Maximal cocliques in the Kneser graph on point-plane flags in $\mathrm{PG}(4, q)$

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Abstract

We determine the maximal cocliques of size $\geq 4q^2 + 5q + 5$ in the Kneser graph on point-plane flags in $\mathrm{PG}(4, q)$. The maximal size of a coclique in this graph is $(q^2 + q + 1)(q^3 + q^2 + q + 1)$.

1 Introduction

Let Γ be the Kneser graph on the point-plane flags (incident point-plane pairs) in $\mathrm{PG}(4, q)$: two flags (P, A) and (Q, B) are adjacent when they are in general position, i.e., when $A \cap B$ is a single point R distinct from P and Q. In this note we determine the large maximal cocliques in Γ. In particular we show that the largest cocliques have size $(q^2 + q + 1)(q^3 + q^2 + q + 1)$ and consist of all flags (P, A) with the plane A in a fixed hyperplane (solid).

For a maximal coclique $\mathcal{C}$ in Γ, let a *heavy* plane be a plane occurring in at least two $\mathcal{C}$-flags (flags in $\mathcal{C}$). We will see in the next section that a heavy plane is in fact in $q^2 + q + 1$ $\mathcal{C}$-flags.

Theorem 1 *Let $\mathcal{C}$ be a maximal coclique in the graph Γ. Then either*

(A) $\mathcal{C}$ has $q^3 + q^2 + q + 1$ heavy planes, and has structure as described in Proposition 7 (example (i)), or

(B) $\mathcal{C}$ has $q^2 + q + 1$ heavy planes, and has structure as described in Propositions 8–10 (examples (ii)–(v)), or

(C) $\mathcal{C}$ has $q + 1$ heavy planes, and has structure as described in Propositions 11, 12, 14 (examples (vi)–(xi)), or

(D) $\mathcal{C}$ has at most one heavy plane, and $|\mathcal{C}| \leq 4q^2 + 5q + 4$.

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The sizes of the examples are listed in the table below.

(i)	$q^5+2q^4+3q^3+3q^2+2q+1$	(v)	$q^4+3q^3+4q^2+2q+1$	(ix)	q^3+5q^2+3q+1
(ii)	$2q^4+3q^3+4q^2+2q+1$	(vi)	q^3+6q^2+2q+1	(x)	q^3+4q^2+4q+1
(iii)	$2q^4+3q^3+4q^2+2q+1$	(vii)	q^3+5q^2+3q+1	(xi)	q^3+2q^2+8q+1
(iv)	$q^4+3q^3+4q^2+2q+1$	(viii)	q^3+5q^2+3q+1		

This is a variation on the Erdős-Ko-Rado and Hilton-Milner theme. These authors characterized the largest and second largest cocliques in the classical Kneser graph $K(n,k)$ that has as vertices the k-subsets of an n-set (where $2k \le n$), adjacent when disjoint.

The q-analog $K_q(n,k)$ of $K(n,k)$ has as vertices the k-subspaces of an n-dimensional vector space over the field with q elements (with $2k \le n$), where two k-spaces are adjacent when they have trivial intersection. Large cocliques in $K_q(n,k)$ were studied in [1, 2, 4].

These graphs $K_q(n,k)$ can be viewed as graphs with vertex set G/P for a Chevalley group G (namely, $GL(n,q)$) and a maximal parabolic subgroup P (namely, the stabilizer in G of a subspace). More generally one can study G/P for not necessarily maximal P and look at graphs of which the vertices are flags in a finite spherical building. Such a building has a W-valued distance function, and in this setting the Kneser graph is defined as the graph where two vertices are adjacent when they have the largest possible distance (cf. [5, 6]). The present note investigates the case of point-plane flags in $PG(4,q)$, and [3] handles the case of point-hyperplane flags in $PG(n-1,q)$.

2 Closure properties

In this section we show that maximal cocliques have certain closure properties. Consider a maximal coclique $\mathcal{C}$. Flags in $\mathcal{C}$ will be called *$\mathcal{C}$-flags*, and planes occurring in some $\mathcal{C}$-flag, *$\mathcal{C}$-planes*. If (P,A) is a $\mathcal{C}$-flag we will call P the *top* of the flag, and also a top of the plane A.

Lemma 2 *If, for some plane A, the coclique $\mathcal{C}$ contains both (P,A) and (Q,A), with $P \neq Q$, then it contains (R,A) for all $R \subseteq A$.*

Proof. A flag adjacent to (R,A) is adjacent to at least one of (P,A) or (Q,A), so it does not belong to $\mathcal{C}$. Since $\mathcal{C}$ is maximal, $(R,A) \in \mathcal{C}$. □

A $\mathcal{C}$-plane A is called *light* (*heavy*) when $(P,A) \in \mathcal{C}$ for a unique point (for all q^2+q+1 points) P in A. If there are h heavy planes, and l light planes, then $|\mathcal{C}| = (q^2+q+1)h + l$. We shall call a flag *light* when its plane is light.

Lemma 3 *If A and B are distinct heavy planes, then they meet in a line L, and all planes C in the pencil of planes $L \subseteq C \subseteq A+B$ are heavy.*

Proof. Two heavy planes share a line, because both form a flag with a point not in the other plane. Suppose $L \subseteq C \subseteq A+B$, and $(P,C) \notin \mathcal{C}$ for some point $P \subseteq C$. Then there is a $(Q,D) \in \mathcal{C}$ such that $C \cap D$ is a single point different from P, Q. In particular, $Q \not\subseteq L$. Since (Q,D) is not adjacent to (P_A, A) or (P_B, B) for any points $P_A \subseteq A$, $P_B \subseteq B$, each of the planes A, B either meets D in the single point Q or meets D in a line. It follows that D lies in $A+B$ and hence meets C in a line, a contradiction. □

As a consequence we find the following:

Proposition 4 *The configuration of heavy planes is one of the following:*
A: the q^3+q^2+q+1 planes in a solid S;
B_1: the q^2+q+1 planes on a line L;
B_2: the q^2+q+1 planes in a solid S on a fixed point P;
C: the $q+1$ planes on a line L in a solid S;
D: at most one heavy plane.

