Orbits on Points and Lines in Finite Linear and Quasilinear Spaces

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10 Oct 1984

Abstract

Given two positive integers π and λ, we prove that there exists a finite linear space whose automorphism group has exactly π orbits on points and λ orbits on lines if and only if $\pi \leq \lambda$.

1 Introduction

A *linear* (resp. *quasilinear*) space is an incidence structure of *points* and *lines* such that any two points are incident with exactly one (resp. at most one) line, with the nondegeneracy conditions that any point is incident with at least two lines and any line is incident with at least two points. Without any confusion, a line may be identified with the set of its incident points. Note that, if S is a linear space, the dual incidence structure S^* is quasilinear.

Let Γ be any subgroup of the automorphism group Aut S of a linear space S and let π_Γ (resp. λ_Γ) denote the number of orbits of Γ on the set of points (resp. on the set of lines) of S. If S is finite, a classical result of Block [1] asserts that $\pi_\Gamma \leq \lambda_\Gamma$. This is no longer true if S is infinite, as shown by the following example due to Valette (see Buekenhout [3]): let D be a closed disc in the Euclidean plane and let S be the linear space whose points are the points of D and whose lines are the intersections of D with the lines of the plane having more than one point in common with D. The group Aut S is clearly transitive on the lines of S, on the points inside D and on the points on the boundary of D. Actually Aut S has two orbits on the points of S and only one orbit on the lines. Indeed, for any point x inside D, there exist two lines L_1 and L_2 not containing x, such that any line of S passing through x intersects $L_1 \cup L_2$. On the contrary, for any point x on the boundary of D and for any pair of lines L_1 and L_2 not containing x, there is a line of S passing through x and disjoint from $L_1 \cup L_2$. Therefore the interior of D and the boundary of D are two orbits of Aut S.

This paper is concerned with the following two problems: for which pairs (π, λ) of positive integers does there exist

1. a finite linear space S
2. a finite quasilinear space S

such that Aut S has exactly π orbits on the points and λ orbits on the lines of S? A pair (π, λ) of positive integers satisfying condition (1) (resp. condition (2)) will be called *linearly* (resp. *quasilinearly*) realizable.

Theorem 1 *A pair (π, λ) is linearly realizable if and only if $\pi \leq \lambda$.*

Theorem 2 *All pairs (π, λ) are quasilinearly realizable.*

2 Proof of Theorem 1

Given a prime power q and an integer $n \geq 2$, put $q_i = q^{2^{i-1}}$ $(1 \leq i \leq n)$ and consider the projective space $P = \mathrm{PG}(n, q_n)$ with homogeneous coordinates $(x_1, x_2, \ldots, x_{n+1})$. The points of P such that $x_{n+1} = 0$ will be called *points at infinity*. Let $\mathcal{P}$ be the set of points of P whose coordinates satisfy $x_i \in GF(q_i)$ for $1 \leq i \leq n$ and $x_{n+1} = 1$. Let $\mathcal{L}$ be the set of lines of P having at least two points in $\mathcal{P}$ and let $\mathcal{P}_\infty$ be the set of points at infinity of all the lines in $\mathcal{L}$.

Now let S denote the linear space whose set of points is $\mathcal{P} \cup \mathcal{P}_\infty$ and whose set of lines is $\mathcal{L} \cup \{\mathcal{P}_\infty\}$, a point being incident with a line iff it belongs to that line. Since the lines of S have $n+1$ distinct sizes, namely

$$q_1 + 1, q_2 + 1, \ldots, q_n + 1$$

and

$$|\mathcal{P}_\infty| = 1 + q_n + q_{n-1}q_n + \cdots + q_2q_3 \cdots q_n \ ,$$

Aut S has at least $n+1$ line-orbits. Similarly, Aut S has at least n orbits on the points of the line $\mathcal{P}_\infty$ because the points at infinity of lines of different sizes are necessarily in different orbits, and so Aut S has at least $n+1$ point-orbits. Let G be the group of projectivities of P defined by the matrices

$$\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 & b_1 \\ a_{21} & a_{22} & 0 & \cdots & 0 & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & b_n \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

where $a_{ij},\ b_i \in \mathrm{GF}(q_i)$ and $a_{ii} \neq 0$ for $1 \leq j \leq i \leq n$. The group G induces on S an automorphism group which is easily checked to be transitive on $\mathcal{P}$ and also on the lines of S having the same size. It follows that Aut S has exactly $n+1$ point-orbits and $n+1$ line-orbits. Thus the pair $(n+1, n+1)$ is linearly realizable for every $n \geq 2$.

