On the chromatic number of q-Kneser graphs

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Abstract

We show that the q-Kneser graph $qK_{2k:k}$ (the graph on the k-subspaces of a $2k$-space over $GF(q)$, where two k-spaces are adjacent when they intersect trivially), has chromatic number $q^k + q^{k-1}$ for $k = 3$ and for $k < q \log q - q$. We obtain detailed results on maximal cocliques for $k = 3$.

1 Introduction

Let $qK_{n:k}$ denote the q-analog of the Kneser graph. The vertices of this graph are the k-dimensional subspaces of an n-dimensional vector space V over $GF(q)$, two vertices are adjacent if the corresponding subspaces intersect trivially. In what follows a d-dimensional subspace will be simply called a

d-space or just a $[d]$. We use projective terminology, so a point is a [1], a line a [2], a plane a [3], a solid a [4] and a hyperplane an $[n-1]$ in V. Two $[k]$'s are adjacent in the Kneser graph if their intersection is [0]. We are interested in the chromatic number, and hence in the size of large cocliques (intersecting families of k-spaces), in particular for the extremal case $n = 2k$ (for $n < 2k$ the graph itself is a coclique). The case $n > 2k$ is studied in [1, 5]. For $n = 2k$, the case $k \leq 3$ is considered in Tim Mussche's thesis [6], but will be treated in much greater detail here.

From now on, let $n = 2k$. The largest cocliques in the q-Kneser graph are the Erdős-Ko-Rado families and their duals. An EKR-family P^* is the point pencil consisting of the $[k]$'s on the point P. A dual EKR-family H^* consists of the $[k]$'s contained in the hyperplane H. Both have size $\left[{2k-1 \atop k-1}\right]$. The bound on the size comes from Frankl and Wilson [3], the classification is due to Godsil and Newman [4].

The second largest cocliques are conjectured to be the Hilton-Milner families and their duals. An HM-family $(P, S)^*$ is defined by an incident pair (P, S), where P is a point and S a $[k+1]$ containing P. It consists of the $[k]$'s contained in S together with the $[k]$'s on P intersecting S in at least a [2]. The dual $(L, H)^*$ of this is defined by an incident pair (L, H), where H is a hyperplane, and L a $[k-1]$ contained in H, and consists of the $[k]$'s containing L together with the $[k]$'s contained in H that have a nontrivial intersection with L. The size of an HM-family or its dual equals $\left[{2k-1 \atop k-1}\right] - q^{k(k-1)} + q^k$, which is slightly more than q^{k^2-k-1}. More information on q-Kneser graphs can be found in [6].

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In Spring 2004 the first author visited Günther Ziegler in Berlin. One of the problems he suggested to look at was determining the chromatic number of the q-Kneser graph.

2 Very small k

Let us consider the case that k is very small first. For $k = 1$ the Kneser graph $qK_{2:1}$ is a clique of size $q + 1$, maximal cocliques are singletons which are of EKR type as well as dual EKR type. Its chromatic number is $q + 1$.

For $k = 2$ the situation is slightly more interesting. Cocliques in the Kneser graph $qK_{4:2}$ consist of mutually intersecting lines and such a family either consists of concurrent lines, and then it is contained in an EKR-family, or consists of coplanar lines, and then is contained in a dual EKR-family. From this it follows, as we will see later, that the chromatic number of $qK_{4:2}$ is $q^2 + q$, a result due to Eisfeld, Storme and Sziklai [2].

The case $k = 3$ is treated in detail in Section 6.

3 A weak Hilton-Milner bound for $n = 2k$, q large enough

Our aim is to show that, for sufficiently large q, cocliques not contained in a point pencil or its dual have size less than $\frac{1}{2}q^{k^2-k} < \frac{1}{2}\left[{2k-1 \atop k-1}\right]$, in fact at most $c\,q^{k^2-k-1}$ where c is a constant slightly larger than 1. We'll then use this bound to determine the chromatic number of the Kneser graph $qK_{2k:k}$ for $k < q\log q - q$.

Theorem 3.1 *Let $\mathcal{F}$ be a maximal coclique in $qK_{2k:k}$ of size*

$$|\mathcal{F}| > (1 + \frac{1}{q}) \begin{bmatrix} k \\ 1 \end{bmatrix}^{k-1} \begin{bmatrix} k-1 \\ 1 \end{bmatrix}.$$

Then $\mathcal{F}$ is an EKR family P^* or a dual EKR family H^*.

In this section we prove this theorem for $k > 3$. The case $k < 3$ is trivial (every coclique is contained in an EKR or dual EKR family), and a much stronger result for $k = 3$ is proved in Theorem 6.1 below. For $k >> q\log q$ the statement in the theorem is empty, since then the right hand side is larger than the size of an EKR-family.

In the proof below we use the concept of *covering number* $\tau(\mathcal{F})$ of a family $\mathcal{F}$. This is the minimal dimension of a *covering subspace*, that is, a subspace intersecting every $\mathcal{F}$-set nontrivially. Families with covering number 1 are precisely the Erdős-Ko-Rado families. Hilton-Milner families have covering number 2. The dual EKR and HM families both have covering number k.

