Counting families of mutually intersecting sets

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To Cecile, on the occasion of her 65th birthday

Abstract

We determine the number of maximal intersecting families on a 9-set and find 423295099074735261880. We determine the number of independent sets of the Kneser graph $K(9,4)$ and find 366996244568643864340. We determine the number of intersecting families on an 8-set and on a 9-set and find 14704022144627161780744368338695925293142507520 and 12553242487940503914363982718112298267975272720808010757809032705650591023015520462677475328 (roughly $1.255 \cdot 10^{91}$), respectively.

1 Introduction

Let X be a finite set, and $\mathcal{F}$ a collection of subsets of X. We call $\mathcal{F}$ *linked* (or *intersecting*) when any two (not necessarily distinct) members of $\mathcal{F}$ have nonempty intersection. We call $\mathcal{F}$ a *maximal linked system* (mls) when $\mathcal{F}$ is linked, but no strictly larger collection of subsets of X is linked. Let $|X| = n$. If $n > 0$, then a maximal linked system has size 2^{n-1} (since it contains precisely one of A, $X \setminus A$ for each subset A of X).

For a family $\mathcal{F}$ of subsets of X, let $\mathcal{F}^{\min}$ be the collection of inclusion-wise minimal elements of $\mathcal{F}$. Then $\mathcal{F}^{\min}$ is an antichain. Let $\mathcal{F}^{\uparrow}$ be the collection of subsets of X containing some element of $\mathcal{F}$. If $\mathcal{F}$ is an mls, then $\mathcal{F} = \mathcal{F}^{\min\uparrow}$.

The Kneser graph $K(n,r)$ is the graph with as vertices the r-subsets of a fixed n-set, two vertices being adjacent when they are disjoint. It follows that a coclique (independent set) in $K(n,r)$ is a collection of mutually intersecting r-subsets of the given n-set.

Let $\lambda(n)$ and $\Lambda(n)$ be the number of maximal linked systems and linked systems on n points, respectively. In this note we determine $\lambda(9)$ and $\Lambda(8)$. The numbers $\lambda(n)$ (and, to a lesser degree, $\Lambda(n)$) play a rôle in various areas of mathematics. The description in terms of maximal linked systems is from topology (giving the size of the superextension of a finite space). In this setting, $\lambda(n)$ with $n \le 6$ was determined by G. A. Jensen in 1966, $\lambda(7)$ was found in [1], and $\lambda(8)$ in [10]. An equivalent formulation comes from the area of Boolean functions (see below) where $\lambda(n)$ is the number of self-dual monotone Boolean functions of n variables. Knuth [6] computes $\lambda(n)$ for $n \le 8$. Hoşten & Morris [4] found that the order dimension of the complete graph K_n is the smallest t for which $\lambda(t-1) \ge n$, and computed $\lambda(n)$ for $n \le 6$. Conway and Loeb studied $\lambda(n)$ in the context of multi-player coalitions and determined $\lambda(n)$ for $n \le 8$, cf. [8, 9]. The value of $\Lambda(n)$ for $n \leq 7$ was found in [11]. The sequences $\lambda(n)$ and $\Lambda(n)$ are given in Sloane's Encyclopedia of Integer Sequences under A001206 and A051185, respectively.

1.1 Description in terms of Boolean functions

A *Boolean function* in n variables is a function $f : \{0,1\}^n \to \{0,1\}$. There is a 1-1 correspondence between Boolean functions f and set systems $\mathcal{F}$ obtained by letting f be the characteristic function of $\mathcal{F}$. The Boolean function f is called *monotone* when it cannot decrease (become false) when some variables are increased (made true). The equivalent property for $\mathcal{F}$ is that $\mathcal{F}^\uparrow = \mathcal{F}$. The Boolean function f is called *self-dual* when $f(1-x_1, ..., 1-x_n) = 1-f(x_1, ..., x_n)$. The equivalent property for $\mathcal{F}$ is that $\mathcal{F}$ contains precisely one element from every complementary pair $\{A, X \setminus A\}$. Counting maximal linked systems is therefore equivalent to counting self-dual monotone Boolean functions.

1.2 Counting

Erdős [3] (p. 79) writes: *It does not seem easy to determine $\lambda(n)$. We could not even get an asymptotic formula.* It is an easy exercise to give asymptotic formulas for $\log_2 \lambda(n)$ and $\log_2 \Lambda(n)$.