Let $\mathcal{P}_\infty^1$, $\mathcal{P}_\infty^2$,…,$\mathcal{P}_\infty^n$ denote the orbits of Aut S of size 1, q_n, $q_{n-1}q_n$, …, $q_2q_3 \cdots q_n$ respectively on the points of $\mathcal{P}_\infty$. Consider the linear space S^r $(1 \leq r \leq n-1)$ obtained from S by deleting the points of $\mathcal{P}_\infty^1 \cup \cdots \cup \mathcal{P}_\infty^r$, and also the linear space S^n obtained from S be deleting the points of $\mathcal{P}_\infty$ and the line $\mathcal{P}_\infty$.

Clearly, Aut S^r has $n+1-r$ point-orbits and $n+1$ line-orbits, while Aut S^n has just one point-orbit and n line-orbits. This shows that every pair (π, λ) with $\pi \leq \lambda$ is linearly realizable, except perhaps $(1,1)$ and $(2,2)$. But any finite Desarguesian projective plane gives an example where $\pi = \lambda = 1$, and any finite degenerate projective plane having a line of size $k \geq 3$ and k lines of size 2 gives an example where $\pi = \lambda = 2$.

3 Proof of Theorem 2

Since any linear space is also quasilinear, every pair (π, λ) with $\pi \leq \lambda$ is quasilinearly realizable by Theorem 1. On the other hand, if the automorphism group Aut S of a linear space S has π orbits on the points and λ orbits on the lines of S, then the automorphism group Aut S^* of the dual quasilinear space S^* has λ orbits on the points and π orbits on the lines of S^*. It follows that every pair (π, λ) with $\pi \geq \lambda$ is also quasilinearly realizable.

4 Remarks

(1) The problems discussed in this paper may of course be restricted to particular classes of finite linear or quasilinear spaces, for instance the 2–$(v, k, 1)$ designs (i.e., the finite linear spaces all of whose lines have the same size), but very little information seems to be available. Saxl [11, 12] has proved that if $\pi = \lambda$ in a 2–$(v, 3, 1)$ design with $v > 7$, then $\pi \leq 3$.

In the particular case of finite projective planes, it is well known that π is always equal to λ. We do not know for which positive integers π there exists a finite projective plane whose automorphism group has exactly π orbits on points (and so π orbits on lines). Here is some partial information: $\pi = 1$ in the Desarguesian planes, $\pi = 2$ in the Hughes planes [7, pp. 246–247]. $\pi = 3$ in the Figueroa planes [8], $\pi = 4$ in the Hering and Schaeffer planes [9, pp. 261–263], $\pi = 5$ in the Narayana Rao and Satyanarayana planes [10], $\pi = 10$ in the Capursi plane [6]; moreover, for any odd prime power $q > 3$, $\pi = q + 2$ is realized in a generalized André plane of order q^2, as shown in [2].

(2) The situation is worse if we remove the finiteness assumption in our original problem. For instance, in the introduction we have described an infinite linear space for which $(\pi, \lambda) = (2, 1)$, but we do not even know whether there exists an infinite linear space for which $\pi > 2$ and $\lambda = 1$ (an interesting idea about this problem was formulated by Cameron [5]).

(3) Similar problems arise in a natural way for all incidence structures. For example, in the case of undirected graphs (without loops and multiple edges), Buset [4] has proved that, given two integers $v > 0$ and $\varepsilon \geq 0$, there exists a finite graph (resp. a finite connected graph) whose automorphism group has exactly v orbits on vertices and ε orbits on edges if and only if $v \leq 2\varepsilon + 1$ (resp. $v \leq \varepsilon + 1$).

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