Locally Paley graphs

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Let Π be a graph. A graph Γ is called *locally* Π if the neighbourhood of every vertex in Γ is isomorphic to Π. For the classification it suffices to classify connected locally Π graphs.

Let $q = 4t + 1$ be a prime power. Let $\Pi = \Pi^{(q)}$ be the Paley graph of order q. The locally Π graphs have been classified. If $q = 9$ there are two examples, on 16 and 20 vertices, while for all other q there is a unique example, on $2q + 2$ vertices.

Where is the proof of this statement? For $q = 5$ the Paley graph is the pentagon, and the unique locally pentagon graph is the icosahedron. For $q = 9$ the Paley graph is the 3×3 grid, which is a generalized quadrangle. The classification was given by Buekenhaut & Hubaut [2]. Dominique Buset told me [3] that she had classified the possibilities for $13 \le q \le 41$, and that the details would appear in her thesis. For $q > 41$, I did the classification in [1], modulo a hypothesis on the automorphism group of the local graph of Π, that was subsequently proved by Muzychuk & Kovács [5]. So, the classification is complete. Except that Sergey Goryainov provided me with a copy of Buset's thesis [4] yesterday, and the case $q = 41$ is not treated there. So perhaps one should check this case, just to be sure.

Read [1] until the point in Section 5 where the case is considered in which the graph Γ has four distinct points ∞, x, a, a_x, where $a_x \sim \infty \sim x \sim a$ and $a \not\sim \infty$ and $a_x \not\sim x$, with the property that $\{\infty, a, x\}^\perp = \{\infty, a_x, x\}^\perp$.

(Notation: $\sim$ denotes adjacency, and if A is a set of vertices then $A^\perp$ denotes the set of vertices adjacent to each vertex in A.)

Pick $p \in \{\infty, a, x\}^\perp$. Its neighbourhood $\Gamma(p)$ is a Paley graph $\Pi^{(q)}$ in which the triples of vertices $\{\infty, a, x\}$ and $\{\infty, a_x, x\}$ have precisely the same neighbours, so that no vertex of $\Gamma(p)$ is adjacent to ∞, a, x but not to a_x. But that is impossible for $q > 81$ and also for $q = 49, 61, 73, 81$, see Lemma 3.2 of [1] and the subsequent discussion. If $q = 53$ or $q = 41$ or $q = 37$ or $q = 25$ this situation occurs only for 4-sets $\{\infty, a, a_x, x\}$ with precisely 4, 3, 3, 2 common neighbours, respectively, so that the μ-graph $\{\infty, x, a_x\}^\perp$ in $\Gamma(\infty)$ is regular of this valency. But $\Pi^{(q)}$ does not have regular μ-graphs. It follows that $q \in \{13, 17, 29\}$.

So, the same argument that was used in [1] for $q = 53$ also works for $q = 41$ (and for $q = 37$), and there is no missing case.

References

[1] A. E. Brouwer, *Locally Paley graphs*, Des. Codes Cryptogr. **21** (2000) 69–76.

[2] F. Buekenhaut & X. Hubaut, *Locally polar spaces and related rank 3 groups*, J. Algebra **45** (1977) 391–434.

[3] D. Buset, letter dd 1996-08-11.

[4] D. Buset, *Quelques conditions locales et extrémales en théorie des graphes*, Ph. D. Thesis, Université Libre de Bruxelles, December 1997.

[5] M. Muzychuk & I. Kovács, *A solution of a problem of A. E. Brouwer*, Des. Codes Cryptogr. **34** (2005) 249–264.