Proposition 1.1 ([1])
(i) Let $\alpha(n)$ be the number of antichains on n points. Then

$$\log_2 \lambda(n) \sim \log_2 \alpha(n-1) \sim \frac{2^n}{\sqrt{2\pi n}}.$$

(ii) Let $i(n)$ be the number of families on n points with nonempty intersection. Then

$$\log_2 \Lambda(n) \sim \log_2 i(n) \sim 2^{n-1}.$$

(A proof is given in the next section.)

The currently known values of $\lambda(n)$ and $\Lambda(n)$ are as follows.

Proposition 1.2 *The values of $\lambda(n)$ and $\Lambda(n)$ for $n \leq 9$ are as given in the table below.*

n	$\lambda(n)$	$\Lambda(n)$
0	1	1
1	1	2
2	2	$2 \cdot 3$
3	4	$2^3 \cdot 5$
4	12	$2^5 \cdot 43$
5	81	$2^{12} \cdot 321$
6	2646	$2^{22} \cdot 217633$
7	1422564	$2^{49} \cdot 517277329$
8	229809982112	$2^{93} \cdot 1484726812083249435$
9	423295099074735261880	$2^{200} \cdot 781190197891493647924238476 4953$

That is, the sequence $\Lambda(n)$ starts out 1, 2, 6, 40, 1376, 1314816, 912818962432, 291201248266450683035648, 1470402214462716178074436833869592529314250 7520, 1255324248794050391436398271811229826797527272080801075780903270565059102301552046267747 5328,

2 Easy bounds

Before one starts counting, it helps to have some idea about the size of the result, so that one can pick a suitable algorithm. Below we give some rough estimates.

Lemma 2.1 *Let* $n \geq 1$. *Then* $\log_2 \lambda(n) \geq \binom{n-1}{[n/2]-1}$.

Proof. If n is even, say $n = 2m$, then pick arbitrarily one element from each pair $\{A, X \setminus A\}$ of complementary sets of size m. This gives 2^e linked systems, where $e = \frac{1}{2}\binom{n}{m} = \binom{n-1}{m-1}$. Extend each of these linked systems to a maximal linked system. The mls's obtained will be pairwise distinct.

If n is odd, say $n = 2m + 1$, then pick arbitrarily one element from each pair $\{A, X \setminus A\}$ of complementary sets where A has size m and contains a fixed element $x_0 \in X$. This gives 2^e linked systems, where $e = \binom{2m}{m-1}$, and the same conclusion follows. □

Lemma 2.2 *Let* $n \geq 1$. *Then* $\lambda(n) < \alpha(n-1)$.

Proof. Fix $x_0 \in X$. The map $\mathcal{F} \mapsto \{A \in \mathcal{F}^{\min} \mid x_0 \notin A\}$ is a bijection from mls's on n points to linked antichains on $n-1$ points (and $\{\emptyset\}$ is an antichain that is not linked). □

Since Kleitman [5] shows that $\log_2 \alpha(n) \sim \binom{n}{[n/2]}$, these two lemmas imply part (i) of Proposition 1.1. More precise results were given by Korshunov [7].

For small n the value of $\alpha(n)$ was determined by various authors. One finds 2, 3, 6, 20, 168, 7581, 7828354, 2414682040998, 56130437228687557907788 for $0 \leq n \leq 8$ ([12]). This is Sloane's sequence A000372.

Lemma 2.3 *Let* $n \geq 1$. *Then*

(i)

$$\lambda(n)\, 2^{2^{n-1}-\binom{n}{[n/2]}} \leq \Lambda(n) \leq \lambda(n)\, 2^{2^{n-1}},$$

(ii)

$$n 2^{2^{n-1}}\left(1 - \frac{n-1}{2 \cdot 2^{2^{n-2}}}\right) \leq \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{2^{n-k}} = i(n) \leq \Lambda(n).$$

Proof. (i) Let $\mathcal{F}$ be an mls. Then $\mathcal{F}$ has size 2^{n-1}, and hence contains $2^{2^{n-1}}$ linked subfamilies. This shows that $\Lambda(n) \leq \lambda(n).2^{2^{n-1}}$. Next, $\mathcal{F}^{\min}$ is an antichain, and, by Sperner's Lemma, has size at most $\binom{n}{[n/2]}$. Each of the at least $2^{2^{n-1}-\binom{n}{[n/2]}}$ linked families $\mathcal{G}$ with $\mathcal{F}^{\min} \subseteq \mathcal{G} \subseteq \mathcal{F}$ determines $\mathcal{F} = \mathcal{G}^{\uparrow}$.

