# Two distance-regular graphs

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### Abstract

We construct two families of distance-regular graphs, namely the subgraph of the dual polar graph of type $B_3(q)$ induced on the vertices far from a fixed point, and the subgraph of the dual polar graph of type $D_4(q)$ induced on the vertices far from a fixed edge. The latter is the extended bipartite double of the former.

## 1 The extended bipartite double

We shall use $\sim$ to indicate adjacency in a graph. For notation and definitions of concepts related to distance-regular graphs, see [BCN]. We repeat the definition of extended bipartite double.

The *bipartite double* of a graph $\Gamma$ with vertex set $X$ is the graph with vertex set $\{x^+, x^- \mid x \in X\}$ and adjacencies $x^\delta \sim y^\epsilon$ iff $\delta\epsilon = -1$ and $x \sim y$. The bipartite double of a graph $\Gamma$ is bipartite, and it is connected iff $\Gamma$ is connected and not bipartite. If $\Gamma$ has spectrum $\Phi$, then its bipartite double has spectrum $(-\Phi) \cup \Phi$. See also [BCN], Theorem 1.11.1.

The *extended bipartite double* of a graph $\Gamma$ with vertex set $X$ is the graph with vertex set $\{x^+, x^- \mid x \in X\}$, and the same adjacencies as the bipartite double, except that also $x^- \sim x^+$ for all $x \in X$. The extended bipartite double of a graph $\Gamma$ is bipartite, and it is connected iff $\Gamma$ is connected. If $\Gamma$ has spectrum $\Phi$, then its extended bipartite double has spectrum $(-\Phi - 1) \cup (\Phi + 1)$. See also [BCN], Theorem 1.11.2.

## 2 Far from an edge in the dual polar graph of type $D_4(q)$

Let $V$ be a vector space of dimension 8 over a field $F$, provided with a nondegenerate quadratic form of maximal Witt index. The maximal totally isotropic subspaces of $V$ (of dimension 4) fall into two families $\mathcal{F}_1$ and $\mathcal{F}_2$, where the dimension of the intersection of two elements of the same family is even (4 or 2 or 0) and the dimension of the intersection of two elements of different families is odd (3 or 1).
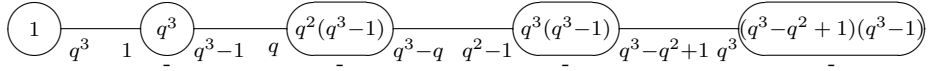
The geometry of the totally isotropic subspaces of $V$, where $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$ are incident when $\dim A \cap B = 3$ and otherwise incidence is symmetrized inclusion, is known as the geometry $D_4(F)$. The bipartite incidence graph on

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the maximal totally isotropic subspaces is known as the dual polar graph of type $D_4(F)$.

Below we take $F = \mathbf{F}_q$, the finite field with $q$ elements, so that graph and geometry are finite. We shall use projective terminology, so that 1-spaces, 2-spaces and 3-spaces are called points, lines and planes. Two subspaces are called disjoint when they have no point in common, i.e., when the intersection has dimension 0.

**Proposition 2.1** *Let $\Gamma$ be the dual polar graph of type $D_4(\mathbf{F}_q)$. Fix elements $A_0 \in \mathcal{F}_1$ and $B_0 \in \mathcal{F}_2$ with $A_0 \sim B_0$. Let $\Delta$ be the subgraph of $\Gamma$ induced on the set of vertices disjoint from $A_0$ or $B_0$. Then $\Delta$ is distance-regular with intersection array $\{q^3, q^3 - 1, q^3 - q, q^3 - q^2 + 1;\ 1, q, q^2 - 1, q^3\}$.*
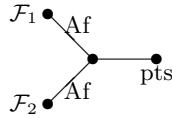
The distance distribution diagram is



**Proof:** There are $q^6$ elements $A \in \mathcal{F}_1$ disjoint from $A_0$ and the same number of $B \in \mathcal{F}_2$ disjoint from $B_0$, so that $\Delta$ has $2q^6$ vertices.

