Characterization of the Patterson graph

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Abstract

In this note we show that there is a unique distance-regular graph with intersection array $\{280, 243, 144, 10;\ 1, 8, 90, 280\}$.

1 Summary

One distance-transitive graph with intersection array $\{280, 243, 144, 10;\ 1, 8, 90, 280\}$ is known, namely the Patterson graph on 22880 vertices with automorphism group Suz.2 and point stabilizer $3.U_4(3).2^2$. For a description, see [2], §13.7. In this note we show that it is the only one with this intersection array.

The proof goes by observing that the graph is tight and hence has strongly regular local graphs, then identifying the local graph as the unique generalized quadrangle $GQ(9, 3)$, then reconstructing the graph.

This case is a bit more subtle than similar local characterizations, mainly because the point stabilizer $3.U_4(3).2^2$ on the one hand is not faithful on the local graph (the central 3 is invisible) and on the other hand does not induce the full automorphism group ($U_4(3).D_8$) of the local graph.

2 Introduction

For a graph Γ, and a vertex x of this graph, let $\Gamma(x)$ be the set of neighbours of x in Γ, and, more generally, let $\Gamma_i(x)$ be the set of vertices of Γ at distance i from x (so that $\Gamma(x) = \Gamma_1(x)$).

A finite connected graph Γ without loops or multiple edges is called *distance-regular* when for all i there are constants c_i, a_i, b_i such that for any two vertices x, y at mutual distance i we have $|\Gamma_{i-1}(x) \cap \Gamma(y)| = c_i$ and $|\Gamma_i(x) \cap \Gamma(y)| = a_i$ and $|\Gamma_{i+1}(x) \cap \Gamma(y)| = b_i$.

If Γ is distance-regular of diameter d (so that c_i, a_i, b_i are well-defined for $0 \le i \le d$), then we call the array $\{b_0, b_1, \ldots, b_{d-1};\ c_1, c_2, \ldots, c_d\}$ the *intersection array* of Γ. One uses the notation $k := b_0$, $\lambda := a_1$, $\mu := c_2$.

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Note that all parameters c_i, a_i, b_i can be found from the intersection array, since $c_i + a_i + b_i = k$ for all i, and $c_0 = b_d = 0$.

For example, the Petersen graph is distance-regular with diameter 2 and intersection array $\{3, 2;\ 1, 1\}$. More generally, a connected graph is called *strongly regular* when it is distance-regular of diameter 2.

The number of vertices v of a distance-regular graph can be computed from the parameters. With $k_i := |\Gamma_i(x)|$ one finds $k_0 = 1$ and $k_{i+1} = b_i k_i / c_{i+1}$, and $v = k_0 + k_1 + \cdots + k_d$.

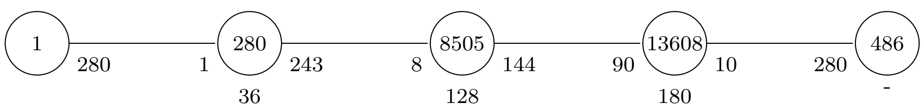
Also the spectrum of a distance-regular graph can be computed from the parameters. The *spectrum* of a graph with 0-1 adjacency matrix A is by definition the spectrum (eigenvalues with multiplicities) of the matrix A. For details, see, e.g., [2].

A partition Π of the vertex set of a finite graph Γ is called *regular* when for $A, B \in \Pi$ and $a \in A$, the number $e_{AB} = |\Gamma(a) \cap B|$ does not depend on the choice of $a \in A$. For example, the distance partition $\{\Gamma_i(x) \mid 0 \le i \le d\}$ of the vertex set of a distance-regular graph is regular.