Proof. A collection of planes that pairwise meet in a line, is the dual of a collection of lines pairwise contained in a plane, that is, a collection of pairwise intersecting lines, and the lines in such a collection are all contained in one plane, or all pass through one point. For the heavy planes this means that all contain a fixed line, or all are contained in a fixed solid. If we are in the former case, we have one of the cases B_1, C, D. If not, the heavy planes form a dual subspace in the fixed solid, and we have case A or B_2. □

Maximal cocliques with at least q^2+q+1 heavy planes will be called *large*, the other ones *small*.

Lemma 5 *Let (P,A) and (P,B) be two $\mathcal{C}$-flags, with $A \cap B = L$ a line. Then $(P,C) \in \mathcal{C}$ for all planes C with $L \subseteq C \subseteq A+B$.*

Proof. Clearly $P \subseteq L$. Suppose $L \subseteq C \subseteq A+B$ and $(Q,D) \in \mathcal{C}$ with (Q,D) adjacent to (P,C). Then $C \cap D$ is a single point different from P, Q. In particular, $P \not\subseteq D$ and $Q \not\subseteq L$. Since (Q,D) is not adjacent to (P,A) or (P,B), each of the planes A, B either meets D in the single point Q or meets D in a line. It follows that D lies in $A+B$ and meets C in a line, a contradiction. □

We see that locally in a point P the planes with top P form a collection of lines that is pencil-closed. Locally in a line L the planes (with fixed top

P on L) form a collection of points that is linearly closed, and therefore has size $(q^i - 1)/(q - 1)$ for some i, $0 \le i \le 3$. Locally in P the planes inside a solid S on P form a dual subspace, and again there are $(q^i - 1)/(q - 1)$ of them for some i, $0 \le i \le 3$.

Proposition 6 *Fix a point P. We have one of the following possibilities:*

(1) The point P is top of all $q^3 + q^2 + q + 1$ planes containing it.

(2) There are a solid S and a line L, where $P \subseteq L \subseteq S$, such that the point P is top of all $2q^2 + q + 1$ planes in S or on L containing it.

(3) There is a solid S on P such that the point P is top of the $q^2 + q + 1$ planes in S on P.

(4) There is a line L on P such that the point P is top of the $q^2 + q + 1$ planes on L.

(5) The point P is top of at most $2q + 1$ planes, at most $q + 1$ on a line and at most $q + 1$ in a solid.

In these five cases, we call the point P (1) *purple*, (2) *red*, (3) *orange*, (4) *yellow*, (5) *white*. In cases (1)–(4) we call P *colored*.

Proof. Fix a point P. If all $\mathcal{C}$-planes contain P, we have case (1). Otherwise, there is a flag $(Q, B) \in \mathcal{C}$ with $P \not\subseteq B$. Put $L = P + Q$ and $S = P + B$. For a $\mathcal{C}$-flag (P, A) either A intersects B in a line not on Q and hence $A \subseteq S$, or $Q \subseteq A$ and hence $L \subseteq A$ (or both). By the remark above there are 0, 1, $q + 1$ or $q^2 + q + 1$ such flags of each type. If $q^2 + q + 1$ occurs, we have case (2), (3) or (4). If not, case (5). Note that if there are $q + 1$ flags of each type, then there is a flag of both types. □

3 Proof of the theorem—the large examples

We now start classifying the maximal cocliques with various properties. In the rest of this paper we find examples (i)–(xviii). For each example, we give the size c and the number h of heavy planes. Each time it is straightforward to check that a given example is in fact a maximal coclique, and this is not mentioned separately.

Proposition 4 shows that the number of heavy planes is as is claimed in the theorem. Case A (of theorem and proposition) is now settled by

Proposition 7 *Let $\mathcal{C}$ be a maximal coclique as in case A. Then we have*

(i) $c = (q^2 + q + 1)(q^3 + q^2 + q + 1)$, $h = q^3 + q^2 + q + 1$, and $\mathcal{C}$ consists of all flags (P, A) with A contained in a fixed solid S_0. □

Next we handle the case that all $\mathcal{C}$-planes contain a given point. We are in this case when there is a purple point.

Proposition 8 *If all $\mathcal{C}$-planes contain a fixed point P_0, then we have $c = (q^2+q+1)(2q^2+q+1)$, $h = q^2+q+1$, and $\mathcal{C}$ is as described in (ii) or (iii) below:*

(ii) $\mathcal{C}$ consists of all flags (P, A) with P coinciding with a fixed point P_0, or A containing a fixed line L_0 on P_0.

(iii) $\mathcal{C}$ consists of all flags (P, A) with P coinciding with a fixed point P_0 or A containing P_0 and contained in a fixed solid S_0.

Proof. In this case $\mathcal{C}$ consists of all flags (P_0, A), together with a collection of flags (Q, B) with $Q \neq P_0$, where the (necessarily heavy) planes B pairwise meet in a line on P_0. Locally in P_0 these planes form a collection of pairwise intersecting lines, and hence either all lines contain a fixed point, or all lines are contained in a fixed plane. For the planes B this means that they either all contain a fixed line L_0, or are contained in a fixed solid S_0. □

Case B_1 of Proposition 4 is settled by

Proposition 9 *A maximal coclique with q^2+q+1 heavy planes all on a fixed line L_0 is of type (ii) or (iv).*

(iv) $c = q^4+3q^3+4q^2+2q+1$, $h = q^2+q+1$, and $\mathcal{C}$ consists of all flags (P, A) with A containing a fixed line L_0 or A contained in a fixed solid S_0 on L_0, where $P = A \cap L_0$.

Proof. The heavy planes on L_0 give a non-maximal coclique. The (necessarily light) flags that extend it must have their top on the line L_0, and two extending flags either have the same top, or intersect in a line. If all extending flags have the same top P_0 then we are in the situation of Proposition 8 and have example (ii).