Proof. Since every $\mathcal{F}$-set is a covering subspace of dimension k, we have $\tau(\mathcal{F}) \leq k$. Let $\tau(\mathcal{F}) = \ell$, and let L be a covering $[\ell]$. For any subspace

M we denote by $\mathcal{F}_M$ the collection of $\mathcal{F}$-sets containing M. Let f_i denote the maximum cardinality of $\mathcal{F}_M$ over all i-dimensional M, so that $f_0 = |\mathcal{F}|$. Since L meets every $\mathcal{F}$-set, some 1-space L_1 (in L) is contained in $|\mathcal{F}|/\left[{\ell \atop 1}\right]$ (or more) $\mathcal{F}$-sets, so $f_1 \geq f_0/\left[{\ell \atop 1}\right]$. If $\ell > 1$ then there is an $\mathcal{F}$-set disjoint from L_1, and we find a 2-space L_2 containing L_1 and a point of this $\mathcal{F}$-set contained in $|\mathcal{F}|/(\left[{\ell \atop 1}\right]\left[{k \atop 1}\right])$ sets. Continuing like this we find for every $i \leq \ell$ that $f_i \geq f_0/(\left[{\ell \atop 1}\right]\left[{k \atop 1}\right]^{i-1})$.

We first consider the case $\tau(\mathcal{F}) = k$. As above we find a $[k-1]$, say $M = L_{k-1}$ with $|\mathcal{F}_M| = f_{k-1} > \left[{k-1 \atop 1}\right]$. Take an $\mathcal{F}$-set not meeting M. Then F and M generate a hyperplane H and the elements of $\mathcal{F}_M$ are all contained in H and since there are more than $\left[{k-1 \atop 1}\right]$ they generate H. Therefore H contains all $\mathcal{F}$-sets not meeting M. One possibility is that $\mathcal{F}$ is contained in the dual Erdős-Ko-Rado-family $\left[{H \atop k}\right]$. If this is not the case then there is an $F_1 \in \mathcal{F}$ such that $F_1 \cap H = M_1$ is $(k-1)$-dimensional. As F and F_1 intersect we have that $M_1 \setminus M$ is non-empty and all $\mathcal{F}$-sets not meeting M must meet M_1. Now $\dim(M_1) < \tau(\mathcal{F})$, so there is an $\mathcal{F}$-set F_2 disjoint from it. This F_2 can not be contained in H, because then F_1 and F_2 would be disjoint. So we find M_2 in H disjoint from M_1 and all elements of $\mathcal{F}$ not meeting M must also meet M_2. Hence the number of $\mathcal{F}$-sets not meeting M is at most $f := \left[{k-1 \atop 1}\right]^2 f_2$. It follows that the number of $\mathcal{F}$-sets meeting M is at least $f_0 - f$, hence some point of M is contained in a lot of them, so $f_1 \geq (f_0 - f)/\left[{k-1 \atop 1}\right]$. From $f_2 \geq f_1/\left[{k \atop 1}\right]$ we now get:

$$f_2\left(1 + \frac{q^{k-1}-1}{q^k-1}\right) \geq f_0 \Big/ \left(\begin{bmatrix} k-1 \\ 1 \end{bmatrix}\begin{bmatrix} k \\ 1 \end{bmatrix}\right),$$

from which it follows that $f_k > 1$, which is a contradiction.

We conclude that $\tau(\mathcal{F}) = \ell < k$. As before we find an $[\ell-1]$, say again $M = L_{\ell-1}$ with $|\mathcal{F}_M| > f_0/(\left[{\ell \atop 1}\right]\left[{k \atop 1}\right]^{\ell-2})$. Since $\dim(M) < \tau(\mathcal{F})$, we find as before an $\mathcal{F}$-set F disjoint from M. For every point P in this F, either $\langle M, P\rangle$ is a covering space for $\mathcal{F}$, or it is contained in at most $\left[{k \atop 1}\right]\left[{2k-\ell-1 \atop k-\ell-1}\right]$ $\mathcal{F}$-sets. Let N be the number of covering subspaces among the $\langle M, P\rangle$, then

$$\begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} 2k-\ell-1 \\ k-\ell-1 \end{bmatrix} + N\left(\begin{bmatrix} 2k-\ell \\ k-\ell \end{bmatrix} - \begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} 2k-\ell-1 \\ k-\ell-1 \end{bmatrix}\right) \geq f_0 \Big/ \left(\begin{bmatrix} \ell \\ 1 \end{bmatrix}\begin{bmatrix} k \\ 1 \end{bmatrix}^{\ell-2}\right).$$

