Buekenhout-Tits geometries and chain calculus

Abstract

We say some introductory words on Buekenhout-Tits geometries and describe chain calculus in A_n, D_n, E_6, E_7, E_8.

1 Buekenhout-Tits geometries

A *Buekenhout-Tits geometry* is a set X of *objects* provided with a symmetric relation $*$ called *incidence* and a function $t : X \to I$ that assigns a *type* to each object, such that two objects of the same type are never incident. The set I is the set of types. The cardinality $|I|$ is called the *rank* of the geometry.

Let us call the geometry $\Gamma(X, *, I, t)$.

A Buekenhout-Tits geometry can be viewed as a multipartite graph Γ with vertex set X and partition $\{X_i \mid i \in I\}$ (with $X_i = t^{-1}(i)$), with incidence taken as adjacency.

(Also common is the slightly different definition that adds a loop at every vertex of Γ, so that two objects of the same type are incident if and only if they coincide. It will not make any difference whether we draw a loop at every vertex or at no vertex.)

The geometry Γ is called *connected* when the graph Γ is connected. (Note that the graph without vertices is not connected: a connected graph has precisely 1 connected component, while the graph without vertices has no connected component.)

A *flag* F in Γ is a clique, a complete subgraph, a set of mutually incident objects. No two elements of a flag have the same type. The *rank* of F is $|t(F)|$ (that is, $|F|$). The *corank* of F is $|I \setminus t(F)|$.

The *residue* $\mathrm{Res}(F)$ (also written Γ_F) is the geometry with set of objects $Y = \{y \in X \setminus F \mid F \cup \{y\} \text{ is a flag}\}$, incidence inherited from Γ, set of types $I \setminus t(F)$, and type function inherited from Γ.

The geometry Γ is called *residually connected* when every residue of rank at least two is connected (and hence nonempty), and every residue of rank one is nonempty.

The intuition that belongs to a Buekenhout-Tits geometry is that of a collection of geometrical objects of various types (points, lines, planes, circles, ...) together with some incidence between them. An axiom system is imposed by giving a Buekenhout-Tits diagram.

1.1 Buekenhout-Tits diagrams

Let $\mathcal{D}$ be a labelled graph on I, where for $i, j \in I$ the label $\mathcal{D}_{ij}$ is a class of rank 2 geometries. We say that $\mathcal{D}$ is a Buekenhout-Tits diagram for the geometry

$\Gamma = (X, *, I, t)$ when for every flag F of Γ of corank 2, say $t(F) = I \setminus \{i, j\}$, the residue Γ_F belongs to the class of geometries $\mathcal{D}_{ij}$.

This is a recursive definition of the meaning of a diagram in terms of what the labelled edges mean for rank 2 geometries.

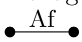
There is a dictionary of traditional labels.

 : Every i-object is incident to every j-object.

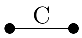
 : The i-objects and j-objects form the points and lines of a projective plane.

 : The i-objects and j-objects form the points and lines of a generalized quadrangle.

 : The i-objects and j-objects form the points and lines of a generalized hexagon.



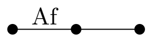
: The i-objects and j-objects form the points and lines of an affine plane.



: The i-objects and j-objects form the points and edges of a complete graph.

Etc.

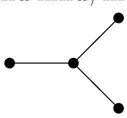
Examples The geometry of points, lines and planes in a 3-dimensional projective space satisfies the axioms given by the diagram .

The geometry of points, lines and planes in a 3-dimensional affine space satisfies the axioms given by the diagram 

.

The geometry of 8 corners, 12 edges and 6 faces of a cube satisfies the axioms given by the diagram .

The geometry of totally singular points, lines, planes and solids in a geometry of type $O_8^+(F)$ satisfies the axioms given by the diagram .

The geometry of totally singular points, lines, solids of the first kind, and solids of the second kind, in a geometry of type $O_8^+(F)$ satisfies the axioms given by the diagram  .

1.2 Simple properties

Proposition 1.1 *Let Γ be a residually connected Buekenhout-Tits geometry of finite rank.*

(i) For any two distinct types $i, j \in I$ the subgraph induced on $X_i \cup X_j$ is connected.

(ii) If the types i, j belong to different connected components of the Buekenhout-Tits diagram, then each i-object is incident with each j-object. □

Proof: (i) Induction on the rank. The case of rank at most 2 holds by definition. Since Γ is connected, we can join two objects in $X_i \cup X_j$ by a chain $x_0 * x_1 * ... * x_l$. Next, for each x_h in this chain with a type different from i and j, we can replace x_h by a chain in $X_i \cup X_j$ in $\mathrm{Res}(x_h)$ (by the induction hypothesis and residual connectedness).

(ii) Induction on the rank. The case of rank at most 2 holds by definition. Using part (i) we can join two objects $x \in X_i$ and $y \in X_j$ by a chain $x = x_0 * x_1 * ... * x_l = y$ contained in $X_i \cup X_j$ (so that the types alternate between

i and j). Let the length l be chosen minimal, and suppose that $l > 1$. Let k be a third type different from i and j. We may suppose that j and k belong to different connected components of the Buekenhout-Tits diagram. In $\text{Res}(x_1)$ we can replace $x_0 * x_1 * x_2$ by a path $x_0 = x'_0 * x'_1 * ... * x'_m = x_2$ using only types i and k. Now x_3 and its two predecessors in the chain have types k-i-j, and by the induction hypothesis we can omit the middle object (of type i). Then x_3 and its two predecessors have types i-k-j, and again we can omit the middle object. It follows after m steps that $x_0 * x_3$, so that l was not minimal. □

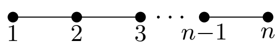
After this preparation, it is an easy exercise to prove the Veblen-Young axiom from the A_n diagram, so that a (thick) geometry satisfying the A_n diagram is a projective space.