If there are extending flags (P, A) and (Q, B) with $P \neq Q$ and A and B meeting in a line M, then all extending flags live in the solid $L_0 + M$. Indeed, if the top is different from P then the plane intersects A in a line, otherwise the plane intersects B in a line. In this case we have a coclique of size $(q^2+q+1)^2 + q^3 + q^2$, since there are q^3+q^2 light planes, the planes in the solid $L_0 + M$ not containing L_0. □

Case B_2 of Proposition 4 (and case B of the theorem) is settled by

Proposition 10 *A maximal coclique with q^2+q+1 heavy planes all on a fixed point P_0 inside a fixed solid S_0, is of type (iii) or (v).*

(v) $c = q^4+3q^3+4q^2+2q+1$, $h = q^2+q+1$, and $\mathcal{C}$ consists of all flags (P, A) with A containing a fixed point P_0 and contained in a fixed solid S_0, or A contained in S_0 but not containing P_0, and $P = A \cap L_0$, where L_0 is a fixed line, or A containing L_0 and P coinciding with P_0. Here $P_0 \subseteq L_0 \subseteq S_0$.

Proof. Any light flag (Q, B) must have $Q = P_0$ or $B \subseteq S_0$. If always $B \subseteq S_0$, we have example (i), impossible. If always $Q = P_0$, we have example (iii). If $(P_0, B_1), (Q_2, B_2) \in \mathcal{C}$, with $B_1 \not\subseteq S_0$ and $P_0 \not\subseteq B_2$, then $B_1 \cap S_0$ is a line L_0 on P_0 and Q_2 must be on L_0. We find that the light planes are the q^2 planes on L_0 not in S_0, and the q^3 planes in S_0 not on L_0. □

4 The case of a red point

Recall that maximal cocliques with at most $q+1$ heavy planes are called *small*. As we saw in the above, small cocliques do not have all planes in a fixed solid, or on a fixed point. The case where a red point occurs is settled by

Proposition 11 *Let $\mathcal{C}$ be a small maximal coclique in the graph Γ. Assume that $\mathcal{C}$ contains a red point P. Then we have one of the cases (vi), (vii), (viii), (xii) or (xiii) below.*

(vi) $c = q^3+6q^2+2q+1$, $h = q+1$, and $\mathcal{C}$ is constructed as follows. Let $P \subseteq L$ and let R be a point not on L. Put $M = P+R$. Let S be a solid on $L+M$. Take (a) all flags (P, A) with $P \subseteq A \subseteq S$, (b) all flags (P, A) with $L \subseteq A$, (c) all flags (Q, B) with $Q \subseteq L$, $P \not\subseteq B$, $Q+R \subseteq B \subseteq S$, (d) all flags (R, C) with $M \subseteq C$, $C \not\subseteq S$, (e) all flags (R', D) with $M \subseteq D \subseteq S$.

(vii) $c = q^3+5q^2+3q+1$, $h = q+1$, and C is constructed as follows. Let $P \subseteq L$ and let R be a point not on L. Put $M = P+R$ and $D = L+M$. Let S, T be two solids meeting in the plane D. Take (a) all flags (P, A) with $P \subseteq A \subseteq S$, (b) all flags (P, A) with $L \subseteq A$, (c) all flags (P', A) with $L \subseteq A \subseteq T$, (d) all flags (Q, B) with $Q \subseteq L$, $P \not\subseteq B$, $Q+R \subseteq B \subseteq S$, (e) all flags (R, C) with $M \subseteq C$, $C \subseteq T$.

(viii) $c = q^3+5q^2+3q+1$, $h = q+1$, and $\mathcal{C}$ is constructed as follows. Let S, T be two solids meeting in the plane D. Let K, L be two lines in D meeting in the point Q. Let P be a point of L other than Q. Take (a) all flags (P, A) with $P \subseteq A \subseteq S$, (b) all flags (P, A) with $L \subseteq A$, (c) all flags

(P', A) with $L \subseteq A \subseteq T$, $P' \neq P$, (d) all flags (Q, B) with $K \subseteq B \subseteq S$, $P \not\subseteq B$, (e) all flags (R, C) with $P \subseteq C \subseteq T$, $L \not\subseteq C$, $R = C \cap K$.

(xii) $c = 4q^2 + 4q + 1$, $h = 1$, and $\mathcal{C}$ is constructed as follows. Let S, T be two solids meeting in the plane D. Let L be a line in S not in D, and $P = L \cap D$. Let K be a line in D not on P. Take (a) all flags (P, A) with $P \subseteq A \subseteq S$, (b) all flags (P, A) with $L \subseteq A$, (c) all flags (Q, B) with $K \subseteq B \subseteq S$, $P \not\subseteq B$, $Q = B \cap L$, (d) all flags (R, C) with $P \subseteq C \subseteq T$, $C \neq D$, $R = C \cap K$, (e) all flags (R', D).

(xiii) $c = 4q^2 + 3q + 2$, $h = 1$, and C is constructed as follows. Let P be a point, L, N lines, B a plane, S a solid, with $P = N \cap S$, $P \subseteq L$, $P \not\subseteq B$. Take (a) all flags (P, A) with $P \subseteq A \subseteq S$, (b) all flags (P, A) with $L \subseteq A$, (c) the flag (Q, B) with $Q = B \cap L$, (d) all flags (R, C) with $N \subseteq C$, $R = C \cap B$, (e) all flags (P', D) where $D = L + N$.

Proof. Let (Q, B) be a $\mathcal{C}$-flag with $P \not\subseteq B$. Put $L := P + Q$ and $S := P + B$. Then $(P, A) \in \mathcal{C}$ for all planes A on L and all planes A on P in S. If (Q', B') is another $\mathcal{C}$-flag with $P \not\subseteq B'$, then $B' \subseteq S$ and $Q' = B' \cap L$.

(a) Suppose that every $\mathcal{C}$-plane not in S contains L. Then $\mathcal{C}$ contains all flags (Q', B') with $B' \subseteq S$ and $Q' \subseteq B' \cap L$. Since $\mathcal{C}$ is small, not all planes on L are heavy. It follows that there is a $\mathcal{C}$-plane not containing L and with top not on L, necessarily on P in S. Then every $\mathcal{C}$-plane not in S has top P, and all planes on P in S are heavy, a contradiction.