Given $A \in \mathcal{F}_1$, there are $q^3 + q^2 + q + 1$ elements $B \in \mathcal{F}_2$ incident to it. Of these, $q^2 + q + 1$ contain the point $A \cap B_0$ and hence are not vertices of $\Delta$. So, $\Delta$ has valency $q^3$.

Two vertices $A, A' \in \mathcal{F}_1$ have distance 2 in $\Delta$ if and only if they meet in a line, and the line $L = A \cap A'$ is disjoint from $B_0$. If this is the case, then $L$ is in $q + 1$ elements $B \in \mathcal{F}_2$, one of which meets $B_0$, so that $A$ and $A'$ have $c_2 = q$ common neighbours in $\Delta$.

Given vertices $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$ that are nonadjacent, i.e., that meet in a single point $P$, the neighbours $A'$ of $B$ at distance 2 to $A$ in $\Delta$ correspond to the lines $L$ on $P$ in $A$ disjoint from $B_0$ and nonorthogonal to the point $A_0 \cap B$. There are $q^2 + q + 1$ lines $L$ on $P$ in $A$, $q + 1$ of which are orthogonal to the point $A_0 \cap B$, and one further of which meets $B_0$. (Note that the points $A_0 \cap B$ and $A \cap B_0$ are nonorthogonal since neither point is in the plane $A_0 \cap B_0$ and $V$ does not contain totally isotropic 5-spaces.) It follows that $c_3 = q^2 - 1$, and also that $\Delta$ has diameter 4, and is distance-regular. $\square$

The geometry induced by the incidence relation of $D_4(F)$ on the vertices of $\Delta$, together with the points and lines contained in the planes disjoint from $A_0 \cup B_0$, has Buekenhout-Tits diagram (cf. [P])



that is, the residue of an object $A \in \mathcal{F}_1$ is an affine 3-space, where the objects incident to $A$ in $\mathcal{F}_2$ play the rôle of points. Similar things hold more generally for $D_n(F)$ with arbitrary $n$, and even more generally for all diagrams of spherical type. See also [BB], Theorem 6.1.

Let $P$ be a nonsingular point, and let $\phi$ be the reflection in the hyperplane $H = P^\perp$. Then $\phi$ is an element of order two of the orthogonal group that fixes $H$ pointwise, and consequently interchanges $\mathcal{F}_1$ and $\mathcal{F}_2$. For each $A \in \mathcal{F}_1$ we

have $\phi(A) \sim A$. The quotient $\Gamma/\phi$ is the dual polar graph of type $B_3(q)$, and we see that more generally the dual polar graph of type $D_{m+1}(q)$ is the extended bipartite double of the dual polar graph of type $B_m(q)$. The quotient $\Delta/\phi$ is a new distance-regular graph discussed in the next section. It is the subgraph consisting of the vertices at maximal distance from a given point in the dual polar graph of type $B_3(q)$. For even $q$ we have $B_3(q) = C_3(q)$, and it follows that the symmetric bilinear forms graph on $\mathbf{F}_q^3$ is distance-regular, see [BCN] Proposition 9.5.10 and the diagram there on p. 286.

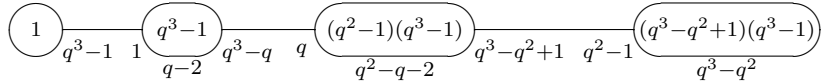# 3 Far from a point in the dual polar graph of type $B_3(q)$

First a very explicit version of the graph of this section.

