DIRECTED STRONGLY REGULAR GRAPHS FROM $1\frac{1}{2}$-DESIGNS

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Abstract. Some families of directed strongly regular graphs with $t = \mu$ are constructed by using antiflags of $1\frac{1}{2}$-designs.

1. Introduction

A *finite incidence structure* consists of a finite set P of *points*, a set $\mathcal{B}$ of *blocks*, and an *incidence relation* $\in$ between points and blocks. An incident point-block pair is called a *flag*, and a non-incident point-block pair is called an *antiflag*. A *tactical configuration with parameters* $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r})$ is a finite incidence structure $\mathcal{T} = (P, \mathcal{B}, \in)$ with $|P| = \mathbf{v}$, $|\mathcal{B}| = \mathbf{b}$ such that every block contains $\mathbf{k}$ points and every point belongs to exactly $\mathbf{r}$ blocks.

A $1\frac{1}{2}$-*design* (Neumaier [14]) or *partial geometric design* (Bose, Shrikhande & Singhi [1]) with parameters $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r}; a, b)$ is a tactical configuration $\mathcal{T} = (P, \mathcal{B}, \in)$ with parameters $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r})$ satisfying the property:

> For every point $x \in P$ and every block $B \in \mathcal{B}$, the number of flags (y, C) such that $y \in B$ and $C \ni x$ is a if $x \notin B$ and b if $x \in B$.

Examples of $1\frac{1}{2}$-designs include 2-designs, complete bipartite graphs $K_{n,n}$, transversal designs, and partial geometries. The dual of a $1\frac{1}{2}$-design is again a $1\frac{1}{2}$-design. (cf. [14])

A *directed strongly regular graph* (Duval [4]) with parameters (v, k, t, λ, μ) is a directed graph Γ on v vertices without loops such that (i) every vertex has in-degree and out-degree k, (ii) every vertex x has t out-neighbors that are also in-neighbors of x, and (iii) the number of directed paths of length two from a vertex x to another vertex y is λ if there is an edge from x to y, and is μ if there is no edge from x to y. We often denote Γ a DSRG(v, k, t, λ, μ) in short.

Let I denote the identity matrix, and J the all-1 matrix (not necessarily square), with sizes that are clear from the context. The adjacency matrix of a directed strongly regular graph is a square (0,1)-matrix A with zero diagonal such that the $\mathbb{Z}$-linear span of I, J and A is closed under matrix multiplication. Equivalently, a square (0,1)-matrix A with zero diagonal such that for certain constants k, t, λ, μ we have $AJ = JA = kJ$ and $A^2 = tI + \lambda A + \mu(J - I - A)$.

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The incidence matrix of a $1\frac{1}{2}$-design is a (0,1)-matrix N such that for certain constants $\mathbf{k}, \mathbf{r}, a, b$ we have $JN = \mathbf{k}J$, $NJ = \mathbf{r}J$, and $NN^\top N = (b-a)N + aJ$.

In this note we observe the following: Given a $1\frac{1}{2}$-design with incidence matrix N, define a matrix A, with rows and columns indexed by the point-block pairs (p, B) for which $N_{pB} = 0$, by $A_{(p,B),(q,C)} = N_{pC}$. Then A is a directed strongly regular graph. This yields directed strongly regular graphs with previously unknown parameters.

2. Construction

We show that the set of antiflags of a $1\frac{1}{2}$-design gives rise to a directed strongly regular graph with parameters $t = \mu$.

Theorem 2.1. *Let $\mathcal{T} = (P, \mathcal{B}, \in)$ be a tactical configuration with parameters $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r})$. Let $\Gamma = \Gamma(\mathcal{T})$ be the directed graph defined by*

$$V(\Gamma) = \{(p, B) \in P \times \mathcal{B} : p \notin B\}$$

and

$$(p, B) \to (q, C) \text{ if and only if } p \in C.$$

Then Γ is directed strongly regular if and only if $\mathcal{T}$ is a $1\frac{1}{2}$-design.