We'll show that this implies that $N > \left[{k-2 \atop 1}\right]$, so that we can find at least $k-1$ M-independent points P determining a covering subspace, so that there are at most $q^{(\ell-1)(k-1)}\left[{k+1 \atop 1}\right]$ $\mathcal{F}$-sets not meeting M. We first note that for $\ell \leq k-1$:

$$\begin{bmatrix}2k-\ell\\ k-\ell\end{bmatrix} = \frac{q^{2k-\ell}-1}{q^{k-\ell}-1}\begin{bmatrix}2k-\ell-1\\ k-\ell-1\end{bmatrix} \leq \begin{bmatrix}k+1\\ 1\end{bmatrix}\begin{bmatrix}2k-\ell-1\\ k-\ell-1\end{bmatrix}.$$

Now using $\left[{k+1 \atop 1}\right] - \left[{k \atop 1}\right] = q^k$ we conclude that N satisfies:

$$h(\ell) := \left(\begin{bmatrix}k\\ 1\end{bmatrix}^2 + q^k\, N\right)\begin{bmatrix}2k-\ell-1\\ k-\ell-1\end{bmatrix}\begin{bmatrix}\ell\\ 1\end{bmatrix}\begin{bmatrix}k\\ 1\end{bmatrix}^{\ell-2} \geq f_0.$$

Now for fixed N and $\ell \leq k-1$ the left hand side is maximal if $\ell = k-1$. To see this we show that $h(\ell+1)/h(\ell) \geq 1$ if $1 \leq \ell \leq k-2$:

$$\frac{h(\ell+1)}{h(\ell)} = \frac{(q^{k-\ell-1}-1)(q^{\ell+1}-1)(q^k-1)}{(q^{2k-\ell-1}-1)(q^\ell-1)(q-1)} > 1.$$

So, since $\ell \leq k-1$ we have:

$$\left(\begin{bmatrix}k\\ 1\end{bmatrix}^2 + q^k\, N\right)\begin{bmatrix}k-1\\ 1\end{bmatrix}\begin{bmatrix}k\\ 1\end{bmatrix}^{k-3} > (1+\frac{1}{q})\begin{bmatrix}k\\ 1\end{bmatrix}^{k-1}\begin{bmatrix}k-1\\ 1\end{bmatrix}.$$

From this we quickly get that $N > \left[{k-2 \atop 1}\right]$ and there are at most $q^{(\ell-1)(k-1)}\left[{k+1 \atop 1}\right]$ $\mathcal{F}$-sets not meeting M.

As before we get new estimates for f_1 and $f_{\ell-1}$:

$$f_1 \geq \frac{f_0 - q^{(\ell-1)(k-1)}\left[{k+1 \atop 1}\right]}{\left[{\ell-1 \atop 1}\right]},$$

and

$$f_{\ell-1} \geq \frac{f_0 - q^{(\ell-1)(k-1)}\left[{k+1 \atop 1}\right]}{\left[{\ell-1 \atop 1}\right]\left[{k \atop 1}\right]^{\ell-2}} > \begin{bmatrix}k\\ 1\end{bmatrix}\begin{bmatrix}2k-\ell\\ k-\ell\end{bmatrix},$$

contradicting the fact that $\tau(\mathcal{F}) = \ell$. The last inequality can be seen as follows: First rewrite it as

$$\begin{bmatrix}k\\ 1\end{bmatrix}^{\ell-1}\left((1+\frac{1}{q})\begin{bmatrix}k-1\\ 1\end{bmatrix}\begin{bmatrix}k\\ 1\end{bmatrix}^{k-\ell} - \begin{bmatrix}\ell-1\\ 1\end{bmatrix}\begin{bmatrix}2k-\ell\\ k-\ell\end{bmatrix}\right) > q^{(\ell-1)(k-1)}\begin{bmatrix}k+1\\ 1\end{bmatrix}.$$

Since $\begin{bmatrix}k\\1\end{bmatrix} > q^{k-1}$ and $\begin{bmatrix}2k-\ell\\k-\ell\end{bmatrix} < \begin{bmatrix}k+1\\1\end{bmatrix}^{k-\ell}$ it suffices to show

$$\begin{bmatrix}k-1\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^{k-\ell} > \begin{bmatrix}\ell-1\\1\end{bmatrix}\begin{bmatrix}k+1\\1\end{bmatrix}^{k-\ell},$$

and

$$\frac{1}{q}\begin{bmatrix}k-1\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^{k-\ell} > \begin{bmatrix}k+1\\1\end{bmatrix}.$$

The last inequality is obvious, except for the case $k = 4$, $\ell = 3$, but also then it is easy to verify. (Recall that we are assuming $k > 3$.) The first inequality is proved by (repeatedly) using that if $a \leq b$, then $\begin{bmatrix}a-1\\1\end{bmatrix}\begin{bmatrix}b+1\\1\end{bmatrix} < \begin{bmatrix}a\\1\end{bmatrix}\begin{bmatrix}b\\1\end{bmatrix}$. $\square$

Corollary 3.2 *Let $k < q \log q - q$ and let $\mathcal{F}$ be a maximal coclique in $qK_{2k:k}$ of size*

$$|\mathcal{F}| > q^{k(k-1)}/2.$$

Then $\mathcal{F}$ is contained in an Erdős-Ko-Rado family or its dual.

