The eigenvalues of oppositeness graphs in buildings of spherical type

Andries E. Brouwer

Department of Mathematics, Technological University Eindhoven,
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

2009-06-17

To Reza Khosrovshahi on the occasion of his 70th birthday

Abstract

Consider the graph Γ obtained by taking as vertices the flags in a finite building of spherical type defined over $\mathbb{F}_q$, where two flags are adjacent when they are opposite. We show that the squares of the eigenvalues of Γ are powers of q.

1 Introduction

Let G be a finite group of Lie type with Borel subgroup B and Weyl group W, so that one has the Bruhat decomposition $G = \bigcup_w BwB$. Let (W, S) be a Coxeter system, and let w_0 be the longest element of W w.r.t. the set of generators S. Then conjugation by w_0 induces a diagram automorphism on the Coxeter diagram of W (with vertex set S).

Let a *type* be a nonempty subset of S. Call two types J, K *opposite* when $K = J^{w_0}$ (so that $J = K^{w_0}$).

For $J \subset S$, let $W_J := \langle J \rangle$ and $P_J := BW_JB$. Let an *object of type* $S \setminus J$, or *of cotype* J, be a coset gP_J in G. Call two objects gP_J and hP_K *opposite* when their cotypes J, K are opposite, and moreover $P_K h^{-1} g P_J = P_K w_0 P_J$.

Let $\Gamma_{J,K}$, with $K = J^{w_0}$, be the bipartite graph with as vertices in one part the objects of cotype J and in the other part the objects of cotype K, where two vertices in different parts are adjacent when they are opposite. If $J = K$, let Γ_J be the graph with as vertices the objects of cotype J, adjacent when opposite.

Theorem 1.1 *Let G be defined over $\mathbb{F}_q$. Let J be a proper subset of S, and let $K = J^{w_0}$. Let θ be an eigenvalue of $\Gamma_{J,K}$ or, if $J = K$, of Γ_J. Then $\theta^2 = q^e$ for some integer e.*

The exponents e can be determined explicitly.

2 Examples

We give diagrams, with the nodes in the type (outside the cotype) circled, so that at least one node is circled. The action of w_0 on the diagram is the identity everywhere, except in the cases A_n ($n > 1$), D_n (n odd), and E_6, where w_0 induces the unique diagram automorphism of order 2.

2.1 The projective line

Consider the diagram A_1: $\odot$. The geometry is the projective line, with $q+1$ points. The graph Γ on these points, adjacent when distinct, is the complete graph K_{q+1}, with eigenvalues q and -1.

2.2 The projective plane

Consider the diagram A_2, with $J = \odot\!-\!\bullet$ and $K = \bullet\!-\!\odot$. The graph Γ is the bipartite point-line nonincidence graph of the projective plane $PG(2,q)$. It has eigenvalues $\pm q^2$, $\pm\sqrt{q}$.

Consider the diagram A_2, with $J = K = \odot\!-\!\odot$. The graph Γ is the graph on the flags of $PG(2,q)$, adjacent when in general position. It has eigenvalues q^3, $\pm q\sqrt{q}$, -1.

2.3 Projective 3-space

Consider the diagram A_3.

(i) $J = K = \bullet\!-\!\odot\!-\!\bullet$. The graph Γ is the graph on the lines of $PG(3,q)$, adjacent when skew. It has eigenvalues q^4, $-q^2$, q.

(ii) $J = K = \odot\!-\!\bullet\!-\!\odot$. The graph Γ is the graph on the point-plane flags of $PG(3,q)$, adjacent when in general position. It has eigenvalues q^5, $\pm q^3$, q^2, $-q$.

(iii) $J = K = \odot\!-\!\odot\!-\!\odot$. The graph Γ is the graph on the chambers (point-line-plane flags) of $PG(3,q)$, adjacent when in general position. It has eigenvalues q^6, $\pm q^4$, q^3, $\pm q^2$, 1.

(iv) $J = \odot\!-\!\bullet\!-\!\bullet$ and $K = \bullet\!-\!\bullet\!-\!\odot$. The graph Γ is the bipartite nonincidence graph on the points and planes of $PG(3,q)$. It has eigenvalues $\pm q^3$, $\pm q$.

(v) $J = \odot\!-\!\odot\!-\!\bullet$ and $K = \bullet\!-\!\odot\!-\!\odot$. The graph Γ is the bipartite graph on the point-line and line-plane flags of $PG(3,q)$, adjacent when in general position. It has eigenvalues $\pm q^5$, $\pm q^3$, $\pm q^2$, $\pm q$.

2.4 Projective space

Consider the diagram A_n.

(i) $J = K = \odot\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\odot$. The graph Γ is the graph on the point-hyperplane flags of $PG(n,q)$, adjacent when in general position. If $n > 2$, it has eigenvalues q^{2n-1}, $\pm q^{3(n-1)/2}$, q^{n-1}, $-q^{n-2}$.