Proof: Let Γ have adjacency matrix A. Write pB for an antiflag (p, B). Then

$$(A^2)_{pB,qC} = \sum_{rD} A_{pB,rD} A_{rD,qC} = \sum_{r,D} N_{pD} N_{rC} (1 - N_{rD}) = \mathbf{kr} - (NN^\top N)_{pC}.$$

If $\mathcal{T}$ is a $1\frac{1}{2}$-design with parameters $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r}; a, b)$, then $NN^\top N = (b-a)N + aJ$, and hence $A^2 = (\mathbf{kr} - a)J - (b-a)A$, so that Γ is a directed strongly regular graph with parameters

$$v = \mathbf{b}(\mathbf{v} - \mathbf{k}), \quad k = \mathbf{r}(\mathbf{v} - \mathbf{k}), \quad t = \mu = \mathbf{kr} - a, \quad \lambda = \mathbf{kr} - b.$$

Conversely, suppose Γ is a DSRG(v, k, t, λ, μ). Then $A^2 = (t-\mu)I + (\lambda - \mu)A + \mu J$ and we find $\mathbf{kr} - (NN^\top N)_{pC} = (t-\mu)\delta_{pB,qC} + (\lambda-\mu)N_{pC} + \mu$ for all antiflags pB, qC (where $\delta_{pB,qC}$ is 1 when $pB = qC$ and 0 otherwise). If $t \neq \mu$ then $\delta_{pB,qC}$ is determined by p, C and independent of q, B. This can hold only for $\mathbf{v} = \mathbf{k}+1$, $\mathbf{b} = \mathbf{r}+1$, and $A = J - I$ so that μ is undefined. Therefore, we may assume that $t = \mu$, so that $NN^\top N = (\mu - \lambda)N + (\mathbf{kr} - \mu)J$. □

Similarly, the set of flags of a $1\frac{1}{2}$-design gives a directed strongly regular graph with $t = \lambda + 1$.

Theorem 2.2. *Let $\mathcal{T} = (P, \mathcal{B}, \in)$ be a tactical configuration with parameters $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r})$. Let Γ be the directed graph defined by*

$$V(\Gamma) = \{(p, B) \in P \times \mathcal{B} : \ p \in B\}$$

and

$$(p, B) \to (q, C) \text{ if and only if } (p, B) \neq (q, C) \text{ and } p \in C.$$

Then Γ is directed strongly regular if and only if $\mathcal{T}$ is a $1\frac{1}{2}$-design.

Proof: This time, write pB for a flag (p, B). Let Γ have adjacency matrix A and put $M = A + I$ so that $M_{pB,qC} = N_{pC}$. Then

$$(M^2)_{pB,qC} = \sum_{rD} M_{pB,rD} M_{rD,qC} = \sum_{r,D} N_{pD} N_{rC} N_{rD} = (NN^\top N)_{pC}.$$

If $\mathcal{T}$ is a $1\frac{1}{2}$-design with parameters $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r}; a, b)$, then $NN^\top N = (b - a)N + aJ$, and $M^2 = (b-a)M + aJ$, so that $A^2 = (M - I)^2 = (b - a - 2)A + (b - a - 1)I + aJ$ and it follows that Γ is a directed strongly regular graph with parameters

$$v = \mathbf{vr}, \quad k = \mathbf{rk} - 1, \quad t = b - 1, \quad \lambda = b - 2, \quad \mu = a.$$

Conversely, suppose Γ is a DSRG(v, k, t, λ, μ). Then $A^2 = (t - \mu)I + (\lambda - \mu)A + \mu J$, so that $M^2 = (\lambda - \mu + 2)M + \mu J + (t - \lambda - 1)I$, and therefore $(NN^\top N)_{pC} = (\lambda - \mu + 2)N_{pC} + \mu + (t - \lambda - 1)\delta_{pB,qC}$. If $t \neq \lambda + 1$, then $\delta_{pB,qC}$ is determined by p, C and independent of q, B. This can hold only for $\mathbf{k} \leq 1$, $\mathbf{r} \leq 1$ and λ is undefined. Therefore, we may assume that $t = \lambda + 1$, so that $NN^\top N = (\lambda - \mu + 2)N + \mu J$. □

3. Examples

In this section we give some concrete examples of new directed strongly regular graphs that are constructed by the Theorem 2.1.