**Proposition 3.1** *(i) Let $W$ be a vector space of dimension 3 over the field $\mathbf{F}_q$, provided with an outer product $\times$. Let $Z$ be the graph with vertex set $W \times W$ where $(u, u') \sim (v, v')$ if and only if $(u, u') \neq (v, v')$ and $u \times v + u' - v' = 0$. Then $Z$ is distance-regular of diameter 3 on $q^6$ vertices. It has intersection array $\{q^3 - 1, q^3 - q, q^3 - q^2 + 1; 1, q, q^2 - 1\}$ and eigenvalues $q^3 - 1$, $q^2 - 1$, $-1$, $-q^2 - 1$ with multiplicities $1$, $\frac{1}{2}q(q+1)(q^3 - 1)$, $(q^3 - q^2 + 1)(q^3 - 1)$, $\frac{1}{2}q(q-1)(q^3 - 1)$, respectively.*

*(ii) The extended bipartite double $\hat{Z}$ of $Z$ is distance-regular with intersection array $\{q^3, q^3 - 1, q^3 - q, q^3 - q^2 + 1; \ 1, q, q^2 - 1, q^3\}$ and eigenvalues $\pm q^3$, $\pm q^2$, $0$ with multiplicities $1$, $q^2(q^3 - 1)$, $2(q^3 - q^2 + 1)(q^3 - 1)$, respectively.*

*(iii) The distance-1-or-2 graph $Z_1 \cup Z_2$ of $Z$, which is the halved graph of $\hat{Z}$, is strongly regular with parameters $(v, k, \lambda, \mu) = (q^6, q^2(q^3 - 1), q^2(q^2 + q - 3), q^2(q^2 - 1))$.*

The distance distribution diagram of $Z$ is



**Proof:** Note that the adjacency relation is symmetric, so that $Z$ is an undirected graph. The computation of the parameters is completely straightforward. Clearly, $Z$ has $q^6$ vertices. For $a, b \in W$ the maps $(u, u') \mapsto (u+a, u'+(a \times u)+b)$ are automorphisms of $Z$, so $\mathrm{Aut}(Z)$ is vertex-transitive.

The $q^3 - 1$ neighbours of $(0, 0)$ are the vertices $(v, 0)$ with $v \neq 0$. The common neighbours of $(0, 0)$ and $(v, 0)$ are the vertices $(cv, 0)$ for $c \in \mathbf{F}_q$, $c \neq 0, 1$. Hence $a_1 = q - 2$.

The $(q^3 - 1)(q^2 - 1)$ vertices at distance 2 from $(0, 0)$ are the vertices $(u, u')$ with $u, u' \neq 0$ and $u' \perp u$. The common neighbours of $(0, 0)$ and $(u, u')$ are the $(v, 0)$ with $v \times u = u'$, and together with $(v, 0)$ also $(v + cu, 0)$ is a common neighbour, so $c_2 = q$. Vertices $(u, u')$ and $(v, v')$, both at distance 2 from $(0, 0)$ are adjacent when $0 \neq v \perp u'$ and $v \neq u$ and $v \times u \neq u'$ and $v' = u \times v + u'$, so that $a_2 = q^2 - q - 2$.

The remaining $(q^3 - 1)(q^3 - q^2 + 1)$ vertices have distance 3 to $(0, 0)$. They are the $(w, w')$ with $w \not\perp w'$ or $w = 0 \neq w'$. The neighbours $(u, u')$ of $(w, w')$ that lie at distance 2 to $(0, 0)$ satisfy $0 \neq u \perp w'$ and $(0 \neq)u' = w \times u + w'$,

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so that $c_3 = q^2 - 1$. This shows that $Z$ is distance-regular with the claimed parameters. The spectrum follows.

The fact that the extended bipartite double is distance-regular, and has the stated intersection array, follows from [BCN], Theorem 1.11.2(vi).

The fact that $Z_3$ is strongly regular follows from [BCN], Proposition 4.2.17(ii) (which says that this happens when $Z$ has eigenvalue $-1$). $\qquad\square$

For $q = 2$, the graphs here are (i) the folded 7-cube, (ii) the folded 8-cube, (iii) the halved folded 8-cube. All are distance-transitive. For $q > 2$ these graphs are not distance-transitive.

When $q$ is a power of two, the graphs $\hat{Z}$ have the same parameters as certain Kasami graphs, but for $q > 2$ these are nonisomorphic.

Next, a more geometric description of this graph.