(ii) $J = K = \bullet\!-\!\bullet\!-\!\bullet\!-\!\odot\!-\!\bullet\!-\!\bullet\!-\!\bullet$. For $n = 2d-1$ we can pick the middle node. Now the graph Γ is the graph on the d-spaces in a $2d$-space, adjacent when disjoint. (Here an i-space is a $PG(i-1,q)$.) This graph has eigenvalues $(-1)^i q^{d^2-di+i(i-1)/2}$ ($0 \le i \le d$).

Here, and in several other cases, there is a distance-regular graph Δ of diameter d, and our graph Γ is the distance-d graph of Δ. (That is, the adjacency matrices of Δ and Γ are the matrices A_1 and A_d, respectively.) Now A_i has the same eigenvalues as L_i, where L_i is the matrix of order $d+1$ defined by

$(L_i)_{kj} = p^k_{ij}$. In particular, A_d has the same eigenvalues as L_d. Now L_d is lower-right triangular (indeed, $p^k_{dj} = 0$ for $j + k < d$ by the triangle inequality), so the product of the eigenvalues of Γ equals $\det L_d = (-1)^{d(d-1)/2} \prod_i p^i_{d,d-i}$. It follows that here the $p^i_{d,d-i}$ must be powers of q. In this particular case we have $p^i_{d,d-i} = q^{d^2-i^2}$.

Formulas for the eigenvalues of metric and cometric distance-regular graphs are given in [1], 8.3.3 and 8.4.2. As a special case one gets the eigenvalues for the graphs on the m-spaces in an n-space, adjacent when they have an $(m-1)$-space in common. Eigenvalues for other relations can be computed from these. See also Eisfeld [4].

2.5 Generalized quadrangles

Here the two generating reflections of W are not conjugate, and two prime powers are involved.

: The non-collinearity graph on the points of $GQ(s,t)$ has eigenvalues s^2t, t, $-s$.

: The graph on the flags of $GQ(s,t)$, adjacent when in general position, has eigenvalues s^2t^2, s^2, t^2, 1, $-st$.

2.6 Generalized hexagons

: The collinearity graph of a generalized hexagon $GH(s,t)$ is distance-regular of diameter 3. The distance-3 graph on the points has eigenvalues s^3t^2, $\pm s\sqrt{st}$, $-t^2$. (The flag graph of $PG(2,q)$ is the case $(s,t) = (q,1)$.)

The P-matrix is

$$P = \begin{pmatrix} 1 & s(t+1) & s^2t(t+1) & s^3t^2 \\ 1 & s-1+\sqrt{st} & -s+(s-1)\sqrt{st} & -s\sqrt{st} \\ 1 & s-1-\sqrt{st} & -s-(s-1)\sqrt{st} & s\sqrt{st} \\ 1 & -t-1 & t(t+1) & -t^2 \end{pmatrix}.$$

2.7 Generalized octagons

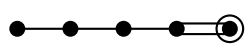
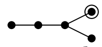
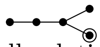
The situation for the generalized octagon $GO(s,t)$ is interesting in that the collinearity graph has five distinct eigenvalues, while the distance-4 graph on the points only has four distint eigenvalues (namely, s^4t^3, s^2t, t^3, $-s^2t$). It follows that A_4 does not generate the Bose-Mesner algebra.

The P-matrix is

$$P = \begin{pmatrix} 1 & s(t+1) & s^2t(t+1) & s^3t^2(t+1) & s^4t^3 \\ 1 & s-1+\sqrt{2st} & st-s+(s-1)\sqrt{2st} & s^2t-st-s\sqrt{2st} & -s^2t \\ 1 & s-1 & -st-s & -s^2t+st & s^2t \\ 1 & s-1-\sqrt{2st} & st-s-(s-1)\sqrt{2st} & s^2t-st+s\sqrt{2st} & -s^2t \\ 1 & -t-1 & t(t+1) & -t^2(t+1) & t^3 \end{pmatrix}.$$

2.8 Polar spaces and dual polar spaces

: The noncollinearity graph of a polar space has eigenvalues q^{2d+e-2}, q^{d+e-2}, $-q^{d-1}$, with d, e as in [1] (9.4.1), so that the corresponding dual polar space has diameter d, and the final double stroke corresponds to a generalized quadrangle $GQ(q, q^e)$.

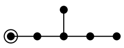
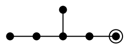
 : The graph on the maximal totally isotropic subspaces in a polar space, adjacent when disjoint, has eigenvalues $(-1)^i q^{d(d-1)/2+de-i(d+e-i)}$ $(0 \leq i \leq d)$ with e as above. Here $p^i_{d,d-i} = q^{(d-i)(d+i+2e-1)/2}$. If $e = 0$ (the case of D_d, that is, O^+_{2d}) then for even d it is the disjoint union of two copies of the oppositeness graph for the half dual polar space  , while for odd d this graph is the bipartite graph found for $J =$  , $K =$  .

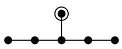
More generally, Eisfeld [4] determined the eigenvalues for all relations between subspaces of (vector space) dimension m, $1 \leq m \leq d$. Vanhove [6] evaluated Eisfeld's formulas for the oppositeness relation (where m-spaces A and B are opposite when $A^\perp \cap B = 0$) and found that the eigenvalues are

$$(-1)^{i+j} q^{m(4d-3m-1)/2+e(m+j-i)-i(d-i)-j(i+1-j)}$$

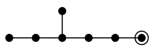
where $0 \leq i \leq m$ en $0 \leq j \leq \min(i, d-m)$.