(b) Let $(R, C) \in \mathcal{C}$ with $C \not\subseteq S$ and $M := C \cap S \neq L$. Now M is a line on P, so meets $\mathcal{C}$-planes in S not on P in a single point, which must be R. If all $\mathcal{C}$-flags (R', C') with C' not contained in S and $R' \neq P$ do have $C' \cap S = M$, then they must also all have the same point $R' = R = B \cap M$. We have case (vi).

(c) If all $\mathcal{C}$-flags (R', C') with C' not contained in S have $C' \cap S = L$ or M, but there are $\mathcal{C}$-flags (R', C'), necessarily on L, with $R' \not\subseteq S$, then C' is heavy, and all planes C and planes C' must meet in a line, and hence all are contained in a solid $T = C + L = C' + M$. We have case (vii).

(d) Now let also $(R', C') \in \mathcal{C}$ with $R' \neq P$ and $C' \not\subseteq S$ and $M' := C' \cap S \neq L, M$. We have $R' = M' \cap B$. The planes C and C' meet in a line on P. If all such lines M' lie in the plane $L + M$, we have case (viii).

(e) If all lines $M = C \cap S$ (for $\mathcal{C}$-planes C not in S and with top other than P) are coplanar, where this plane D containing these lines does not contain L, then we have case (xii).

(f) Finally, if the lines $M = C \cap S$ (for $\mathcal{C}$-planes C not in S and with top other than P) are not coplanar, then (Q, B) is uniquely determined, and we have case (xiii). □

5 The case of an orange point

The case where an orange point occurs is settled by

Proposition 12 *Let $\mathcal{C}$ be a small maximal coclique without red points, but with an orange point P. Then we have one of the cases (x), (xiv), (xv), (xvi).*

(x) $c = q^3 + 4q^2 + 4q + 1$, $h = q + 1$, and $\mathcal{C}$ is constructed as follows. Let K be a line not on P, and S, T solids containing both P and K. Take (a) all flags (P, A) with $P \subseteq A \subseteq S$, (b) all flags (Q, B) with $K \subseteq B \subseteq S$, (c) all flags (R, C) with $P \subseteq C \subseteq T$, $C \not\subseteq S$, $R = K \cap C$.

(xiv) $c = 3q^2 + 3q + 3$, $h = 1$, and $\mathcal{C}$ is constructed as follows. Let N be a line on P, B a plane disjoint from N, and $S = P + B$. Take (a) all flags (P, A) with $P \subseteq A \subseteq S$, (b) all flags (Q, B), (c) all flags (R, C) with $N \subseteq C$, $R = B \cap C$.

(xv) $c = 2q^2 + 8q + 1$, $h = 1$, and $\mathcal{C}$ is constructed as follows. Let S, T be two solids on P, and let $D = S \cap T$. Let R_1, R_2, R_3 be three noncollinear points in D, distinct from P and such that the three lines $P + R_i$ $(i = 1, 2, 3)$ are distinct. Take (a) all flags (P, A) with $P \subseteq A \subseteq S$, (b) all flags (Q, D), (c) all flags (R_i, C), where C is a plane on $P + R_i$ in T, (d) all flags (Q, B) where $Q = B \cap (P + R_i)$ and $R_j + R_k \subseteq B$, where $\{i, j, k\} = \{1, 2, 3\}$.

(xvi) $c = q^2 + 2q + 6$, $h = 0$, and $\mathcal{C}$ is constructed as follows. Let S be a solid on P, and let N be a line on P not in S. Let K be a line in S not on P. Let R_1, R_2 be two points of S not in the plane $P + K$, and such that $P + R_1 \neq P + R_2$, and $R_1 + K \neq R_2 + K$. Take (a) all flags (P, A) with $P \subseteq A \subseteq S$, (b) all flags $(R, R + N)$ where $R \subseteq K$ or $R = R_1$ or $R = R_2$, (c) the two flags (Q, B) where $Q = (P + R_i) \cap B$, $B = R_j + K$, where $\{i, j\} = \{1, 2\}$.

Proof. Let $(P, A) \in \mathcal{C}$ for every plane A on P in the solid S, where there are no further $\mathcal{C}$-flags with top P. For every $\mathcal{C}$-flag (Q, B) with $P \not\subseteq B$ we have $B \subseteq S$. Since P is not purple, there are such (Q, B).

If (R, C) is a $\mathcal{C}$-flag with C not in S, and we put $M := C \cap S$, then for each $\mathcal{C}$-flag (Q, B) as above we have $R \subseteq B$ or $Q \subseteq C$, that is, $R = M \cap B$ or $Q = M \cap B$. Since $\mathcal{C}$ is small, there are such (R, C).

If some point R lies outside S, then always $Q = M \cap B$ so that M is a fixed line. For all $\mathcal{C}$-planes not containing M, the top must lie on M, so all planes on M are heavy, but this is impossible since $\mathcal{C}$ is small.

We see that for each $\mathcal{C}$-flag (Q, B) as above we have $R \subseteq B$ or $R \subseteq P+Q$. Since P is not in any further $\mathcal{C}$-flags, the planes C are not heavy, so that all lines $P + R$ are distinct. Therefore, the planes C meet pairwise in a line, so they all contain a common line N or are all contained in the same solid T. In the latter case, all points R lie in the plane $S \cap T$ on P.

If all points R lie on a line K in S, not on P, then all planes on K in S are heavy, and all planes C meet pairwise in a line, so are contained in the same solid $C + K$. This is case (x).

If the points R span a plane B in S not on P, then B is heavy, and all planes C contain the same line N. This is case (xiv).

Fix one of the points R, say R_1. Since $P + R_1$ does not contain any point $R \neq R_1$, there is a $\mathcal{C}$-flag (Q_1, B_1) with $R_1 \subseteq B_1$, $R_1 \not\subseteq P + Q_1$. Now that not all points R are in B_1, there is a unique R_2 on $P + Q_1$, not in B_1, and we find (Q_2, B_2) where B_2 contains R_2 and all other points R except one. Thus, the line $K = B_1 \cap B_2$ contains all points R except two.