Proof. By the theorem it suffices to have $q/2 > (1 + 1/q)(1 - 1/q)^{-k}$ to get the desired conclusion. This is implied by $k < q \log q - q$. Note that $q^{k(k-1)} < \begin{bmatrix}2k-1\\k-1\end{bmatrix}$. $\square$

4 The chromatic number of $qK_{2k:k}$

We conjecture that the chromatic number χ of $qK_{2k:k}$ equals $q^k + q^{k-1}$, for all q and k. This is certainly an upper bound. For example, fix a $(k+1)$-subspace T and a cover of T with points and k-subspaces. A proper coloring of $qK_{2k:k}$ is obtained by taking all families P^* where P is one of the points in this cover, and all families H^* where H is a hyperplane that contains some k-subspace in this cover. If we fix a $(k-1)$-subspace S in T and take s k-subspaces on S, where $1 \leq s \leq q$, and cover the rest with points, then we have $(q+1-s)q^{k-1}$ colors of type P^* and sq^{k-1} colors of type H^* where H does not contain T, and these suffice, so that $\chi \leq (q + 1 - s)q^{k-1} + sq^{k-1} = q^k + q^{k-1}$.

Unfortunately we will only be able to prove that $\chi \geq q^k + q^{k-1}$ when k is not large compared to q.

Theorem 4.1 *If $k < q \log q - q$ then the chromatic number of $qK_{2k:k}$ equals $q^k + q^{k-1}$.*

Proof. Suppose we have colored part of the $[k]$'s using a set $\mathcal{P}$ of point pencils and a set $\mathcal{H}$ of hyperplanes. Suppose $|\mathcal{P}|+|\mathcal{H}| = q^k + q^{k-1} - \varepsilon$ where $\varepsilon > 0$. Let K be a $[k]$ not contained in a hyperplane from $\mathcal{H}$, and containing a unique $P \in \mathcal{P}$. Count $[k]$'s intersecting K in a $[k-1]$ not containing P: $q^{k-1}\left(\begin{bmatrix} k+1 \\ 1 \end{bmatrix} - 1\right)$. A point Q (not in K) is contained in q^{k-1} of them, a hyperplane contains $\begin{bmatrix} k \\ 1 \end{bmatrix}$, a second (or third) hyperplane on the same $[k-1]$ an additional q^{k-1}, so we find that at most

$$(|\mathcal{P}|-1)\, q^{k-1} + |\mathcal{H}| q^{k-1} + q^{k-1}\begin{bmatrix} k-1 \\ 1 \end{bmatrix}$$

$[k]$'s (intersecting K in a $[k-1]$ not containing P) are colored by point or hyperplane colors. From $|\mathcal{P}| + |\mathcal{H}| = q^k + q^{k-1} - \varepsilon$ we get that at least $\varepsilon\, q^{k-1}$ of these $[k]$'s are uncolored. In particular this finishes the proof of the conjecture for $k = 2$, because in that case every coclique is necessarily of point or hyperplane type, so there are no uncolored $[k]$'s. Define a bipartite graph on $A \cup B$, where A is the set of uniquely colored $[k]$'s and B the set of uncolored $[k]$'s, joining them if they intersect in a $[k-1]$. Let $b = |B|$ be the number of uncolored $[k]$'s. The size of a non-EKR coclique is at most $f = f(k,q)$, so we may assume $b < \varepsilon f$. We start with an estimate for $a = |A|$: The total number of colored $[k]$'s equals $\begin{bmatrix} 2k \\ k \end{bmatrix} - b$, we have a multiset of $(q^k + q^{k-1} - \varepsilon)\begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix}$ colored $[k]$'s, so the number of uniquely colored spaces is at least $2(.) - (.)$, therefore:

$$a \geq 2\begin{bmatrix} 2k \\ k \end{bmatrix} - 2b - (q^k + q^{k-1} - \varepsilon)\begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix}.$$

Counting in two ways the number of edges in this graph we get

$$\left(2\begin{bmatrix} 2k \\ k \end{bmatrix} - 2b - (q^k + q^{k-1} - \varepsilon)\begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix}\right) \varepsilon\, q^{k-1} \leq |E| \leq b\begin{bmatrix} k \\ k-1 \end{bmatrix}\begin{bmatrix} k+1 \\ 1 \end{bmatrix}.$$

Using our bound for b in terms of f and simplifying the left hand side:

$$(q^k - q^{k-1} + \varepsilon)\begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix} q^{k-1} < f\left(\begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} k+1 \\ 1 \end{bmatrix} + 2\varepsilon\, q^{k-1}\right).$$

Reordering terms we get:

$$(q^k - q^{k-1})\begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix} q^{k-1} - f\begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} k+1 \\ 1 \end{bmatrix} \leq \varepsilon\left(2f - \begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix}\right) q^{k-1}.$$

Since $k < q \log q - q$ we have $f < q^{k(k-1)}/2$ and the left hand side is positive (since $q \geq 5$) while the right hand side is negative, so we get a contradiction. $\square$

5 Using only point and hyperplane cocliques

We show that a minimal coloring of $qK_{2k:k}$ that only uses color classes of type P^* and H^*, must be one of the examples given at the start of the previous section. For $k = 2$ this was shown already in [2].

