The number of dominating sets of a finite graph is odd

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To Reza Khosrovshahi on the occasion of his 70th birthday

Let Γ be a finite graph with vertex set $V = V\Gamma$. A subset D of V is called *dominating* when each vertex in $V \setminus D$ has a neighbour in D. The *domination polynomial* $p(\Gamma, X)$ is defined by $p(\Gamma, X) = \sum_D X^{|D|}$ where the sum is over all dominating sets D.

S. Akbari asked whether $p(\Gamma, -1)$ is always nonzero, and it is:

Theorem 0.1 *The number of dominating sets of a finite graph is odd.*

Proof: Let us write S^+ for the set of vertices in S or with a neighbour in S. By induction on $|V|$, and for fixed $|V|$ on $|S|$, we prove the following two claims for $S \subseteq V$:

(i) $\#\{D \mid S \subseteq D \subseteq V,\ D^+ = V\} \equiv \#\{E \mid E \subseteq V,\ E^+ = V \setminus S\} \pmod 2$,
(ii) $\#\{D \mid D \subseteq V \setminus S,\ D^+ = V\} \equiv \#\{E \mid E \subseteq V,\ V \setminus S \subseteq E^+\} \pmod 2$.

Indeed, if $S = \emptyset$ both (i) and (ii) are trivial. Assume $S \neq \emptyset$.

Let $U = S^+ \setminus S$ and $W = V \setminus S$. Then (i) is equivalent to

(i') $\#\{D \mid D \subseteq W,\ W \setminus U \subseteq D^+\} \equiv \#\{E \mid E \subseteq W \setminus U,\ E^+ = W\} \pmod 2$.

for $U \subseteq W$. But this is precisely (ii), with W instead of V, and since $|W| < |V|$ this holds by induction. This proves (i).

If we sum the equality (ii) over all $S \subseteq T$, where $T \subseteq V$, the left hand side counts pairs (D, S) with $D^+ = V$ and $S \subseteq T \setminus D$, so that each D is seen $2^{|T \setminus D|}$ times, which is 0 (mod 2) except when $T \subseteq D$. The right hand side counts pairs (E, S) with $V \setminus T \subseteq V \setminus S \subseteq E^+$, so that each E is seen $2^{|E^+ \setminus (V \setminus T)|}$ times, which is 0 (mod 2) except when $E^+ = V \setminus T$. The result is

$$\#\{D \mid T \subseteq D \subseteq V,\ D^+ = V\} \equiv \#\{E \mid E \subseteq V,\ E^+ = V \setminus T\} \pmod 2$$

which is precisely (i), but using the variable T instead of S. Since (i) holds, and by induction (ii) holds for all proper subsets S of T, it follows that (ii) also holds for $S = T$. This completes the proof of (i) and (ii).

Now we can prove the theorem. If $V = \emptyset$ then there is precisely one dominating set. Otherwise, let $x \in V$ and put $W = V \setminus x$ and $S = N(x)$, the set of neighbours of x. The dominating sets in V are the dominating sets D in W that intersect S, and the sets $E \cup \{x\}$ where $E \subseteq W$ with $W \setminus S \subseteq E^+$. By induction, the number of dominating sets (of the graph $\Gamma \setminus x$) in W is odd. Adding equation (ii) (with W instead of V) yields the desired conclusion. $\square$