(ii) Since a family with nonempty intersection is linked, $i(n) \leq \Lambda(n)$. The formula for $i(n)$ follows by inclusion-exclusion. Since $\binom{n}{k} 2^{2^{n-k}}$ decreases with k, the first two terms give a lower bound for the sum. □

This lemma implies part (ii) of Proposition 1.1.

Finally, let us note a 1-1 correspondence. Above we saw a 1-1 correspondence between mls's on n points and linked antichains on $n-1$ points. There is also a 1-1 correspondence between linked antichains on $n-1$ points and linked antichains on n points 'in the bottom half $\mathcal{H}$ of the Boolean lattice', where $\mathcal{H}$ consists of the subsets of X of size less than $n/2$, together with an arbitrary choice of precisely one element from each complementary pair $\{A, X \setminus A\}$ of sets of size $n/2$ (in case n is even). Indeed, we can let $\mathcal{F}$ correspond with $(\mathcal{F} \cap \mathcal{H}) \cup \{X \setminus A \mid A \in \mathcal{F} \setminus \mathcal{H}\}$.

3 Computation of $\lambda(8)$

In Section 5, $\lambda(8)$ is found as a side result of the computation of $\Lambda(8)$. The algorithm used in [10] enumerated upwardly closed linked systems $\mathcal{L}$ of sets of size at most 3, and for each $\mathcal{L}$ counted the number a of complementary pairs $\{A, X \setminus A\}$ of 4-sets, such that both A and $X \setminus A$ meet all elements of $\mathcal{L}$. Now $\mathcal{L}$ is contained in precisely 2^a mls's. One finds $\lambda(8) = \sum_{\mathcal{L}} 2^{a(\mathcal{L})} = 229809982112$.

4 Computation of $\lambda(9)$

We count linked antichains $\mathcal{A}$ on 9 points, of which all elements have size at most 4. Let $\mathcal{A}'$ be the subcollection of $\mathcal{A}$ consisting of the sets of size at most 3.

Classifying all linked antichains $\mathcal{B}$ on 9 points, of which all elements have size at most 3, under the action of the symmetric group Sym(9) shows that there are 15952 orbits. For each orbit, pick a representative $\mathcal{B}$ and count extensions to linked antichains $\mathcal{A}$ for which $\mathcal{A}' = \mathcal{B}$. For example, if $\mathcal{B}$ contains a singleton $\{x\}$, then it contains nothing else, and $\mathcal{A} = \mathcal{B}$. (There are 9 such $\mathcal{A}$.) A similar conclusion holds when $\mathcal{B}$ contains three pairs $\{x, y\}$, $\{x, z\}$ and $\{y, z\}$. For largish $\mathcal{B}$ it is very easy to find all extensions $\mathcal{A}$. The worst case is that where $\mathcal{B}$ is empty, so that $\mathcal{A}$ is a coclique in the Kneser graph $K(9, 4)$.

4.1 Counting independent sets in a sparse graph

Counting independent sets in a sparse graph is a popular topic. People mostly prove complexity results: for regular graphs of degree at most 4, approximate counting is easy, for degree at least 6 it is difficult (cf. [2]). Here we have degree 5, and we want the exact count.

A recursive algorithm that keeps a current set of vertices S, and when S contains an edge xy calls itself with $S \setminus \{x\}$ and with $S \setminus (\{x\} \cup N(x))$, where $N(x)$ is the set of neighbours of x, can count $2^{|S|}$ independent sets when S is independent. This is good, but too slow.

A better version of the algorithm will go down the recursion when some vertex x has degree at least 2 in S, and count $2^a 3^b$ when S contains a isolated points and b isolated edges. This is still too slow.

The algorithm used in practice uses recursion when some vertex x has degree at least 3 in S, and counts the proper number when all vertices in S have degree at most 2, so that S induces a union of paths and cycles. (Let $p(n)$ be the number of cocliques in the path P_n on n vertices, and $c(n)$ the number of cocliques in the cycle C_n on n vertices. Then $p(0) = 1$, $p(1) = 2$, $p(m) = p(m-1) + p(m-2)$ for $m \geq 2$, and $c(m) = p(m-1) + p(m-3)$ for $m \geq 3$. The proper number is the product of numbers $p(m)$ and $c(m)$, one for each connected component P_m or C_m.)