Proposition 5.1 *Let $\mathcal{P}$ be a set of points and $\mathcal{H}$ a set of hyperplanes such that $\{P^* \mid P \in \mathcal{P}\} \cup \{H^* \mid H \in \mathcal{H}\}$ is a coloring of $qK_{2k:k}$ where $k \geq 2$. Then $|\mathcal{P}| + |\mathcal{H}| \geq q^k + q^{k-1}$. If equality holds, then $\mathcal{P}$ and $\mathcal{H}$ are nonempty, no $H \in \mathcal{H}$ contains a $P \in \mathcal{P}$, and there are a $(k-1)$-space S and a $(k+1)$-space T containing S, such that $\mathcal{P} \subset T \setminus S$ and $\bigcap \mathcal{H} \supseteq S$.*

Proof. Consider a minimal coloring as described. Let us call a point $P \in \mathcal{P}$ a c-point, and a hyperplane $H \in \mathcal{H}$ a c-hyper. By point/hyperplane duality we may assume that there is a c-point. Since every c-point P is needed, there is always at least one uniquely colored $[k]$ containing it. We repeat the counting argument from the previous section in some more detail. Let K be a uniquely colored $[k]$ with c-point P, and count $[k]$'s intersecting K in a $[k-1]$ not on P. If there are h c-hypers not on P, and m $[k-1]$'s in K, not on P, that are contained in a c-hyper, then

$$q^{k-1}(\begin{bmatrix} k+1 \\ 1 \end{bmatrix} - 1) \leq (|\mathcal{P}| - 1)\, q^{k-1} + hq^{k-1} + m \begin{bmatrix} k-1 \\ 1 \end{bmatrix}$$

and $h \leq |\mathcal{H}|$ and $m \leq q^{k-1}$ so that $|\mathcal{P}| + |\mathcal{H}| \geq q^k + q^{k-1}$. Suppose equality holds. Then $\mathcal{H} \neq \emptyset$, and $h = |\mathcal{H}|$ so that no c-hyper contains P.

We see: If a $[k-1]$, say S, is in a uniquely colored $[k]$, with c-point $P \notin S$, then all $[k]$'s containing S and a c-point Q are uniquely colored. There is at least one c-hyper that contains S, and all c-hypers on S contain a fixed (as soon as there are at least two) $[2k-2]$, say M. If there are s c-hypers on S, then $|\mathcal{P}| = (q + 1 - s)q^{k-1}$, so that s is independent of the choice of S.

In this situation, consider the hyperplane H_0 on M containing P. Then $H_0 \notin \mathcal{H}$, and every $[k]$ on S contained in H_0 but not in M contains a (unique) point of $\mathcal{P}$. So $\mathcal{P}$ has exactly q^{k-1} points in $H_0 \setminus M$. If S' is a $[k-1]$ in M, and P' a c-point in $H_0 \setminus M$ such that $\langle S', P' \rangle$ is uniquely colored, then also S' together with any of the q^{k-1} c-points in H_0 is uniquely colored. The role played by any of the q^{k-1} c-points in H_0 is the same as that of the original P.

Now consider T, a $[k+1]$ contained in H_0 and containing $K = \langle S, P \rangle$. All $[k]$'s containing a $[k-1]$ in K not containing P contain at most one c-point, hence the c-points in T are on a line through P. The number of choices for T is $\begin{bmatrix} k-1 \\ 1 \end{bmatrix}$, and each line on P has at most $q-1$ c-points distinct from P since its intersection with M is not a c-point, so that H_0 contains at most $(q-1)\begin{bmatrix} k-1 \\ 1 \end{bmatrix} + 1 = q^{k-1}$ c-points. Since equality holds, every line in H_0 that contains at least two c-points, contains precisely q c-points. Since P plays the same role as the other c-points in H_0, it follows that every line joining two c-points in H_0 contains exactly q, and the c-points in H_0 form the affine part of a $(k-1)$-dimensional projective space A, intersecting M in a $[k-1]$, say D. Let $H \in \mathcal{H}$. Since no c-hyper contains a c-point, we have $H \cap A = D$, so that $\bigcap \mathcal{H} \supseteq D$. Dually, there is a $[k+1]$-space E containing all c-points, and then of course also D. $\square$

6 The case $k = 3$

The case $k = 3$ is a bit like the general case, but has to be investigated separately. Here it is possible to examine the situation in much more detail and describe all large cocliques. We'll conclude later that the chromatic number of $qK_{6:3}$ is $q^3 + q^2$.