If the points R span a plane D on P, then D is heavy. Every two planes C meet in a line, so all are contained in the same solid $C + D$. The line K meets D is a single point, so that there are three points R. This is case (xv).

Finally, if the points R span all of S, then all planes C contain a common line N and we find case (xvi). □

6 The case of a yellow point

Proposition 13 *Let $\mathcal{C}$ be a small maximal coclique without red or orange points, but with a yellow point. Then we have case (xvii) or (xviii).*

(xvii) $c = 4q^2 + 4q + 1$, $h = 1$, and C is constructed as follows. Let P_1, P_2, P_3 be three noncollinear points. Put $L_1 = P_1 + P_2$, $L_2 = P_2 + P_3$, $L_3 = P_3 + P_1$ and $C = P_1 + P_2 + P_3$. Take (a) all flags (P_i, A) with $L_i \subseteq A$ $(i = 1, 2, 3)$, (b) all flags (R, C).

(xviii) $c = 2q^2 + 6q + 1$, $h = 1$, and $\mathcal{C}$ is constructed as follows. Let P, Q, R, R' be four distinct points, $L = P + Q$, $K = P + R$, where R' is on K, $D = K + L$ and S, S' two solids on the plane D. Take (a) all flags (P, A) with $L \subseteq A$, (b) all flags (Q, B) with either $B \subseteq S$, $B \cap K = R$, or $B \subseteq S'$, $B \cap K = R'$, (c) all flags (R', C) with $K \subseteq C \subseteq S$, and all flags (R, C) with $K \subseteq C \subseteq S'$, (d) all flags (P', D).

Proof. Let P be a point of a line L such that $(P, A) \in \mathcal{C}$ for every plane A on L. Then there are no further flags with top P. For every $\mathcal{C}$-flag (Q, B) with B not on P we have $Q \subseteq L$. Since $(P, D) \notin \mathcal{C}$ for $L \not\subseteq D$, for any such D there is a $\mathcal{C}$-plane B not on P such that $B \cap D$ is a single point. It follows that the $\mathcal{C}$-planes B not on P are not contained in a common solid on P.

If $(Q, B), (Q', B')$ are two $\mathcal{C}$-flags with distinct Q, Q' and B, B' not on P, then $B \cap B'$ is a line K, and $B + L = B' + L$. It follows that all $\mathcal{C}$-planes not on P are in the same solid (on L), a contradiction. So, all $\mathcal{C}$-planes B not on P have the same top Q, and there are $\mathcal{C}$-planes B, B' not on P such that $P + B \neq P + B'$.

Consider the $\mathcal{C}$-flags (R, C) with $P \subseteq C$, $R \neq P$. Any two such planes C must meet in a line, so all are contained in one solid, or all contain a fixed line. First consider the former case: there is a solid S on P such that all $\mathcal{C}$-flags (R, C) with $P \subseteq C$, $R \neq P$ have $C \subseteq S$. If $L \subseteq S$, then every flag (Q, D) with $D \subseteq S$ is in $\mathcal{C}$, a case considered already. So $L \not\subseteq S$. For any two $\mathcal{C}$-flags (Q, B) and (R, C) (with notation as before), we have $R \subseteq M$ or $C = P + M$, where $M = B \cap S$. In all cases $R \subseteq P + M$. Pick B, B' where $P + B \neq P + B'$. Then the planes $P + M$ and $P + M'$ are distinct, and meet in a line K on P that contains all points R. Now the plane $K + L$ is heavy but not contained in S, contradiction.

Consequently, there is a line K on P such that all $\mathcal{C}$-flags (R, C) with $P \subseteq C$, $R \neq P$ have $K \subseteq C$. If $K = L$, then all planes on L are heavy, a case considered already. So let $K \neq L$. The plane $K + L$ is heavy. Let C denote planes on P other than $K + L$. For each $\mathcal{C}$-flag (R, C) either $B \cap C$ is a line, or $R \subseteq B$. In both cases $(P + B) \cap C$ contains the point R. This holds for each $\mathcal{C}$-plane B not on P. Let $D = (P + B) \cap (P + B')$, then D is a plane on L containing all points R. The C do not all lie in a solid, but contain K and a line in D. It follows that $K \subseteq D$, so that K meets all planes B, and $D = K + L$. The planes C are not heavy (since there are no further planes with top P), so each plane C has a unique top R, and since $C \neq D$, this top lies on K. If all tops R coincide, we are in case (xvii).

Suppose $(R, C), (R', C') \in \mathcal{C}$ with $R \neq R'$. Write $S(B) = B + P$ and $S(C) = C + Q$. If $(B, Q), (C, R) \in \mathcal{C}$ and $S(B) \neq S(C)$, then B and C do not meet in a line, so that $R = B \cap C = B \cap K$. If $S(B) \neq S(C), S(C')$, then $R = R'$, contrary to assumption. So $S(C) \neq S(C')$, and the planes B are all in one of these two solids. If $S(C'') \neq S(C), S(C')$, then $R'' = B \cap C'' = B \cap K = R$. And with $S(B') \neq S(C')$ also $R'' = B' \cap C'' = B' \cap K = R'$, contradiction again. So also the planes C are in one of two solids. This is case (xviii). □

7 Proof of the theorem—Case C

We continue the proof of the theorem, assuming that we are not in one of the cases settled already. So all points are white, that is, top of at most $2q+1$ planes, at most $q+1$ on a line, and at most $q+1$ in a solid. Recall that in case C we have $q+1$ heavy planes A with $L \subseteq A \subseteq S$.
Also in this case we are able to give a full classification.