The result of doing this on $K(9,4)$ was 366996244568643864340, and together with the 56298854506091397540 extensions of nonempty $\mathcal{B}$ we find that $\lambda(9) = 423295099074735261880$.

5 Computation of $\Lambda(8)$

We follow the setup of Pogosyan, Miyakawa & Nozaki [11]. Let 2^X be the power set of X, of size 2^n, where $n \geq 1$. If n is even, fix an element $x_0 \in X$. Let $\mathcal{H}$, the bottom half of 2^X, consist of the subsets of X of size less than $n/2$, or of size precisely $n/2$ and not containing x_0. Then for each complementary pair $\{A, X \setminus A\}$ precisely one element is in $\mathcal{H}$. Let there be $k(r,n)$ linked antichains of size r contained in $\mathcal{H}$. Let $m = \binom{n-1}{[n/2]-1}$.

Theorem 5.1 ([11]) *$\lambda(n) = \sum_{r=0}^{m} k(r,n)$ and $\Lambda(n) = \sum_{r=0}^{m} 2^{2^{n-1}-r} k(r,n)$. The precise power of 2 that divides $\Lambda(n)$ is $2^{2^{n-1}-m}$.*

Proof. The largest intersecting antichain in $\mathcal{H}$ for even n is the collection of elements of $\mathcal{H}$ of size $n/2$. (This collection is linked since the sets do not contain x_0.) For odd n the largest intersecting antichains are the collections of sets of size $(n-1)/2$ containing some fixed element $x \in X$. In both cases the size of a largest intersecting antichain equals $m := \binom{n-1}{[n/2]-1}$.

Since each mls $\mathcal{F}$ is uniquely determined by the linked antichain $\mathcal{F}^{\min} \cap \mathcal{H}$, we see that $\lambda(n) = \sum_r k(r,n)$.

For a linked system $\mathcal{F}$, consider the linked antichain $\mathcal{G} = \mathcal{F}^{\min} \cap \mathcal{H}$ and put $r = |\mathcal{G}|$. The number of $\mathcal{F}$ giving rise to the same $\mathcal{G}$ equals $2^{2^{n-1}-r}$. Indeed, there are $2^{n-1} - r$ pairs $\{A, X \setminus A\}$ with $A \in \mathcal{H} \setminus \mathcal{G}$. For such a pair, it possible that $A \in \mathcal{F}$ only if there is a $B \in \mathcal{G}$ with $B \subset A$. In this case $(X \setminus A) \cap B = \emptyset$, so $X \setminus A \notin \mathcal{F}$, while we can freely choose whether $A \in \mathcal{F}$. On the other hand, if there is no such B, then $A \notin \mathcal{F}$ while we can freely choose whether $X \setminus A \in \mathcal{F}$. (Note that $2^X \setminus \mathcal{H}$ is linked, and adding sets that properly contain a set that is present already cannot invalidate the property of being linked.) This shows that $\Lambda(n) = \sum_{r=0}^{m} 2^{2^{n-1}-r} k(r,n)$.

Since $k(m,n) = 1$ if n is even, and $k(m,n) = n$ if n is odd, so that $k(m,n)$ is odd in all cases, it follows that the precise power of 2 that divides $\Lambda(n)$ is $2^{2^{n-1}-m}$. □

For $n = 8$ we found the following values:

r	0	1	2	3	4	5
$k(r,8)$	1	127	5103	110901	1442910	12564636
r	6	7	8	9	10	11
$k(r,8)$	78501094	365924948	1302838180	3609216800	7932407952	14155324680
r	12	13	14	15	16	17
$k(r,8)$	21054328876	26807793040	29932703320	29875293476	27014411074	22319717630
r	18	19	20	21	22	23
$k(r,8)$	16932275290	11821639550	7598222786	4489816356	2432135090	1202614280
r	24	25	26	27	28	29
$k(r,8)$	539687680	218192464	78745884	25082260	6952300	1647520
r	30	31	32	33	34	35
$k(r,8)$	326312	52416	6545	595	35	1

In this particular case we see that the numbers given add up to $\lambda(8) = 229809982112$, a good check. It follows that $\Lambda(8) = \sum_r 2^{128-r} k(r, 8) = 2^{93} \cdot 1484726812083249435 = 14704022144627161780744368338695925293142507520$.