2 Chain calculus

2.1 Existence of chains

We shall talk about chains $x_0 * x_1 * ... * x_l$ (in some residually connected Buekenhout-Tits geometry satisfying a given diagram) by just giving the sequence of types t_0-t_1-...-t_l, where the object x_i is of type t_i.

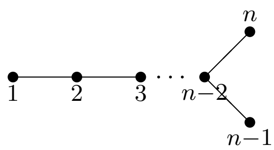
A sequence of types given as a statement, denotes the claim that arbitrary objects x_0 and x_l of the types occurring first and last can be joined by a chain of objects of the indicated types, each incident with the preceding and following. In the proofs we shall modify chains, but always keep the ends fixed.



A_n.

Proposition A_n: *For* $2 \leq i \leq n$ *we have* 1-i-$(i-1)$. *In particular, for* $n \geq 2$, *we have* 1-2-1.

Proof: If $i < n$, then by induction we find that if 1-2-1-i-$(i-1)$, then 1-2-$(i+1)$-i-$(i-1)$, hence 1-$(i+1)$-$(i-1)$, hence 1-i-$(i-1)$, so chains 1-(2-1-)^j i-$(i-1)$ can be shortened to 1-i-$(i-1)$, and by residual connectedness we are done. By definition of A_2 we have 1-2-1 in A_2. Remains the case $i = n \geq 3$. But there we have 1-2-1-$(n-1)$, so 1-2-n-$(n-1)$, so 1-n-$(n-1)$, by induction and since 1-2-1 holds. □



D_n.

Proposition D_n: *Let* $n \geq 2$. *Then the following hold.*

(a) 1-$(n-1)$-n.

(b) 1-i-$(i-1)$-i *for* $2 \leq i \leq n-2$. *In particular:* 1-2-1-2.

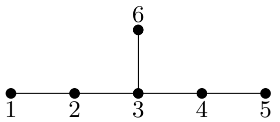
(c) If n *is even, then* $(n-1)$-1-n. *If* n *is odd, then* n-1-n.

Proof: In D_2 we have 1-2, implying all our claims. For $n = 3$ everything follows from Proposition A_3. Now use induction on n. For part (a) we find by induction and Proposition A_n: if 1-2-1-$(n-1)$-n then 1-2-n-$(n-1)$-n so 1-n-$(n-2)$-n so 1-$(n-1)$-$(n-2)$-n so 1-$(n-1)$-n, proving part (a).

For part (b): 1-$(n-1)$-n-i, so 1-$(n-1)$-$(i-1)$-i, so 1-i-$(i-1)$-i.

For part (c): if n is even, then (by induction): $(n-1)$-n-1-n, so $(n-1)$-n-2-n, so $(n-1)$-1-2-n, so $(n-1)$-1-n, and if n is odd, then n-1-$(n-1)$-n, so n-2-$(n-1)$-n, so n-2-1-n, so n-1-n. □

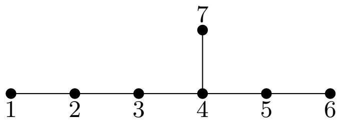
For E_6, E_7, E_8 we shall omit the '-' in type sequences.



E_6.

Proposition E_6: *(a)* 151*,*
(b) 12165*,*
(c) if 1215 *then* 165*.*

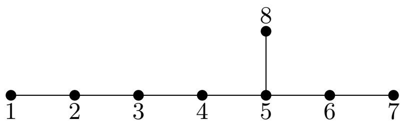
Proof: (c) 1215 yields 1265 and then 165.
(a) 12151 yields 1651, 1641, 1541, 151.
(b) 1515 yields 14615, 14625, 121625, 121325, 121365, 12165. □



E_7.

Proposition E_7: *(a)* 1216*,*
(b) 1612*,*
(c) if 12121 *then* 161*.*

Proof: (c) 12121 yields 12621 and then 161.
(a) 121216 yields 1616, 1626, 12126, 1216.
(b) 12121212 yields 161212, 161262, 16162, 16262, 121262, 12162, 121212, 1612. □



E_8.

Proposition E_8: *(a)* 17171*,*
(b) if 17121 *then* 12121*,*
(c) 121212*.*

Proof: (b) 17121 yields 172321, 17231, 121231, 12131, 12121.
(a) 1717121 yields 1712121, 1212121, 121232721, 1213271, 12723271, 1721271, 17171.
(c) 12121212 yields 1212123272, 121213272, 1212723272, 1217172, (by (b)) 1212172, 12123272, 1213272, 12723272, 1723272, 17172, 171212, 121212. □

For the collinearity graph Γ (vertices: objects of type 1; adjacency: both in the residue of some flag of cotype 1 - in our cases this is equivalent to both incident to some object of type 2) the above means the following:
A_n: Γ is a clique (has diameter 1)
D_n: Γ has diameter 2; any line carries a point at distance at most one from a given point
E_6: Γ has diameter 2 - indeed, any two vertices are in a D_5 subgraph

E_7: Γ has diameter 3; any two vertices at distance 2 are in a D_6 subgraph; any line carries a point at distance at most two from a given point

E_8: Γ has diameter 3; if x and y are two points at distance 2 in a D_7 subgraph, then y has no neighbours at distance 3 from x; any line carries a point at distance at most two from a given point

For the relation between points x and symplecta S (objects of type $m-1$) in E_m, the above implies:

E_6: $x^\perp \cup S$ is either empty or a projective 4-space.

E_7: $x^\perp \cup S$ is either a single point or a projective 5-space.

E_8: $x^\perp \cup S$ is either empty or a line or a projective 6-space.