Example 3.1. *Let P be the set of $2n$ vertices, and $\mathcal{B}$ the set of n^2 edges of the complete bipartite graph $K_{n,n}$. Then the incidence structure $\mathcal{T} = (P, \mathcal{B}, \in)$ is a $1\frac{1}{2}$-design with parameters*

$$\mathbf{v} = 2n,\ \mathbf{b} = n^2,\ \mathbf{k} = 2,\ \mathbf{r} = n;\ a = 1,\ b = n + 1.$$

Therefore the graph $\Gamma(\mathcal{T})$ is a directed strongly regular graph with parameters

$$v = 2n^2(n-1),\ k = 2n(n-1),\ t = \mu = 2n - 1,\ \lambda = n - 1.$$

For $n = 3, 4$ we obtain directed strongly regular graphs with new parameter sets $(36, 12, 5, 2, 5)$ and $(96, 24, 7, 3, 7)$. By Duval [4], if there exists a DSRG(v, k, t, λ, μ) with $t = \mu$, then there also are DSRG$(hv, hk, ht, h\lambda, h\mu)$ for all positive integers h. In particular, we also find directed strongly regular graphs with parameter sets $(72, 24, 10, 4, 10)$ and $(108, 36, 15, 6, 15)$.

Example 3.2. *A partial geometry* pg(κ, ρ, τ) *is a set of points P, a set of lines $\mathcal{L}$, and an incidence relation between P and $\mathcal{L}$ with the following properties:*

1. *Every line is incident with κ points ($\kappa \geq 2$), and every point is incident with ρ lines ($\rho \geq 2$).*
2. *Any two points are incident with at most one line.*
3. *If a point p and a line L are not incident, then there exists exactly τ ($\tau \geq 1$) lines that are incident with p and meet L.*

A partial geometry pg(κ, ρ, τ) *is an $1\frac{1}{2}$-design $\mathcal{T}$ with parameters*

$$(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r}; a, b) = (\kappa c,\ \rho c,\ \kappa,\ \rho;\ \tau,\ \mathbf{r} + \mathbf{k} - 1)$$

where $c = 1 + (\kappa - 1)(\rho - 1)/\tau$.

For example, the partial geometry obtained from an affine plane of order q by considering all q^2 points and taking the lines of l parallel classes is a $pg(q, l, l-1)$ and hence yields a $1\frac{1}{2}$-design $\mathcal{T}$ with parameters $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r}; a, b) = (q^2, ql, q, l; l-1, q+l-1)$ and a directed strongly regular graph $\Gamma(\mathcal{T})$ defined as in Theorem 2.1 with parameters

$$(v, k, t, \lambda, \mu) = (lq^2(q-1),\ lq(q-1),\ lq-l+1,\ (l-1)(q-1),\ lq-l+1).$$

For example, for $q = l = 3$ we find the previously-unknown graph with parameter set $(54, 18, 7, 4, 7)$. Doubling yields the graph with parameter set $(108, 36, 14, 8, 14)$.

Remark 3.3. *The above characterization theorems may be used to show non-existence of $1\frac{1}{2}$-designs with given parameter sets. We give one example. Suppose there exists a $1\frac{1}{2}$-design with parameters $(8, 16, 5, 10; 25, 35)$. Then there is a directed strongly regular graph with parameters $(48, 30, 25, 15, 25)$ according to Theorem 2.1. However, it is known that there is no DSRG $(48m, 30m, 25m, 15m, 25m)$ for any positive integer m by Jørgensen* [11]*. So, although the parameter set $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r}; a, b) = (8, 16, 5, 10; 25, 35)$ satisfies all the necessary conditions imposed in Section 3.3 of Neumaier* [14]*, there is no $1\frac{1}{2}$-design with these parameters.*

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