A(36,10,6) = 37

A. E. Brouwer

2007-11-04

Abstract

We find that $A(36, 10, 6)$, the maximum number of 6-subsets of a 36-set such that no two of these 6-subsets have more than one element in common, is 37. In particular, one cannot schedule a tournament with 24 players in six rounds with six 4-player tables each, where two players never meet twice, and a player never sits at the same table twice.

1 Introduction

The maximum number of binary vectors of length n, weight w and mutual distance at least d is called $A(n, d, w)$.

Finding $A(n, d, w)$ is equivalent to finding the packing number $D(t, k, v)$ (the maximum number of k-subsets of a v-set such that no t-set is covered twice) where $v = n$, $k = w$, and $t = w + 1 - \frac{1}{2}d$. In particular, $A(n, 10, 6) = D(2, 6, n)$.

In case an affine plane $AG(2, n)$ of order n exists, one immediately sees that $A(n^2, 2n - 2, n) = n^2 + n$. However, no $AG(2, 6)$ exists, and determining $A(36, 10, 6)$ is nontrivial.

2 Lower bounds

Using cyclic constructions, one finds $A(35, 10, 6) = 35$ and $A(36, 10, 6) \geq 36$. Indeed, taking all 35 cyclic shifts of the block $\{0, 1, 3, 7, 12, 20\}$ in $\mathbf{Z}_{35}$ shows that $A(35, 10, 6) \geq 35$, while on the other hand $A(35, 10, 6) \leq \lfloor \frac{35}{6} \lfloor \frac{34}{5} \rfloor \rfloor = 35$ by the Johnson bound.

Similarly, taking all 36 cyclic shifts of the block $\{0, 1, 3, 8, 23, 27\}$ in $\mathbf{Z}_{36}$ shows that $A(36, 10, 6) \geq 36$.

With a smaller group one can improve the lower bound by 1. Consider the set of $36 = 30 + 6$ points i, j' with $i \in \mathbf{Z}_{30}$ and $j \in \mathbf{Z}_6$. Consider the set of $37 = 1 + 6 + 30$ blocks, invariant under $\mathbf{Z}_{30}$, given by: (i) $\{0', 1', 2', 3', 4', 5'\}$, (ii) $\{0', 0, 6, 12, 18, 24\} + i$, $i = 0, ..., 5$, (iii) $\{0', 1, 2, a, b, 28\} + i$, $i = 0, ..., 29$. With the two choices $(a, b) = (9, 11)$, $(21, 23)$ we obtain the two nonisomorphic systems with group $\mathbf{Z}_{30}$.

Thus $A(36, 10, 6) \geq 37$.

3 Upper bounds

This investigation was prompted by the question whether it is possible to schedule a tournament with 24 players in six rounds on six tables, four players at each table in each round, such that each player sits at one table each round, no player sits at the same table twice, no two players meet twice.

This is asking for a partial linear space on 36 points (6 rounds, 6 tables, 24 players) with 36 lines of size 6 (round, table, 4 players) and two more lines of size 6 (the 6 rounds, and the 6 tables), such that the latter two 6-lines are disjoint and the former 36 6-lines meet both. A very small computer program very quickly decides that this is impossible.

(The construction of the previous section shows that one can have six rounds with 6 groups of five players each, where no two of these 30 players meet twice.)

Finding $A(36, 10, 6)$ is more work, but still rather straightforward. A standard backtrack search with isomorph rejection on the initial levels took a few hours. No solution better than the above 37 was found. There are two nonisomorphic solutions with 37 lines, and both have automorphism group $\mathbf{Z}_{30}$.