Proposition 14 *Let $\mathcal{C}$ be a small maximal coclique without colored points, having exactly $q+1$ heavy planes A with $L \subseteq A \subseteq S$, for some line L and solid S. Then we have case (ix) or (xi).*

(ix) $c = q^3 + 5q^2 + 3q + 1$, $h = q+1$, and $\mathcal{C}$ is constructed as follows. Let P, Q, R be three noncollinear points, and put $L = P+Q$, $M = P+R$, $N = Q+R$ and $D = P+Q+R$. Let S, T be two solids meeting in the plane D. Take (a) all flags (P', A) with $L \subseteq A \subseteq S$, (b) all flags (P, B) with $M \subseteq B \subseteq T$, $B \neq D$, (c) all flags (Q, B) with $N \subseteq B \subseteq T$, $B \neq D$, (d) all flags (P', B) with $B \subseteq S$ and either $P' \subseteq M$ and $Q \subseteq B$, or $P' \subseteq N$ and $P \subseteq B$, or $P' = R$.

(xi) $c = q^3 + 2q^2 + 8q + 1$, $h = q+1$, and $\mathcal{C}$ is constructed as follows. Let R_1, R_2, R_3 be three noncollinear points, and put $D = R_1 + R_2 + R_3$. Let L be a line in D not on one of the R_i. For $i, j, k = 1, 2, 3$, pairwise distinct, let $M_i = R_j + R_k$ and $P_i = L \cap M_i$. Let S, T be two solids meeting in the plane D. Take (a) all flags (P', A) with $L \subseteq A \subseteq S$, (b) all flags (P_i, B) with $M_i \subseteq B \subseteq T$, $B \neq D$, (c) all flags (R_i, B) with $P_i \subseteq B \subseteq S$, $B \neq D$.

Proof. Since P is white, there are no further $\mathcal{C}$-flags (P, B) with $P \subseteq L \subseteq B$. If a $\mathcal{C}$-flag (P, B) has $P \not\subseteq S$, then $L \subseteq B$, and all $\mathcal{C}$-planes that do not contain L have their top on L, so all planes on L are heavy, by maximality of $\mathcal{C}$, contradiction. It follows that light flags (P, B) have $L \not\subseteq B$ and satisfy (i) $P \subseteq L$; or (ii) $P \not\subseteq L$ but $P \subseteq S$ and (hence) $B \subseteq S$.

Since we are not in case (i), and have seen all flags on L with top on L already, there is a flag (P, B) with $P \subseteq L$, $B \not\subseteq S$, $M := B \cap S \neq L$. Since not all planes on M are $\mathcal{C}$-planes with top P (we handled that case already), there must be further planes not contained in S and meeting L in a point other than P. If Q is a point of L other than P, then any light planes with top Q are in the solid $T := Q + B = L + B$ on a fixed line N_Q where $N_Q \subseteq S \cap T$, and then also the light planes with top P are in the solid T on a fixed line N_P with $M = N_P \subseteq S \cap T$. Since not all tops of planes not on L are on L (otherwise all planes on L are heavy), there is a flag (R, C) with $R \not\subseteq L$, $L \not\subseteq C$, $C \subseteq S$. For each top P on L we must have $R \subseteq N_P$ or $P \subseteq C$.

If all lines N_P are concurrent, and there are at least three, we find example (x), which contains an orange point. If there are only two lines N_P, we find example (ix). Otherwise, since C meets L in only one point, all except one of the lines N_P pass through R, and this contradicts linearity except in the case of precisely three lines N_P, where we can have three points R, and we find example (xi). $\square$

8 Intermezzo—flags in the plane

Consider a collection $\mathcal{A}$ of point-line flags in a projective plane of order q, with a partition $\{\mathcal{A}_1, \ldots, \mathcal{A}_m\}$ into $m \geq 2$ parts, such that if $(P, L) \in \mathcal{A}_i$ and $(Q, M) \in \mathcal{A}_j$, with $i \neq j$, then $P \subseteq M$ or $Q \subseteq L$.

Our aim is to bound $|\mathcal{A}|$ when each point and each line occurs at most once among the flags in a given part $\mathcal{A}_i$, and at most twice in $\mathcal{A}$, while each part $\mathcal{A}_i$ contains two flags (P, L) and (P', L') with $P \not\subseteq L'$ and $P' \not\subseteq L$.

Example 1. Fix a conic in $PG(2, 3)$, and take $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, where $\mathcal{A}_1$ consists of the four flags (P, L) with P on the conic, and L the tangent at P, and $\mathcal{A}_2$ consists of the six flags (Q, M) with Q an exterior point, and M the unique secant on Q.

Example 2. In $PG(2, q)$, fix three noncollinear points P_1, P_2, P_3 and take $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$, where $\mathcal{A}_1$ consists of the $q+1$ flags (P, L) with $P \subseteq P_2 + P_3$ and $L = P + P_1$, and $\mathcal{A}_2$ of the two flags $(P_i, P_i + P_2)$ $(i = 1, 3)$, and $\mathcal{A}_3$ of the two flags $(P_i, P_i + P_3)$ $(i = 1, 2)$.

Let (P, L) and (P', L') be flags in $\mathcal{A}_1$ such that $P \not\subseteq L'$ and $P' \not\subseteq L$. Any flag (Q, M) in $\mathcal{A} \setminus \mathcal{A}_1$ has either $Q = L \cap L'$, or $M = P + P'$, or $Q \subseteq L$ and $P' \subseteq M$, or $Q \subseteq L'$ and $P \subseteq M$. If $m = 2$, then $\mathcal{A}_2$ contains at most $1 + 1 + (q - 1) + (q - 1) = 2q$ flags, and the same holds for $\mathcal{A}_1$, so $|\mathcal{A}| \leq 4q$. If $m \geq 3$, then two flags $(Q, M), (Q', M')$ with $Q, Q' \subseteq L$ and $P' \subseteq M, M'$ must belong to the same part, and we find $|\mathcal{A} \setminus \mathcal{A}_1| \leq 2 + 2 + (q - 1) + (q - 1) = 2q + 2$, so that $|\mathcal{A}| \leq 3q + 3$. Altogether $|\mathcal{A}| \leq \max(3q + 3, 4q)$.

