Szöllősi's equiangular system in $\mathbb{R}^{18}$

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D. S. Asche (cf. [1], Example 5.19) gave a construction for a system of 72 lines in $\mathbb{R}^{19}$ with mutual angles $\arccos(1/5)$. F. Szöllősi [2] discovered that Asche's system contains a subsystem of 54 lines in $\mathbb{R}^{18}$ with mutual angles $\arccos(1/5)$.

The construction in [2] is very explicit. Here we give precisely the same construction but formulated without choosing an explicit Golay code, and an explicit vector m.

1 Asche's construction

Let $(X, \mathcal{B})$ be a Steiner system $S(5, 8, 24)$. (Thus, $|X| = 24$, $|\mathcal{B}| = 759$, members of $\mathcal{B}$ have size 8, and each subset of size 5 of X is contained in a unique member of $\mathcal{B}$.) Call the elements of X *points*. Call the members of $\mathcal{B}$ *octads*. Octads meet in 8, 4, 2, or 0 points.

Let $B_1, B_2 \in \mathcal{B}$ be two octads that meet in 2 points. Let p be a point outside $B_1 \cup B_2$, and let $B_1 \cap B_2 = \{q, r\}$. Let A be the set of 72 octads that contain p but not q, r and meet each of B_1 and B_2 in 2 points. All choices made are unique up to isomorphism, and the resulting set A (for Asche) is a single orbit of the subgroup of size 144 of M_{24} that stabilizes this situation.

For each subset S of X, let $\chi_S \in \mathbb{R}^X$ be its characteristic vector (so that $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ otherwise). With the usual inner product this means that $(\chi_S, \chi_T) = |S \cap T|$. Put $u := \chi_{\{p\}} + \frac{1}{4}\chi_X$. Now the 72 unit vectors $(\chi_B - u)/\sqrt{5}$ have mutual inner products $\pm 1/5$, so that the system Φ of 72 lines they span is equiangular. The five conditions $p \in B$, $q \notin B$, $r \notin B$, $|B \cap B_1| = 2$, $|B \cap B_2| = 2$ force $\langle \Phi \rangle$ to have dimension 19.

2 Szöllősi's construction

Continuing the notation from the previous section, let B_3 be an octad containing p, q, r such that $|B_3 \cap B_1| = 4$ and $|B_3 \cap B_2| = 2$. Let s be a point other than q, r in $B_3 \cap B_1$. Let S (for Szöllősi) be the subset of size 54 of A consisting of the octads B such that $|B \cap B_3| = 4$ if $s \in B$ and $|B \cap B_3| = 2$ otherwise. All choices made are unique up to isomorphism.

The condition given says that $\chi_B - u$ is orthogonal to $3(\chi_{B_3} - 2\chi_{\{s\}}) + 2\chi_{\{p\}} - 4\chi_{\{q,r\}}$, so that the resulting subsystem Ψ of Φ spans a subspace of dimension 18.

References

[1] G. Greaves, J. H. Koolen, A. Munemasa & F. Szöllősi, *Equiangular lines in Euclidean spaces*, J. Combin. Th. A **138** (2016) 208–235.

[2] F. Szöllősi, *A remark on a construction of D. S. Asche*, `arXiv:1703.04505`, Mar 2017.