Theorem 6.1 *Let $V = V(6, q)$ be a 6-dimensional vector space over $GF(q)$. Let $\mathcal{F}$ be a maximal intersecting family of planes in V. Then we have one of the following four cases:*

(i) $|\mathcal{F}| = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1$, and $\mathcal{F}$ is either the collection P^* of all planes on a fixed point P, or the collection H^* of all planes in a fixed hyperplane H of V.

(ii) $|\mathcal{F}| = 1 + q(q^2 + q + 1)^2 = q^5 + 2q^4 + 3q^3 + 2q^2 + q + 1$, and $\mathcal{F}$ is either the collection π^* of all planes that meet a fixed plane π in at least a line, or the collection $(P, S)^*$ of all planes that are either contained in the solid S, or contain the point P and meet S in at least a line (where $P \subset S$), or $\mathcal{F}$ is the collection $(L, H)^*$ of all planes that either contain the line L, or are contained in the hyperplane H and meet L (where $L \subset H$).

(iii) $|\mathcal{F}| = 3q^4 + 3q^3 + 2q^2 + q + 1$ *and* $\mathcal{F}$ *is the collection* $(P, \pi, H)^*$ *of all planes on* P *that meet* π *in a line, and all planes in* H *that meet* π *in a line, and all planes on* P *in* H *(where* $P \subset \pi \subset H$*).*

(iv) $\mathcal{F}$ *is smaller.*

Proof.

The six examples P^*, H^*, π^*, $(P, S)^*$, $(L, H)^*$, $(P, \pi, H)^*$ are indeed maximal collections of mutually intersecting planes with the stated sizes. Assume that we have none of these. We show that $|\mathcal{F}| < 3q^4 + 3q^3 + 2q^2 + q + 1$.

The planes in $\mathcal{F}$ will be called $\mathcal{F}$-planes. The $\mathcal{F}$-planes on a line L form a subspace in the local space: if an $\mathcal{F}$-plane intersects two planes π_1 and π_2 on L it intersects the $q + 1$ planes containing L and contained in $\langle \pi_1, \pi_2 \rangle$. We say the line is red, orange, yellow or white if this subspace is all of V ($q^3 + q^2 + q + 1$ planes), or a hyperplane ($q^2 + q + 1$ planes), or a solid ($q + 1$ planes), or a single plane.

Note that a red line intersects all $\mathcal{F}$-planes, and conversely, a line intersecting all $\mathcal{F}$-planes is red. Two red lines necessarily intersect, and the red lines on a point form a subspace in the local space.

Case A. *At least two red lines.*

Suppose we do not have P^*.

If there is a triangle of red lines, this triangle spans a plane π and each plane in $\mathcal{F}$ meets π in at least a line, so we have π^*.

If there is a tripod (concurrent, not coplanar) of red lines, then we have $(P, S)^*$.

If all red lines pass through a point P and are contained in a plane π then we have $(P, \pi, H)^*$. Indeed, since we do not have π^* there is an $\mathcal{F}$-plane π' that meets π in P only. Put $H = \pi + \pi'$. Every $\mathcal{F}$-plane not on P meets π in a line, and also meets π' so lies in H. Every $\mathcal{F}$-plane not in H meets H in a line on P. This line must meet all $\mathcal{F}$-planes, hence is red. This shows that we have $(P, \pi, H)^*$.

So, if we do not have P^*, π^*, $(P, S)^*$, or $(P, \pi, H)^*$, there is at most one red line.

Case B. *Precisely one red line.*

Now consider the case of precisely one red line, L. The family $\mathcal{F}$ contains all $q^3 + q^2 + q + 1$ planes on L. Let $\mathcal{F}'$ be the collection of planes in $\mathcal{F}$ not containing L. Our aim is to show that $|\mathcal{F}'| < 3q^4 + 2q^3 + q^2$ unless we have $(L, H)^*$. In fact we show $|\mathcal{F}'| \leq 2q^4 + 3q^3 + q^2$.

Suppose that $|\mathcal{F}'| > 2q^4+3q^3+q^2$. There are at most $(q^2+q)^2$ $\mathcal{F}'$-planes on any given point P on L. If all orange lines meeting L do meet it in the same point P, then there are not more than $q\cdot q\cdot(q^2+q)$ $\mathcal{F}'$-planes not on P, a contradiction. If all orange lines meeting L are contained in the same plane π_0 on L, then $|\mathcal{F}'| \le (q+1)(q(q^2+q)+q^2\cdot q) = 2q^4+3q^3+q^2$, contradiction again. So, there exist two disjoint orange lines meeting L.

The hyperplanes belonging to disjoint orange lines coincide.

Let S be the solid spanned by two disjoint orange lines M, N meeting L in the points P and Q, respectively. If the line K intersects both orange lines, then we already see two planes on K, but then the whole pencil of planes on K in S is there. So all planes in S are in $\mathcal{F}$. Now any other plane will hit S in a line, which is then automatically orange (unless it was L), moreover, this line must intersect L, and if this happens in a point different from P and Q, then it will be disjoint from one of the two original orange lines, and hence it will determine the same hyperplane.