Let $H$ be a vector space of dimension 7 over the field $\mathbf{F}_q$, provided with a nondegenerate quadratic form. Let $\Gamma$ be the graph of which the vertices are the maximal totally isotropic subspaces of $H$ (of dimension 3), where two vertices are adjacent when their intersection has dimension 2. This graph is known as the dual polar graph of type $B_3(q)$. It is distance-regular with intersection array $\{q(q^2 + q + 1), q^2(q + 1), q^3; 1, q + 1, q^2 + q + 1\}$. (See [BCN], §9.4.)

**Proposition 3.2** *Let $\Gamma$ be the dual polar graph of type $B_3(q)$. Fix a vertex $\pi_0$ of $\Gamma$, and let $\Delta$ be the subgraph of $\Gamma$ induced on the collection of vertices disjoint from $\pi_0$. Then $\Delta$ is isomorphic to the graph $Z$ of Proposition 3.1. Its extended bipartite double $\hat{\Delta}$ (or $\hat{Z}$) is isomorphic to the graph of Proposition 2.1.*

**Proof:** Let $V$ be a vector space of dimension 8 over $\mathbf{F}_q$ (with basis $\{e_1, \ldots, e_8\}$), provided with the nondegenerate quadratic form $Q(x) = x_1 x_5 + x_2 x_6 + x_3 x_7 + x_4 x_8$. The point $P = (0, 0, 0, 1, 0, 0, 0, -1)$ is nonisotropic, and $P^{\perp}$ is the hyperplane $H$ defined by $x_4 = x_8$. Restricted to $H$ the quadratic form becomes $Q(x) = x_1 x_5 + x_2 x_6 + x_3 x_7 + x_4^2$.

The $D_4$-geometry on $V$ has disjoint maximal totally isotropic subspaces $E = \langle e_1, e_2, e_3, e_4 \rangle$ and $F = \langle e_5, e_6, e_7, e_8 \rangle$. Fix $E$ and consider the collection of all maximal totally isotropic subspaces disjoint from $E$. This is precisely the collection of images $F_A$ of $F$ under matrices $\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}$, where $A$ is alternating with zero diagonal (cf. [BCN], Proposition 9.5.1(i)). Hence, we can label the $q^6$ vertices $F_A \cap H$ of $\Delta$ with the $q^6$ matrices $A$.

Two vertices are adjacent when they have a line in common, that is, when they are the intersections with $H$ of maximal totally isotropic subspaces in $V$, disjoint from $E$, that meet in a line contained in $H$. Let

$$A = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}.$$

Then $\det A = (af - be + cd)^2$, and if $\det A = 0$ but $A \neq 0$, then $\ker A$ has dimension 2, and is spanned by the four vectors $(0, f, -e, d)^{\top}$, $(-f, 0, c, -b)^{\top}$, $(e, -c, 0, a)^{\top}$, $(-d, b, -a, 0)^{\top}$. Writing the condition that matrices $A$ and $A'$ belong to adjacent vertices we find the description of Proposition 3.1 if we take $u = (c, e, f)$ and $u' = (-d, b, -a)$. $\qquad\square$

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# 4    History

In 1991 the second author constructed the graphs from Section 2 and the first author those from Section 3. Both were mentioned on the web page [ac], but not published thus far. These graphs have been called the Pasechnik graphs and the Brouwer-Pasechnik graphs, respectively, by on-line servers.

# References

[BB]    R. J. Blok & A. E. Brouwer, *The geometry far from a residue*, pp. 29–38 in: Groups and Geometries, L. di Martino, W. M. Kantor, G. Lunardon, A. Pasini, M. C. Tamburini (eds.), Birkhaüser Verlag, Basel, 1998.

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`http://www.win.tue.nl/~aeb/drg/index.html`

[BCN]  A. E. Brouwer, A. M. Cohen & A. Neumaier, *Distance-regular graphs*, Springer, Heidelberg, 1989.

[P]    A. Pasini, *Diagram geometries*, Oxford University Press, 1994.

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