 3 & 16 & 20 \end{pmatrix}.$$

(For:

- $p_{33}^1 = 4$: If $(x,y) \in R_1$, $(x,z), (y,z) \in R_3$ then by (7) x,y,z are all in one quad, the unique quad on xy , and in this quad there are 4 points nonadjacent to x and y .
- $p_{33}^2 = 4$: Suppose $\mu(x,y) = L$. There are two quads on x disjoint from L , and if Q is such a quad then $y \notin Q$ and $\Gamma(y) \cap Q = \emptyset$ (otherwise $\Gamma(y) \cap Q$ would be a line and $\mu(x,y)$ would contain at least 4 points); y is in 5 quads and each meets Q in a single point. These 5 points form an ovoid, as follows from a) and hence 3 of them are adjacent to x . Remain 2 possibilities for z with $(x,z) \in R_3$ on each Q , so $p_{33}^2 = 4$.
- $p_{13}^1 = 4$: Suppose $x \sim y \sim z$, $(x,z) \in R_3$. Then $Q(x,z)$ is the unique quad on xy and z is one of the 4 neighbours of y not on xy in this quad.
- $p_{13}^3 = 3$: Suppose $x \sim z$, $(x,y), (y,z) \in R_3$. Then x,y,z are all in one quad $Q(x,y)$ and z is one of the 3 neighbours of x nonadjacent to y in this quad.
- $p_{13}^2 = 6$: Suppose $x \sim z$, $(x,y) \in R_2$, $(y,z) \in R_3$. Let $Q = Q(y,z)$. By b) the line $L = \mu(x,y)$ meets Q , in a point p , say. Now $\Gamma(x) \cap Q$ is the line pz , and Q is the unique quad containing the line py . For p there are 3 choices on L and in each case we find two possibilities for z .
- $p_{33}^3 = 20$: Suppose $(x,y) \in R_3$, $Q = Q(x,y)$. Inside Q there are 4 points nonadjacent to x and y . Any other point z with $(x,z), (y,z) \in R_3$ must be in $\Gamma_2(Q)$. If $z \in \Gamma_2(Q)$ then the five quads on z meet Q in five points forming an oval \circ in Q . Now Q has 6 ovals, and if \circ, \circ' are any two ovals then there are precisely 32 points z determining either \circ or \circ' (for: $\circ \cap \circ' = \{p\}$ and there are 32 points nonadjacent to p in the four quads distinct from Q on p); it follows that any given oval (and in particular the one containing x,y) is determined by 16 points z .
Thus $p_{33}^3 = 4 + 16 = 20$.

All other p_{jk}^i are determined by these (and the parameters of Γ as a strongly regular graph.)

(11) (X, R_3) is isomorphic with the triangular graph $\binom{22}{2}$.

(For: it has the right parameters by (10), and uniqueness follows by CONNOR [4].)

Let us identify the quads in this triangular graph.

LEMMA. Let Δ be a triangular graph $\binom{n}{2}$ and T a noncomplete subgraph isomorphic to $\binom{m}{2}$. If Δ is labelled with $\binom{Y}{2}$ for some n -set Y then this labelling induces a labelling with $\binom{Z}{2}$ on T , where Z is some m -subset of Y . (In other words, there are only canonical ways to embed noncomplete triangular subgraphs.)

PROOF. Let x,y be vertices of T labelled with ab and ac respectively. We prove that some vertex of T is labelled with bc . Choose $z \in T$, $z \sim y$, $z \not\sim x$. Then z is labelled with cd , say. Now $\mu(x,z)$ is a 4-circuit, so there are two vertices, $u,v \in T$ adjacent to each of x,y,z . But these must be labelled ad and bc . \square

(This Lemma reminds me of recent work by J.I. Hall on Kneser graphs - probably it is a special case of some of his results.)

(12) Γ is the graph with as vertices the pairs from a set of 22 symbols, where two pairs are adjacent whenever they are disjoint and their union is contained in a block of a $S(3,6,22)$ design on the set of symbols.

(For: the collinearity graph of a quad $GQ(2,2)$ is the complement of the triangular graph $\binom{6}{2}$, so by the Lemma and (11) we can label X with the pairs from a set Σ of 22 symbols and the quads correspond to certain 6-subsets of Σ . Each triple in Σ determines a unique quad, so these 6-sets form a Steiner system $S(3,6,22)$ on Σ . If two pairs are adjacent in Γ then they are nonadjacent in (X, R_3) , i.e. disjoint, and they are contained in a quad.) \square

REMARK. The association scheme described in (10) corresponds to the group action, i.e., M_{22} acts rank 4 on X with suborbits $1+30+160+40$.

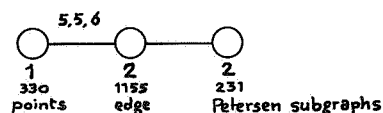
2. UNIQUENESS OF A GRAPH ON 330 VERTICES

Define a graph Γ with as vertices the 330 blocks of the Steiner system $S(5,8,24)$ missing two fixed symbols, where two blocks are adjacent whenever they are disjoint.