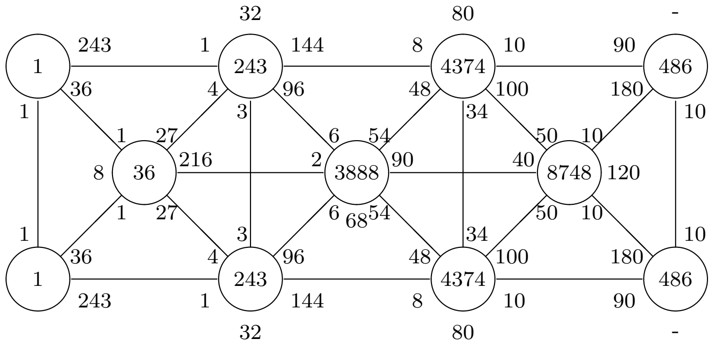
For undefined terminology and notation concerning distance-regular graphs, see [2].

3 The Patterson graph

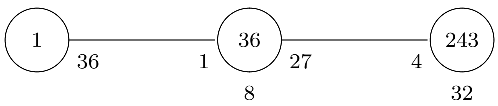
The Patterson graph Γ is a distance-transtive (and, in particular, distance-regular) graph on 22880 vertices with intersection array $\{280, 243, 144, 10;\ 1, 8, 90, 280\}$. It is constructed by taking as vertices the subgroups of order 3 of Suz that are generated by an element of Atlas conjugacy class 3A (the centers of the 3-Sylow subgroups), where two subgroups are adjacent when they commute. The full automorphism group is Suz.2, acting by conjugation. In particular, if x is a vertex, and $g \in x$ an element of order 3, then g fixes x and its neighbours. The distance distribution diagram of Γ is given below.



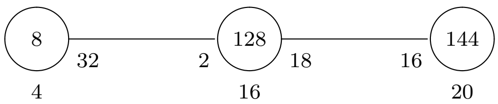
This graph is 1-homogeneous in the sense of Nomura, that is, given an edge xy, the distance distribution partition around x and y (with balloons $D^i_j = \{z \mid d(x,z) = i, d(y,z) = j\}$) is regular, with parameters independent of the choice of the edge xy. The diagram is given below.



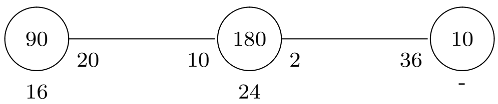
Given two vertices x, y at distance i, the partition $\{\Gamma(y) \cap \Gamma_j(x) \mid j = i-1, i, i+1\}$ of $\Gamma(y)$ is regular for each i. The diagrams for $i = 1, 2, 3$ are given below.



$v = 280$



$v = 280$



$v = 280$

4 Tight graphs

Let Γ be a non-bipartite distance-regular graph with eigenvalues $k = \theta_0 > \cdots > \theta_d$, where $d \geq 3$. In [4] it is shown that the following conditions are equivalent:

(i)

$$(\theta_1 + \frac{k}{\lambda+1})(\theta_d + \frac{k}{\lambda+1}) = -\frac{kb_1\lambda}{(\lambda+1)^2}$$

(ii) each local graph is connected and strongly regular, with nontrivial eigenvalues $-1 - b_1/(1+\theta_1)$ and $-1 - b_1/(1+\theta_d)$

(iii) $a_1 \neq 0$, $a_d = 0$, and the distance partition around a pair of adjacent vertices is regular ('Γ is 1-homogeneous in the sense of Nomura').

Graphs satisfying these properties are called *tight*.

In [3] it is shown that a distance-regular graph with $a_1 \neq 0$ is 1-homogeneous in the sense of Nomura if and only if

(iv) for each i $(1 \leq i \leq d)$ and vertices x, y at distance i, the partition $\{\Gamma(y) \cap \Gamma_j(x) \mid j = i-1, i, i+1\}$ of $\Gamma(y)$ is regular.

5 Determination of the local graph

From now on, let Γ be a distance-regular graph with the intersection array of the Patterson graph. We shall show that Γ is isomorphic to the Patterson graph.