So now the situation is such that we have a pair of points P, Q contained in a line L contained in a solid S contained in a hyperplane H and $\mathcal{F}$ consists of all planes on L, a set of planes contained in H and intersecting L, and maybe some exceptional planes, that are not contained in H and intersect L in P or in Q. Such an exceptional plane on P meets H in an orange line M' contained in S. Since M' is not red, there is a disjoint $\mathcal{F}$-plane, necessarily an exceptional plane on Q that meets H in an orange line N' contained in S. Now M' and N' determine the same hyperplane H' and $H' \neq H$. If K is an orange line meeting L in a point different from P and Q, then K must determine both H and H', contradiction. So all orange lines pass through P or Q and lie in two planes. Every plane not in S is exceptional w.r.t. H or H', hence meets S in an orange line on P or Q, and we find $|\mathcal{F}'| \le (q^3+q^2)+2q(q^2+q)$, contradiction.

Hence we have $(L,H)^*$.

That finishes the case of one red line. Now we may assume that there are no red lines, and dually that for every solid there is an $\mathcal{F}$-plane meeting it in a single point only.

Case C. *An orange line but no red lines or red solids.*

(A red solid is the dual of a red line, a solid S such that $\langle S, \pi\rangle$ is contained in a hyperplane, i.e., such that $S\cap\pi$ contains a line, for every $\mathcal{F}$-plane π. We assume that there is no red solid, so for each solid S there is a plane $\pi \in \mathcal{F}$ meeting it in a single point only.)

Suppose there is an orange line L. Since L is not red, there is a disjoint $\mathcal{F}$-plane π_0. Let $H = \langle L, \pi_0 \rangle$, then H is a hyperplane containing every plane disjoint from L, i.e., every $\pi \in \mathcal{F}$ not in H meets H in a line intersecting L.

If L' is another orange line, disjoint from L, then L' determines the same hyperplane (because the q^2 planes on L' disjoint from L are in the hyperplane of L). Let $S = \langle L, L' \rangle$. Every $\pi \in \mathcal{F}$ not in H meets both L and L' and hence meets S in a line. And every $\pi \subset H$ meets S in at least a line. So S is a red solid, contrary to assumption. Hence, no two orange lines are disjoint.

We do not have H^*, so there is a plane $\pi_1 \in \mathcal{F}$ intersecting H in a line M_1. Such a line M_1 intersects all $\mathcal{F}$-planes contained in H, and hence meets L and all $\mathcal{F}$-planes disjoint from L. Take π_2 disjoint from M_1, then also π_2 intersects H in a line, M_2.

Every $\mathcal{F}$-plane disjoint from L is contained in H and meets both M_1 and M_2. A line intersecting M_1 and M_2 but not L is not orange (since orange lines are not disjoint), and if it is yellow and its solid is in H then its solid intersects L. It follows that the number of $\mathcal{F}$-planes disjoint from L is at most q^3.

If L is the only orange line then on every point of L there are at most $(q+1)(q^2+q)$ planes not containing L, so $|\mathcal{F}| \leq (q^2+q+1) + (q+1)^2(q^2+q) + q^3 = q^4 + 4q^3 + 4q^2 + 2q + 1$.

If there are more orange lines, all going through the same point P, then we have $|\mathcal{F}| \leq 2q^3 + q^2(q+1) + (q^2+q+1)^2 = q^4 + 5q^3 + 4q^2 + 2q + 1$ (consider two intersecting orange lines L and L' and count planes disjoint from L, planes disjoint from L', planes not containing P and intersecting both L and L', and planes containing P).

If not all orange lines go through a point, then they are contained in a plane π. Let π contain $q^2+q+1-k$ orange lines, then the number of $\mathcal{F}$-planes intersecting π in at least a line is at most $(q^2+q+1-k)(q^2+q+1)+k(q+1)$ and if the number of planes intersecting π in at most a point is N, then on the one hand $N \leq 3q^3$ because there are three lines forming a triangle, and a plane intersecting π in a single point is disjoint from at least one of them. On the other hand, counting pairs (orange line, disjoint plane) we have $N(q^2-k) \leq (q^2+q+1-k)q^3$. It follows that $N - kq^2 \leq (q^2+q+1)q$ (if $k \geq 2q-1$ then $N - kq^2 \leq 3q^3 - kq^2 \leq (q^2+q)q$, and if $k \leq 2q-2$ then $q^2 - k > 0$ and $N - kq^2 \leq q^3 + \frac{(q+1)q^3}{q^2-k} - kq^2 \leq (q^2+q+1)q$), so that $|\mathcal{F}| \leq (q^2+q+1)^2 + (q^2+q+1)q = q^4 + 3q^3 + 4q^2 + 3q + 1$.

Case D. *No orange lines or solids.*

(An orange (yellow) solid is the dual of an orange (yellow) line, a solid S containing $q^2 + q + 1$ (resp. $q + 1$) $\mathcal{F}$-planes.)