We have the following correspondence between graph distance and size of intersection:

$ B \cap B' $	$d(B, B')$	
8	0	
0	1	
2	3	
4	2	if the sextet determined by B and B' has both fixed symbols in the same tetrad,

It is easy to check Γ is distance regular (in fact, distance transitive) with intersection array $i(7,6,4,4;1,1,1,6)$. Γ has fully group of automorphisms $M_{22}.2$. The vertices, edges and Petersen subgraphs of Γ form a geometry with Buekenhout diagram

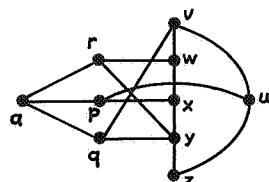


Our aim here is to prove uniqueness of Γ from its parameters. Let us start with a lemma producing the Petersen subgraphs.

LEMMA. Let Γ be a graph with $\mu=c_3=1$ and $\lambda=0, a_2=2$. Then any two vertices at distance two in Γ determine a unique induced Petersen graph.

(For notation, see BIGGS [1]; we do not suppose that Γ is distance regular.)

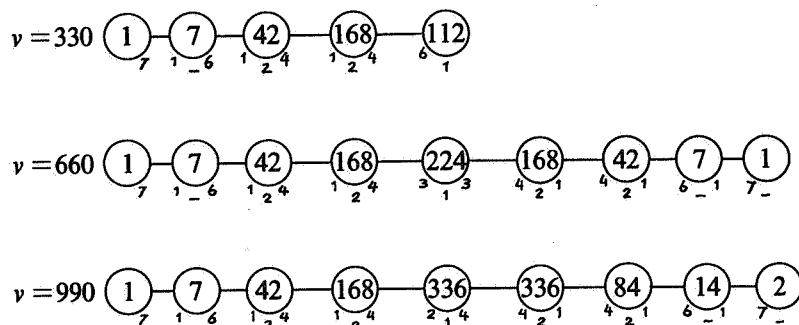
PROOF



Let $d(a,x)=2$, with $a \sim p \sim x$. Let x have neighbours w,y in $\Gamma_2(a)$, with $a \sim r \sim w$, $a \sim q \sim y$. Since Γ has girth 5, all points mentioned are distinct. Since w has two neighbours (r and x) in $\Gamma_2(q)$ we have $d(w,q)=2$, so $q \sim v \sim w$, and similarly $r \sim z \sim y$. Again $d(v,z)=2$ (not $v \sim z$, since $v \sim z \sim r \sim w \sim v$ would be a 4-circuit, and not $d(v,z)=3$ since $\{w,q\} \subset \Gamma(v) \cap \Gamma_2(z)$) so $v \sim u \sim z$ for some vertex u . We must have $d(a,u)=2$ and $d(p,v)=2$ so $p \sim u$, completing our Petersen graph. \square

(We find that there are $k(k-1)/6$ Petersen graphs on a vertex and $vk(k-1)/60$ Petersen graphs altogether, so these numbers must be integers for a graph Γ satisfying the hypotheses of the Lemma.)

This Lemma applies to our graph on 330 vertices as well as to its antipodal 2-covers and 3-covers (these pass all known existence criteria). The three distance distribution diagrams are



In these cases we have $k=7$, each vertex x is in 7 Petersen graphs and the triples induced by these on $\Gamma(x)$ must form the Fano plane; in particular, two Petersen graphs on a vertex x have an edge in common.

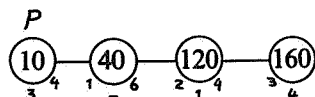
PROPOSITION *There is no distance regular graph on 660 vertices with intersection array $i(7,6,4,4,3,1,1,1;1,1,1,3,4,4,6,7)$.*

PROOF Let Γ be such a graph. Choose vertices x_0, a, b, x_8 with $d(x_0, x_8)=8$, $a \sim b$, $\{a, b\} \subset \Gamma_4(x_0) \cap \Gamma_4(x_8)$. Since $c_4=3$ is odd, there must be a Petersen graph P on the edge ab meeting both $\Gamma_3(x_0) \cap \Gamma(a)$ and $\Gamma_3(x_8) \cap \Gamma(a)$, say $x_3 \in P \cap \Gamma(a) \cap \Gamma_3(x_0)$, $x_5 \in P \cap \Gamma(a) \cap \Gamma_3(x_8)$. Let $x_0 \sim x_1 \sim x_2 \sim x_3 \sim a \sim x_5 \sim x_6 \sim x_7 \sim x_8$ be a geodesic. P must have an edge in common with the Petersen graph determined by x_1 and x_3 , and this edge lies in $\Gamma_3(x_0)$; similarly, P must have an edge in $\Gamma_5(x_0)$ - but one easily sees that this is impossible since $a_4=1$. \square

THEOREM 2 *There is a unique distance regular graph Γ on 330 vertices with intersection array $i(7,6,4,4;1,1,1,6)$.*

PROOF We first show that the graph Δ with as vertices the Petersen subgraphs of Γ where two Petersen graphs are adjacent when they meet, is isomorphic to the Cameron graph. First of all Δ has 231 vertices and valency 30.