The spectrum of Γ is 280^1, 80^{364}, 20^{5940}, -8^{15795}, -28^{780}. Thus $d = 4$, $k = 280$, $b_1 = 243$, $\lambda = 36$, $\theta_1 = 80$ and $\theta_d = -28$. Since $(80 \cdot 37 + 280)(-28 \cdot 37 + 280) = -280 \cdot 243 \cdot 36$ the graph Γ is tight.

A *local graph* in Γ is the graph $\Gamma(x)$ induced on the set of neighbours of a vertex x of Γ.

From the tightness, we conclude that the local graphs in Γ are strongly regular with nontrivial eigenvalues 8 and -4, and hence with parameters $(v, k, \lambda, \mu) = (280, 36, 8, 4)$.

Step 0 *If yz is an edge inside $\Gamma_2(x)$ then x, y, z have precisely 2 common neighbours.*

Indeed, consider the distance partition around y, z. Let $T_i = \Gamma_i(y) \cap \Gamma_i(z)$ $(i = 1, 2)$. Each vertex of T_1 has $1 + 1 + 8 + 27 + 27 = 64$ neighbours at distance at most 1 to y or z, so $280 - 64 = 216$ neighbours in T_2. Now $|T_1| = p^1_{11} = 36$ and $|T_2| = p^1_{22} = 3888$, and the distance partition around y, z is regular, so each vertex of T_2 has $36 \cdot 216/3888 = 2$ neighbours in T_1.

Step 1 *In Γ, the 8 common neighbours of two vertices at distance 2 induce a $K_{4,4}$.*

Indeed, since the local graph has $\mu = 4$, the subgraph M induced on the 8 common neighbours of two vertices p, q at distance 2 is regular of valency 4. Moreover, M does not contain a triangle with pending edge (if x is the endpoint of the pending edge, and yz the opposite side of the triangle, then x, y, z would have more than 2 common neighbours). But then every vertex of M has at least two neighbours in any triangle of M, impossible unless M has no triangles, i.e., is $K_{4,4}$.

Step 2 *In Γ, any two vertices at distance 2 are contained in a unique $K_{4,4,4}$.*

Indeed, this follows immediately from Step 1.

Step 3 *The local graph is the collinearity graph of a generalized quadrangle.*

Indeed, from the previous step we see that the μ-graphs of the local graph are cocliques, so the local graph does not contain subgraphs $K_{1,1,2}$ and hence every edge is contained in a unique clique of size $\lambda + 2 = 10$. With these 10-cliques as lines we get a GQ(9,3). (In a strongly regular graph with v vertices, valency k and smallest eigenvalue s, every clique C has at most $1 + k/(-s)$ vertices, and in case of equality each point outside C is adjacent to precisely $\mu/(-s)$ points of C (cf. [2], 1.3.2 (ii)). In the case of a graph with the parameters of a generalized quadrangle this means that it suffices to find cliques of the right size, one on each edge—then every point outside such a clique is automatically adjacent to a unique point of the clique.)

Step 4 *The local graph is the graph of the isotropic points of $U(4, 3^2)$, the polar space on a vector space of dimension 4 over $\mathbf{F}_9$ provided with a nondegenerate Hermitean form, where points are adjacent when they are orthogonal. The lines of the generalized quadrangle are the totally isotropic lines of this polar space.*

Indeed, according to [5], 5.3.2 (iii), there is a unique generalized quadrangle of order (3,9) (and our generalized quadrangle must be its dual).

6 Determination of the second neighbourhood

Fix a vertex a of Γ, and let $\Delta = \Gamma(a)$. We know the graph Δ, and can label each vertex p of $\Gamma_2(a)$ with the $K_{4,4}$ in Δ it is adjacent to. If p, q are adjacent vertices in $\Gamma_2(a)$, then by Step 0 they have two common neighbours u, v in Δ, and by Step 2 the vertices u, v are adjacent.