Suppose $\mathcal{A}_1$ contains (at least) three flags. For a flag (Q, M) in $\mathcal{A} \setminus \mathcal{A}_1$ we find three conditions, at least two on Q or at least two on M, so that $|\mathcal{A}_2| \leq 6$ and $|\mathcal{A} \setminus \mathcal{A}_1| \leq 12$. It follows that $|\mathcal{A}| \leq \max(12, 2q + 2)$ for $m = 2$ and $|\mathcal{A}| \leq \max(18, 2q + 4)$ for $m > 2$. Combining both estimates, we have in all cases $|\mathcal{A}| \leq 3q + 3$.

9 The subspace spanned by the tops

Not all $\mathcal{C}$-flags can have their top on the same line L, since the planes on L would be heavy. Consider the case where all tops lie in a plane A.

Proposition 15 *Let $\mathcal{C}$ be a maximal coclique without colored points, with a plane A containing all tops, and no further heavy planes. Then all other $\mathcal{C}$-planes intersect A in a line.*

Proof. There is no line containing the tops of all planes different from A, because its points would be colored. Suppose that there is a $\mathcal{C}$-flag (Q, B) with $A \cap B = Q$, a single point. There is a $\mathcal{C}$-plane B' not containing Q. It intersects B in a line M disjoint from A, hence also intersects A in a single point Q'.

The $\mathcal{C}$-planes C that intersect A in a line contain Q and Q', so if there is at least one such plane, then all planes different from A have their top on $Q + Q'$. Contradiction.

If there is no such plane, then all other $\mathcal{C}$-planes intersect A in a single point, and not all of them have their top on $Q+Q'$. It follows that all contain the line M and the coclique is not maximal. □

Proposition 16 *Let $\mathcal{C}$ be a maximal coclique without colored points, with a plane A containing all tops, and no further heavy planes. Then $|\mathcal{C}| \leq 4q^2 + 4q + 1$.*

Proof. All $\mathcal{C}$-planes other than A meet A in a line. So a $\mathcal{C}$-plane $B \neq A$ determines a *long flag* (Q, M, B, S), where Q is the top, $M = A \cap B$ is a line and $S = A + B$ a solid. The collection of long flags gives a collection $\mathcal{A}$ of flags (Q, M) in the plane A, where $\mathcal{A}$ has a partition indexed by S, and $\mathcal{A}_S$ contains the (Q, M) from a long flag (Q, M, B, S). We are in the situation of the previous section:

If (Q, M, B, S) is a long flag, then because of Lemma 5 all (Q, M, B', S) are. Since Q is white, it is in at most two such pencils $(Q, M, *, S)$, and at most one with given S, so that Q occurs at most twice in $\mathcal{A}$, and at most once in $\mathcal{A}_S$. Since all planes other than A are light, there are no long flags (Q', M, B', S) with $Q' \neq Q$, so that M occurs at most once in $\mathcal{A}_S$. If (Q, M, B, S) and (Q', M', B', S') are long flags and $Q \not\subseteq M'$ and $Q' \not\subseteq M$, then $S = S'$, since B and B' must meet in a line. For every long flag (Q, M, B, S) there is such a (Q', M', B', S), for otherwise all flags with top Q and plane containing M are in $\mathcal{C}$, and Q would be colored. If (Q_1, M, B_1, S_1) and (Q_2, M, B_2, S_2) are long flags, with $Q_1 \neq Q_2$, then $S_1 \neq S_2$, and now

(R, N, C, T) with R not on M (which exists since not all tops are on M) must have either $N = Q_1 + R$ and $T = S_2$, or $N = Q_2 + R$ and $T = S_1$. In particular, M cannot contain a 3rd top, so that M occurs at most once in $\mathcal{A}$.

Having verified the assumptions of the previous section, we may use the conclusion that the number of flags (Q, M) is at most $3q + 3$. It follows that $|\mathcal{C}| = q^2 + q + 1 + q|\mathcal{A}| \leq 4q^2 + 4q + 1$. □

Next, consider the case with all tops in a solid.

Proposition 17 *Let $\mathcal{C}$ be a maximal coclique without colored points and with at most one heavy plane, such that all tops lie in a solid S. Then all tops lie in a plane A.*

Proof. Suppose all tops are in S. Not all $\mathcal{C}$-planes are in S, since $\mathcal{C}$ is small. Let $\mathcal{F}$ be the collection of $\mathcal{C}$-flags (Q, B) such that $B \cap S$ is a line, and let $\mathcal{L}$ be the set of those lines. Then $\mathcal{L}$ is intersecting. If $|\mathcal{L}| = 1$ then any top Q on this unique line is colored. Otherwise, if all lines contain a fixed point P, then P is colored. Contradiction in both cases. Hence all lines are in a plane A, and this plane is heavy. Suppose (R, C) is a $\mathcal{C}$-flag with $R \not\subseteq A$. Then A and C meet in a line L. If (Q, B) is a $\mathcal{C}$-flag with $B \not\subseteq S$ then $Q \subseteq L$. It follows that all planes on L in S are heavy, a contradiction. □

10 Proof of the theorem—Case D

Each of the examples (i)–(xi) occurring in the conclusion of the theorem has one or more colored points. So in bounding the size of $\mathcal{C}$, having only white points will play an important part.

We further investigate case D by ignoring a heavy plane, and adding q^2+q to the bound for the number of planes in those cases where Proposition 16 does not apply. Recall that $\mathcal{C}$ is necessarily small and all points are white. We start with a lemma that will be used several times in our estimates.

Lemma 18 *Let U be a subspace, and let $\mathcal{F} \subseteq \mathcal{C}$ be a set of flags (R, C) with $C \cap U = R$. Then either all flags in $\mathcal{F}$ have the same top or the $\mathcal{F}$-planes intersect pairwise in a line.*

Proof. Two $\mathcal{F}$-planes with different top must intersect in a line, because they cannot contain the other one's top. If two $\mathcal{F}$-planes with the same top are otherwise disjoint, then there are no $\mathcal{F}$-planes with another top. □

Proposition 19 *Let $\mathcal{C}$ be a maximal coclique without colored points, containing three flags (P_i, A_i) $(i = 1, 2, 3)$, where the three planes A_i are distinct and contain a common line L but are not all contained in one solid, and the three points P_i are not on L. Then $|\mathcal{C}| \leq 4q^2 + 4q + 2$.*

Proof. We count $\mathcal{C}$-flags (R, C).