Claim The distance distribution around a Petersen graph P is



Indeed: P is geodetically closed so any point in $\Gamma_1(P)$ has a unique neighbour in P . Also, if $x, y \in P$ with $d(x, y) = 2$ then the two neighbours of y at distance two to x are in P and it follows that $\Gamma_1(P)$ is a coclique. If $z \in \Gamma_2(P)$ then at most 7 points of P can be in $\Gamma_2(z)$, so there are points of P in $\Gamma_3(z)$. Now since $c_3 = 1$ we must have that $\Gamma_2(z) \cap P$ is geodetically closed and hence is an edge. If P' is the unique Petersen graph on z meeting P , then P' contains an edge on z in $\Gamma_2(P)$, so z has at most 4 neighbours in $\Gamma_3(P)$. The maximum possible distance to P is 3 since Γ has diameter 4 and $a_4 = 1 < 3$. If some point $u \in \Gamma_3(P)$ had at most two neighbours in $\Gamma_2(P)$ then $|P \cap \Gamma_3(u)| \leq 4$, and removing at most two edges from P we are left with a graph where each vertex has degree at most one - impossible. Thus the number of edges between $\Gamma_2(P)$ and $\Gamma_3(P)$ is both at most and at least $480 = 4 \cdot 120 = 3 \cdot 160$ and we have equality everywhere, proving the claim.

Let us compute λ . If P, P' and P'' are three Petersen graphs that have pairwise nonempty intersection then by the previous $P \cap P' \cap P''$ is nonempty. Let $P \cap P' = \{u, v\}$, then there is one more Petersen graph on $\{u, v\}$, and 4 others on u and on v , so that $\lambda = 1 + 4 + 4 = 9$.

Next look at μ . There are two possibilities (as was to be expected, since the Cameron graph is rank 4, not rank 3): (i) P' meets $\Gamma_1(P)$, and (ii) $d(P, P') > 1$.

In the first case we see from " $a_2(P) = 1$ " that any P'' meeting both P and P' must contain the (unique, since $c_3 = 1$) edge joining P and P' so that P and P' so that P and P' have three common neighbours.

In the second case we see from the distance distribution diagram around P and the fact that any two Petersen graphs on a point have an edge in common, that P' contains 3 edges in $\Gamma_2(P)$.

(Indeed, if $u \in \Gamma_3(P)$ then u is in 6 Petersen graphs meeting $\Gamma_1(P)$, two on each edge uv with $v \in \Gamma_2(P)$, so u is in a unique Petersen graph P' not meeting $\Gamma_1(P)$, and P' contains the three neighbours of u in $\Gamma_2(P)$ so that $P' \cap \Gamma_3(P)$ is a coclique. But the only way to split a Petersen graph into a coclique and a graph where each vertex has degree (at most) one is as $\bar{K}_4 + 3K_2$.)

Thus $\mu = 3$, and by Theorem 1 the graph Δ is isomorphic to the Cameron graph. (Clearly the 3-lines of Δ are the triples of Petersen graphs on a given edge, and the computation of λ also proved the Gamma space property.)

[This gives us a 22-set Σ and a Steiner system $S(3, 6, 22)$ on Σ and a labelling of Δ with $\binom{\Sigma}{2}$ such that Petersen graphs at distance 2 correspond to intersecting pairs and intersecting Petersen graphs correspond to disjoint pairs contained in a block of the Steiner system. We want to let the vertices of Γ correspond to 8-subsets of Σ . This is done as follows:

Given a vertex x , it is in 7 Petersen graphs labelled with 7 pairwise disjoint pairs of symbols. Label x with the set of $22 - 2 \cdot 7 = 8$ remaining symbols. We shall however not use this labelling.]

Each vertex x determines a 7-clique in Δ , and we find 330 7-cliques in Δ in this way; but Δ has only 330 7-cliques, 10 on each vertex of Δ , so we identify Γ as the graph with as vertices the Fano planes in Δ , where two Fano planes are adjacent when they have a line in common. This shows that Γ is uniquely determined. ■

REMARKS

- The Petersen subgraphs arise as follows: let α, β be the two fixed symbols chosen in the symbol set $\Sigma \cup \{\alpha, \beta\}$ in order to define Γ . Any sextet such that α and β lie in the same tetrad T of the sextet has 5 remaining tetrads, and the union of the any two of these is a block of $S(5, 8, 24)$, giving 10 blocks altogether, and these 10 blocks induce a Petersen subgraph in Γ . The pair this Petersen graph is labelled with is $T \setminus \{\alpha, \beta\}$, showing that the labelling proposed above is the correct one.

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Kondrup, 841221