By Step 2, each label occurs three times, so that $\Gamma_2(a)$ (with 8505 vertices) is a 3-cover of the graph E (with 2835 vertices) that has as vertices the $K_{4,4}$'s (pairs of orthogonal hyperbolic lines) in Δ, where two $K_{4,4}$'s are adjacent when they have an edge in common.

Both E and $\Gamma_2(a)$ have valency 128, but they do not have the same local structure. Indeed, if c, c', c'' are three vertices with the same label in $\Gamma_2(a)$ and b is a common neighbour of a, c, c', c'', then we see that c, c', c'' have pairwise distance 3 in $\Gamma(b) \cap \Gamma_2(a)$. Thus, certain triangles in E do not lift to triangles.

6.1 The local graph of E

Let us investigate the local structure of E. Work in the geometry of isotropic points and hyperbolic lines in $PG(3,9)$ provided with a nondegenerate Hermitean form, and view E as the graph on the pairs $\{L, L^\perp\}$ of orthogonal hyperbolic lines, where two such pairs are adjacent when each line of one pair meets a line of the other pair.

Given a pair of orthogonal hyperbolic lines $\{L, L^\perp\}$, fix points P, Q of $L, L^\perp$, respectively. Now $Q^\perp$ is a plane containing Q and L. It has 10 projective lines on P, namely L and (the totally isotropic) $P + Q$ and 8 other (hyperbolic) lines. These 8 lines M all have have an orthogonal mate passing through Q, hence give rise to an 8-clique C_{PQ} in E. Varying P and Q we find 16 8-cliques C_{PQ}, accounting for the valency 128 of E.

Let Q, R be two points of $L^\perp$. Then in the graph induced by E on $C_{PQ} \cup C_{PR}$ each vertex of C_{PQ} is adjacent to two vertices of C_{PR}. (Indeed, if $\{M, M^\perp\} \in C_{PQ}$ and $\{N, N^\perp\} \in C_{PR}$, with M, N containing P, then $M^\perp, N^\perp$ are lines in the plane $P^\perp$, and $\{M, M^\perp\}$ and $\{N, N^\perp\}$ will be adjacent when $M^\perp, N^\perp$ intersect on one of the two totally isotropic lines on P in $P^\perp$ that do not contain Q or R.)

Let P, Q be distinct points of L, and R, S distinct points of $L^\perp$. Then in the graph induced by E on $C_{PR} \cup C_{QS}$ each vertex of C_{PR} is adjacent to one vertex of C_{QS}. (Indeed, if $\{M, M^\perp\} \in C_{PR}$ with M containing P, and $\{N, N^\perp\} \in C_{QS}$ with N containing Q, then if $\{M, M^\perp\}$ and $\{N, N^\perp\}$ are adjacent we cannot have that M, N meet, since a point of intersection would be in $R^\perp \cap S^\perp = L$. So, adjacency happens when M meets $N^\perp$ and $M^\perp$ meets N. But now $\{N, N^\perp\}$ is uniquely determined given P, Q, R, S: the line $M^\perp$ contains a unique point T orthogonal to S, and N must be the line $Q + T$.)

Thus, each local graph of E has 128 vertices, and valency $7 + 12 + 9 = 28$.

6.2 The local graph of $\Gamma_2(a)$

Let b be a vertex of $\Gamma_2(a)$. Then $\Gamma(b) \cap \Gamma_2(a)$ is the subgraph of $\Gamma(b)$ induced on the vertices adjacent to some vertex of a fixed $K_{4,4}$. We see immediately that this is the union of 16 8-cliques (on the 16 totally isotropic lines meeting the $K_{4,4}$ in two points). Let us call these C_{PQ} again. This time there are no edges between C_{PQ} and C_{PR}, and again there is a matching between C_{PR} and C_{QS}.

Thus, each local graph of $\Gamma_2(a)$ has 128 vertices, and valency $7+0+9=16$.