6 Computation of $\Lambda(9)$

Combining the ideas used for $\Lambda(8)$ and $\lambda(9)$ yields a computation of $\Lambda(9)$. We briefly sketch the procedure. Let $n = 9$. Let $f(x) := \sum_{r=0}^{m} k(r, n)x^r$ be the polynomial where the coefficient of x^r is the number of linked antichains of size r (in $\mathcal{H}$, which here is the collection of all subsets of X of size at most 4). By Theorem 5.1 we have $\lambda(9) = f(1)$ and $\Lambda(9) = 2^{256} f(\frac{1}{2})$. The polynomial $f(x)$ is computed just like the number $\lambda(9)$ was earlier, but this time with added bookkeeping of the size of the linked antichains found.

Again consider the graph in which the linked antichains are the cocliques. If, after fixing s vertices of the future coclique, the part of the graph that is compatible with the chosen subset has maximum degree at most two, so that it decomposes into a number of paths and cycles, then the contribution of this choice will be x^s times the product of factors $p_m(x)$ and $c_m(x)$, one for each connected component that is a path P_m or a cycle C_m respectively. The polynomials $p_m(x)$ and $c_m(x)$ are given by $p_0(x) = 1$, $p_1(x) = x + 1$, $p_m(x) = xp_{m-2}(x) + p_{m-1}(x)$ for $m \geq 2$, and $c_m(x) = xp_{m-3} + p_{m-1}$ for $m \geq 3$.

As before (in §4) classify all linked antichains $\mathcal{B}$ on 9 points of which all elements have size at most 3. There are 15952 orbits. The contribution to $f(x)$ of the 15951 orbits with nonempty $\mathcal{B}$ equals

$$g(x) = 129x + 15750x^2 + 912450x^3 + 31596495x^4 + 746247600x^5 + 12916236186x^6 + \cdots + 4221415800x^{45} + 548449482x^{46} + 58674672x^{47} + 4953060x^{48} + 308700x^{49} + 12600x^{50} + 252x^{51},$$

and we find $g(1) = 56298854506091397540$ (as we found before), and $2^{256} g(\frac{1}{2}) = 2^{200} \cdot 300219888152831735513473563225 6$.

It remains to handle cocliques in $K(9, 4)$. Let $h(x)$ be their contribution. Then $h(x) = 1 + 126j(x)$, where $xj'(x)$ is the contribution of the cocliques that contain a fixed chosen vertex. We found

$$j'(x) = 1 + 120x + 6850x^2 + 247740x^3 + 6379125x^4 + 124596364x^5 + 1920563240x^6 + \cdots + 121602100x^{49} + 14255452x^{50} + 1377740x^{51} + 105205x^{52} + 5940x^{53} + 220x^{54} + 4x^{55},$$

so that

$$h(x) = 1 + 126x + 7560x^2 + 287700x^3 + 7803810x^4 + 160753950x^5 + 2616523644x^6 + \cdots + 306437292x^{50} + 35219352x^{51} + 3338370x^{52} + 250110x^{53} + 13860x^{54} + 504x^{55} + 9x^{56},$$

and we find $h(1) = 366996244568643864340$ (as we found before), and $2^{256} h(\frac{1}{2}) = 2^{200} \cdot 480970309738661912410764913269 7$.

Altogether, this yields $f(x) = g(x) + h(x) = 1 + 255x + 23310x^2 + 1200150x^3 + 39400305x^4 + 907001550x^5 + 15532759830x^6 + 205640068950x^7 + \cdots + 2265891210x^{49} + 306449892x^{50} + 35219604x^{51} + 3338370x^{52} + 250110x^{53} + 13860x^{54} + 504x^{55} + 9x^{56}$.

We get $\lambda(9) = f(1) = 423295099074735261880$ (as we found before), and $\Lambda(9) = 2^{256} f(\frac{1}{2}) = 2^{200} \cdot 781190197891493647924238476495 3$.

7 History

Parts of the above are from [1] and [10]. These four authors had agreed to submit a joint paper, but nothing came of it. Today my three coauthors† are no longer alive, and the results of [1, 10] have been published by others. The present note improves all previous results known to me.

Acknowledgment

The referee suggested to add a computation of $\Lambda(9)$.

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