(i) There are at most $q^2 + q + 1$ planes C with $L \subseteq C$.

(ii) There are at most $q^2 + q$ flags (R, C) with $R \subseteq L \not\subseteq C$:
We apply Lemma 18 with $U = L$. If all these flags have the same (white) top then there are at most $2q + 1$. If not, then the planes intersect pairwise in a line. So all are in a solid or on the same line. Since there are at least two points R, and $L \not\subseteq C$, all planes are in a solid on L. Since the points R are white, each R is top of at most $q + 1$ flags in that solid, and in fact of at most q since $L \not\subseteq C$.

(iii) There are at most q^2+q+1 planes C in flags (R, C) where C contains at least two of the points P_i $(i = 1, 2, 3)$ and R is not on L:
If C meets L and contains P_i, P_j, then it is contained in $A_i + A_j$ and meets the third plane in a single point on L only, a contradiction. So C meets each of the A_i in a single point only, either P_i or R. Now the planes C form a collection of planes pairwise intersecting in a line, hence all are on a line or are contained in a solid. If all contain the same line, there are at most $q^2 + q + 1$ planes C. Otherwise, all are contained in a solid S and contain a side of the triangle $P_1P_2P_3$ so there are at most $3q + 1$.

Finally, add $q^2 + q$ to convert the count of planes into a count of flags. □

Proposition 20 *Let $\mathcal{C}$ be a maximal coclique without colored points, with at most one heavy plane (not containing all tops), containing flags (P_0, A_0) and (P_1, A_1) with $P_i \not\subseteq A_{1-i}$ $(i = 0, 1)$. Then $|\mathcal{C}| \leq 4q^2 + 5q + 4$.*

Proof. The planes A_0 and A_1 meet in a line L, and $A_0 + A_1$ is a solid S. By Proposition 17 there is a $\mathcal{C}$-flag (Q, B) with $Q \not\subseteq S$. For any such $\mathcal{C}$-flag, either $B \cap S = L$ or $B \cap S = P_0 + P_1$. (Indeed, if B intersects both A_i in a line, then $B \cap S = L$. Otherwise B must contain a point P_i, but then it cannot intersect the other plane in a line, so B also contains P_{1-i}.) If the former case occurs, we are done by Proposition 19. So we may assume that $B \cap S = P_0 + P_1$ for all flags (Q, B) with top outside S.

If A is a heavy plane, then we can apply the above with $A_0 = A$, choosing for A_1 a plane with top not on A. But then $B \cap A$ is a single point other than Q, a contradiction. Therefore, we may assume that there are no heavy planes.

Again we count (planes C in) $\mathcal{C}$-flags (R, C):

(i) There is at most 1 flag (R, C) with $R \subseteq L \subseteq C$:
The plane C must contain Q, hence $C = L + Q$.

(ii) There are at most $2q + 1$ flags (R, C) with $R \subseteq L \not\subseteq C$:
We apply Lemma 18 with $U = L$. If all C are on the same (white) R then we find at most $2q + 1$. If not, and all C are in a solid T, then T contains L and we find at most $2q$, since if a point R is on $q + 1$ $\mathcal{C}$-planes in T then one of these planes contains L, and this happens for at most one R since the plane has to be $L + Q$. Finally, if all C share a line N then N is disjoint from L (since $L \not\subseteq C$) and we find at most $q + 1$.

(iii) There are at most $q^2 + q$ flags (R, C) with $R \subseteq P_0 + P_1 \not\subseteq C$ (this includes the flags (P_0, A_0) and (P_1, A_1)):
We apply Lemma 18 with $U = P_0 + P_1$. If all C are in a solid then this solid is S and on every point of $P_0 + P_1$ we find at most a pencil of planes on a line, so at most q planes, since one of the planes of the pencil contains $P_0 + P_1$. In the other case the C are all on a line, but this line is L, so they are still in the solid S.

(iv) There are at most $2(q^2 + q + 1)$ flags (R, C) with $R \neq P_i \subseteq C$ for $i = 0$ or $i = 1$ (this includes the cases with $P_0 + P_1 \subseteq C$):
The number of planes on a point P with another top is at most $q^2 + q + 1$ because the planes must pairwise intersect in a line (hence contain a common line, or lie on P inside a common solid).

Altogether we find at most $4q^2 + 5q + 4$ planes, hence flags. □

Finally we consider the case that the above does not occur.

Proposition 21 *Let $\mathcal{C}$ be a maximal coclique without colored points and without heavy planes. If for any pair of flags (P_i, A_i) $(i = 0, 1)$ we have $P_i \subseteq A_{1-i}$ for at least one i, then $|\mathcal{C}| \leq 2q^2 + 6q + 4$.*

The assumption about heavy planes is not essential: if there is a heavy plane A then this plane contains all tops, and we are in a case already dealt with.

Proof. Define a directed graph on $\mathcal{C}$ by $(P, A) \to (Q, B)$ if $P \subseteq B$ (so that in particular $(P, A) \to (P, A)$). Now if $|\mathcal{C}| = l$ then there is a flag with outdegree at least $l/2$ (since there is an arrow between every two vertices), so we have a point P on that many $\mathcal{C}$-planes. Since all points are white, the number of $\mathcal{C}$-flags (P, A) with given P is at most $2q + 1$, but also the number of $\mathcal{C}$-flags (Q, A) with $Q \neq P \subseteq A$ is small, at most $q^2 + q + 1$, since two such flags share a line through P. So we get $l/2 \leq q^2 + 3q + 2$ and $|\mathcal{C}| \leq 2q^2 + 6q + 4$. □

Summing up what we found: We determined all $\mathcal{C}$ with more than one heavy plane. That yielded examples (i)–(xi). In the remaining cases we found upper bounds for $|\mathcal{C}|$ not larger than $4q^2 + 5q + 4$. This proves the theorem.

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