6.3 Construction of the 3-cover

Consider the universal cover of E modulo the triangles that should lift to triangles, that is, modulo all triangles different from the triangles $\{L, L^\perp\}\{M, M^\perp\}\{N, N^\perp\}$ with L, M, N on a common point, and $L^\perp, M^\perp, N^\perp$ pairwise intersecting in an isotropic point.

A computation with GAP (using Soicher's `fundamental.g`, see also [6]) shows that this universal cover is a 9-cover, with elementary abelian fundamental group with generators x_1 and x_2 such that the automorphism group of E can interchange x_1 and x_2, and can fix x_2 while interchanging x_1 and x_1^{-1}. The cover that we need is a 3-cover, with a group that is a quotient of the group of the 9-cover such that x_1 and x_2 stay different from the identity. All such quotients are isomorphic under Aut E, so up to isomorphism, $\Gamma_{\leq 2}(a)$ is uniquely determined.

Remark In the Patterson graph, the full automorhism group of the local generalized quadrangle is $U_4(3).D_8$, but the point stabilizer in Suz.2 induces $U_4(3).(2^2)_{133}$, a subgroup of index 2, on the local graph, since a factor 2 is lost by the choice between $x_2 = x_1$ and $x_2 = x_1^{-1}$.

6.4 Elliptic quadrics in $GQ(9, 3)$

Suppose f is a nondegenerate Hermitean form on a vector space of dimension 4 over $\mathbf{F}_{q^2}$, where q is odd.

If u, v, w are pairwise nonadjacent isotropic points, then $f(u,v)f(v,w)f(w,u)\mathbf{F}_q^*$ is a well-defined coset of $\mathbf{F}_q^*$ in $\mathbf{F}_{q^2}^*$. (Indeed, if we choose a different representative vector αu for the point spanned by u, then $f(u,v)f(v,w)f(w,u)$ is multiplied by $\alpha\bar{\alpha} \in \mathbf{F}_q^*$.)

If u, v, w are collinear, and $f(u,v)f(v,w)f(w,u) = \gamma$, then $\bar{\gamma} = -\gamma$, and in particular $\gamma \notin \mathbf{F}_q^*$.

Suppose $f(u,v)f(v,w)f(w,u) \in \mathbf{F}_q^*$. If the isotropic point x is a common neighbour of u, v, w, then u, v, w lie in the plane $x^\perp$. Choose α, β, γ so that $u + \alpha x, v + \beta x, w + \gamma x$ are collinear, and find a contradiction. Thus, if $f(u,v)f(v,w)f(w,u) \in \mathbf{F}_q^*$ then u, v, w do not have a common neighbour.

We may choose a basis such that f has coefficients in the ground field $\mathbf{F}_q$. (That is, $f(u,v) = \sum c_{ij}\bar{u}_i v_j$, with $c_{ij} \in \mathbf{F}_q$.) Now restricted to the points that have a representing vector with all coordinates in $\mathbf{F}_q$, the form f becomes symmetric bilinear, so that the set of isotropic points becomes a quadric in this subfield subspace. By the above, no isotropic point is collinear to three pairwise nonadjacent points in such a quadric.

We may choose a basis such that f describes an elliptic quadric Q over $\mathbf{F}_q$. A line (of size q^2+1) meeting this elliptic quadric (also of size q^2+1) in a point z contains a neighbour of each point of $Q \setminus \{z\}$ (since we are in a generalized quadrangle), and these neighbours must be distinct. It follows that each point outside an elliptic quadric is adjacent to either 0 or 2 points inside.

For $q = 3$ it is straightforward to check that the 10-cocliques with the property that each point outside is adjacent to either 0 or 2 points inside are just the 10-sets C with the property that for any three distinct $u, v, w \in C$ we have $f(u,v)f(v,w)f(w,u) \in \mathbf{F}_q^*$ (which are just the elliptic quadrics).