Yellow lines and yellow solids are paired: each yellow line L is on $q + 1$ $\mathcal{F}$-planes of which the union is a yellow solid S_L, and conversely each yellow solid contains $q + 1$ $\mathcal{F}$-planes of which the intersection is a yellow line L_S.

If the lines L and M are yellow, and L is disjoint from the solid S_M, then $M \subset S_L$ (for otherwise some $\mathcal{F}$-plane on L is disjoint from M, hence must have a line in S_M, and L would meet S_M). Write $L \to M$ if M is contained in S_L and disjoint from L, that is, if L is disjoint from S_M.

If $L \to M_1$ and $L \to M_2$ then M_1 and M_2 intersect, because otherwise they are both outside the others solid. So the number of yellow M disjoint from L inside S_L is at most q^2.

Every $\mathcal{F}$-plane disjoint from L intersects S_L in a line, and there are q^4 lines in S_L disjoint from L, at most q^2 of which are yellow, so there are at most $q^4 + q^3$ $\mathcal{F}$-planes disjoint from L.

Suppose there are disjoint yellow lines L and M. The number of $\mathcal{F}$-planes disjoint from at least one of them is at most $2q^4+2q^3$, the number intersecting both at most $(q + 1)^3$, so $|\mathcal{F}| \le 2q^4 + 3q^3 + 3q^2 + 3q + 1$.

If every two yellow lines intersect then fix a yellow line L and a disjoint plane π. There are at most q^4 $\mathcal{F}$-planes disjoint from L, $q + 1$ contain L, at most q^2+q+1 yellow lines intersect both L and π, so the number of $\mathcal{F}$-planes intersecting L in a point is at most $(q+1)(q^2+q+1)+(q^2+q+1)q$. Hence $|\mathcal{F}| \le q^4 + 2q^3 + 3q^2 + 4q + 2$.

Case E. *Only white lines*

A point is now on at most q^2+q+1 planes, so $|\mathcal{F}| \le 1+(q^2+q+1)(q^2+q) = q^4 + 2q^3 + 2q^2 + q + 1$. It is of independent interest to determine a (much) better bound in this case. □

7 Chromatic number in case $k = 3$

Theorem 4.1 above shows that the chromatic number of $qK_{6:3}$ equals $q^3 + q^2$ for $q \ge 5$. In fact the restriction on q is superfluous.

Theorem 7.1 *The chromatic number of the graph $qK_{6:3}$ equals $q^3 + q^2$.*

Proof. Let us redo the earlier proof and use slightly sharper estimates. The main improvement is that in a coloring with fewer than $q^k + q^{k-1}$ colors the number of bad colors is less than ε, and hence at most $\varepsilon - 1$. That means that the estimate $b < \varepsilon f$ can be replaced by $b \le (\varepsilon - 1)f$. This yields

$$(q^k - q^{k-1} + 2 + \varepsilon)\begin{bmatrix}2k-1\\k-1\end{bmatrix} q^{k-1} < f\left(\begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}k+1\\1\end{bmatrix} + 2\varepsilon\, q^{k-1}\right)(1 - \frac{1}{\varepsilon}).$$

Using $f = 1 + q(q^2 + q + 1)^2$ we find for $k = 3$, $q = 4$:

$$(50 + \varepsilon).5797.16 < 1765.(21.85 + 32\varepsilon)(1 - \frac{1}{\varepsilon})$$

a contradiction. And for $k = 3$, $q = 3$:

$$(20 + \varepsilon).1210.9 < 508.(13.40 + 18\varepsilon)(1 - \frac{1}{\varepsilon})$$

that is, $1746\varepsilon < 37216 - 264160/\varepsilon$, again a contradiction.

Finally, suppose $q = 2$. Our aim is to show that the chromatic number is 12, so suppose that the 1395 planes in V can be partitioned into 11 intersecting families. Extend the families to maximal intersecting families. Each family of type P^* or H^* covers 155 planes, while each second such family meets the first in 15 planes. Every family of different type covers at most 99 planes. Since $2.155 + 4.140 + 5.99 = 1365$, there can be at most 4 'bad' families. If no plane is in more than three families of type P^* or H^*, then 11 families cover at most $2.(155 + 140 + 126 + 113 + 101) + 99 = 1369$ planes, not enough. So some plane π_0 is in at least 4 families of type P^* or H^*. The 512 planes disjoint from π_0 are colored with at most 7 colors. But a family of type P^* or H^* contains (either 0 or) 64 planes disjoint from π_0, a family of type π^* contains at most 56 planes disjoint from π_0, a family of type $(P, S)^*$ or $(L, H)^*$ contains at most 48 planes disjoint from π_0, while any other family contains at most 83 planes. Since $4.64 + 3.83 < 512$ we need at least 4, and hence precisely 4, of these other other families. But $2.155 + 5.140 + 4.83 = 1342$, which is not enough. This shows that for $q = 2$ the graph $qK_{6:3}$ has chromatic number 12. $\square$

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