Lemma 6.1 *The collinearity graph of $GQ(9,3)$ has 9072 10-cocliques with the property that each point outside has 0 or 2 neighbours inside. These 9072 are equivalent for the full automorphism group of $GQ(9,3)$, but fall into two orbits of size 4536 for $U_4(3)$.*

(Proof: The cocliques are found with a very simple backtrack search.)

7 Determination of Γ

The next step is to reconstruct $\Gamma_3(a)$.

Each point $x \in \Gamma_3(a)$ determines a regular partition of $\Gamma(a)$ into $10 + 180 + 90$, where the part of size 10 is a coclique, and each point from the part of size 180 has two neighbours in this coclique.

We expect to find 10-cocliques of one kind only, each three times.

For each of the 9072 0,2-cocliques in $GQ(9,3)$, the set of 90 nonadjacent points contains 90 subgraphs $K_{4,4}$.

Consider a point $p \in \Gamma_2(a)$ adjacent to $x \in \Gamma_3(a)$. Then $\Gamma_2(a) \cap \Gamma(p) \cap \Gamma(x)$ induces $4K_4$. The 90-set $\Gamma(a) \cap \Gamma_2(x)$ is uniquely determined by the 17 subgraphs $K_{4,4}$ determined by p and the 16 vertices q in these $4K_4$. This means that we can identify the vertex x seen in the local graph of p with the vertex x seen in the local graph of any other vertex $q \in \Gamma_2(a)$.

Since the triples of vertices in $\Gamma_2(a)$ with the same neighbours in $\Gamma(a)$ already see their common neighbours, they have no common neighbours in $\Gamma_3(a)$, and our 90-sets in $\Gamma_2(a)$ project to 90-sets in E. Since each 90-set in E lifts to three 90-sets in $\Gamma_2(a)$, each 10-coclique occurs at most three times.

Since $\Gamma_2(a)$ is connected ($128 > 80$), given a neighbourhood of a single vertex $p \in \Gamma_2(a)$ we find uniquely all vertices of $\Gamma_3(a)$ and their connections to $\Gamma_2(a)$. Since any two adjacent vertices in $\Gamma_3(a)$ have a common neighbour in $\Gamma_2(a)$, this completely determines $\Gamma_{\leq 3}(a)$.

But if $\Gamma_{\leq 2}(a)$ completely determines $\Gamma_{\leq 3}(a)$, then doing the same for the neighbours of a we find all of Γ.

This shows that Γ is determined up to isomorphism, and finishes the proof of our main result.

8 The faraway points

We have $k_4 = 486$, $p^4_{04} = 1$, $p^4_{14} = 0$, $p^4_{24} = 315$, $p^4_{34} = 112$, $p^4_{44} = 58$. Is it possible to relate the points of $\Gamma_4(x)$ to the known structure of $\Gamma(x)$?

Every vertex a of $\Gamma_4(x)$ has distance 3 to every vertex of $\Gamma(x)$. But we can look at $A := \Gamma_4(a) \cap \Gamma_2(x)$. Now A is a coclique of size 315, and each vertex y has 9 neighbours in A, which means that each edge yz in $\Gamma(x)$ has a unique neighbour in A.

8.1 Spreads of $K_{4,4}$'s

The set A determines a set of 315 $K_{4,4}$'s in the $U_4(3)$ generalized quadrangle on $\Gamma(x)$, 9 on each point, 1 on each edge, somewhat similar to a spread. We can investigate such objects on their own right. Let us call them *spreads of $K_{4,4}$'s*.

Proposition 8.1 *(i) The $U_4(3)$ generalized quadrangle with parameters $GQ(9,3)$ possesses precisely 324 spreads of $K_{4,4}$'s, forming a single orbit under $U_4(3).D_8$, the automorphism group of $GQ(9,3)$, which acts rank 5 on the spreads with point stabilizer $L_3(4) : 2^2$. The suborbits have sizes 1, 42, 56, 105, 120. The action is imprimitive, with two blocks of size $162 = 1 + 56 + 105 = 42 + 120$.*

(ii) The stabiliser of one spread is $L_3(4) : 2^2$, acting rank 8. The action is imprimitive, with 105 blocks of size 3.

(For the subgroup $L_3(4)$ of the point stabilizer, the suborbits have sizes 1, 21, 21, 56, 105, 120.)

These spreads fall into two classes of size 162. Spreads in the same class meet in 315, 45 or 27 elements, spreads in different classes meet in 75 or 21 elements. Each spread meets 1 spread in 315, 56 in 45, 105 in 27, 42 in 75 and 120 in 21 elements.

9 The Suzuki tower

In [2], §13.7 the Patterson graph is described in terms of the sporadic Suzuki group. The vertices are the pairs of elements $\{s, s^2\}$ from the conjugacy class of size 45760 (Atlas type 3A), and two vertices have distance 0, 1, 2, 3, 4 in the Patterson graph when their union generates a subgroup 3, 3^2, $SL(2,3)$, Alt(5), Alt(4), respectively. More combinatorial information is visible if we describe this graph using the Suzuki tower.

There are strongly regular graphs on 1782, 416, 100 vertices, with parameters (1782,416,100,96), (416,100,36,20), (100,36,14,12) and automorphism groups Suz.2, $G_2(4).2$, $J_2.2$, respectively, where each is the first subconstituent of the previous. Let us call them the Suzuki, $G_2(4)$ and Hall-Wales graphs, respectively.

Let Σ be the (unique) strongly regular graph with parameters (162,56,10,24), the second subconstituent of the McLaughlin graph, and let T be the (unique) strongly regular graph with parameters (56,10,0,2), the Gewirtz graph. These have automorphism groups $U_4(3).2$ and $L_3(4).2$, and T is the first subconstituent of Σ.

If s is a 3A element of Suz, then s fixes 162 vertices of the Suzuki graph, and this set of 162 vertices induces a copy of Σ. Distance 0, 1, 2, 3, 4 in the Patterson graph corresponds to intersections of size 162, 0, 18, 12, 42, respectively, where the graphs induced on the intersections are Σ, empty, the 2-coclique extension of 3×3, $K_{6,6}$, and the nonincidence graph of points and lines in $PG(2,4)$.

Going down to the neighbourhood of some vertex of the Suzuki graph, we see that the $G_2(4)$ graph contains an orbit of size 2080 of induced copies of T. Distance 0, 2, 3, 4 in the Patterson graph corresponds to intersections of size 56, 8, 6, 16, respectively, where the graphs induced on the intersections are T, $2C_4$, a 6-coclique and a 16-coclique. These relations have valencies 1, 945, 1008, 126, respectively.

Going down to the neighbourhood of some vertex of the $G_2(4)$ graph, we see that the Hall-Wales graph contains an orbit of size 280 of 10-cocliques. Distance 0, 2, 3, 4 in the Patterson graph corresponds to intersections of size 10, 2, 0, 0, respectively. The valencies are 1, 135, 108, 36. The last two relations (both of disjoint 10-cocliques with a regular graph of valency 4 on their union) can be distinguished combinatorially by the fact that the former has a graph of diameter 4 ($k_4 = 1$) and the latter one of diameter 3 on the union.

Since the total number of 10-cocliques in the Hall-Wales graph is 280 ([1]) there are no others than just found. And hence the $G_2(4)$ graph does not contain more copies of T than just found, and the Suzuki graph does not contain more copies of Σ.

Thus, the Patterson graph is found from the Suzuki graph on 1782 vertices by taking as vertices the 22880 induced copies of Σ where two copies are adjacent when they are disjoint. The 11-cliques of Patterson correspond to partitions of the Suzuki graph into 11 copies of Σ.

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