2.9 E_6

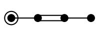
(i) $J =$  and $K =$  : Eigenvalues are $\pm q^{16}$, $\pm q^{10}$, $\pm q^7$.

(ii) $J = K =$  : Eigenvalues are q^{21}, q^{12}, $\pm q^9$, $-q^{15}$.

2.10 E_7

 : Eigenvalues are q^{27}, $-q^{18}$, q^{13}, $-q^{12}$.

2.11 F_4

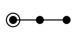
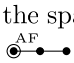
 : Eigenvalues are q^{15}, $\pm q^9$, q^7, $-q^6$.

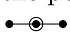
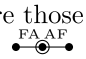
2.12 Disconnected diagrams

If the diagram is disconnected, the graph is the tensor product of the graphs for the components, and the eigenvalues are the products of the eigenvalues of the graphs for the components.

3 Affine space

It is possible to define affine space on objects of type J as the space induced by the set of all such objects opposite to a fixed object of type J^{w_0}. (See also [2].)

For example, inside $PG(d, q)$ affine space on the points is the space of which the points are those not on a fixed hyperplane:  gives 

. Similarly,  gives 

. In the case of A_n such spaces are sometimes called affine Grassmannians.

We shall need that such an affine space has a size that is a power of q. (This is a particular case of the theorem, since this size is the valency of a graph $\Gamma_{J,K}$, and hence one of the eigenvalues.)

Proposition 3.1 *Let J and K be opposite types. The size of the affine space of objects of cotype J, i.e., the number of objects of cotype J opposite to a fixed object of cotype K, is q^a, where the integer a equals the length of the longest word in W minus the length of the longest word in W_J.*

Proof: Two objects gP_J and hP_K are opposite when $P_K h^{-1} g P_J = P_K w_0 P_J$. Thus, the claim is that $|P_K w_0 P_J / P_J| = q^a$.

Let $w_0 = u_0 w_1$ be a reduced expression, with $u_0 \in W_K$ and w_1 left K-reduced. Then $P_K w_0 B = P_K w_1 B$.

Now $P_K = \bigcup_u BuB$, where the union is over $u \in W_K$, and $BuBw_1P_J = Buw_1P_J = Buu_0^{-1}w_0P_J = Bw_0(uu_0^{-1})^{w_0}P_J = Bw_0P_J$ for all $u \in W_K$ since $K^{w_0} = J$. It follows that $P_K w_0 P_J = B w_0 P_J$.

Let $w_0 = wv$ be a reduced expression, with $v \in W_J$ and w right J-reduced. Then v is the longest word in W_J and $l(w) = l(w_0) - l(v) = a$. Now $Bw_0P_J = BwP_J$, and $|BwP_J/P_J| = |U_w^-| = q^{l(w)} = q^a$, as desired. □

For example, in the above two examples $l(w_0) = 6$, and the lengths of the longest word of W_J are 3 and 2, so that in $PG(3,q)$ there are q^3 points outside a given hyperplane, and q^4 lines skew to a given line.

Corollary 3.2 *Let gP_J and hP_K be opposite. Then the number of cosets xB contained in gP_J and opposite to hB equals q^b, where b is the length of the longest word in W_J.* □

Remark All valencies in an association scheme on the cosets of a parabolic in a group of Lie type are of the form

$$k_i = \sum_{w \in S_i} q^{l(w)}$$

for some $S_i \subseteq W$ that has a unique shortest element. It follows that k_i will be a power of q if and only if $|S_i| = 1$, i.e., if and only if $k_i = 1$ in the thin case.

In the cases considered, the thin graphs are ladder graphs: all components are K_2 with eigenvalues ± 1.

3.1 Weights

For the case of twisted Chevalley groups, the Coxeter group is a subgroup of the Coxeter group for the corresponding non-twisted case, and the length function $l(w)$ used here (in $|U_w^-| = q^{l(w)}$) is that of the untwisted group, cf. [1], §10.7. For example, A_n has $l(w_0) = \frac{1}{2}n(n+1)$ with n generators of length 1. And ${}^2A_n$ has the same $l(w_0)$, and $\lceil n/2 \rceil$ generators: all but one of length 2, and if n is odd one of length 1, and if n is even one of length 3. Consequently, the noncollinearity graph of the polar space ${}^2A_n(q^2)$ (that is $U_{n+1}(q)$) has valency q^{2n-1}, where $2n-1 = \frac{1}{2}n(n+1) - \frac{1}{2}(n-2)(n-1)$. And the oppositeness graph of the corresponding dual polar graph has valency q^{d^2} for $n = 2d-1$, and $q^{d(d+2)}$ for $n = 2d$.

More generally, if node s (for $s \in S$) of the diagram has order q^{e_s} (in the Buekenhout sense: a flag of cotype $\{s\}$ is contained in q^{e_s} maximal flags), then the length function $l()$ is weighted, and each generator s has weight e_s. Of course conjugate generators have the same weight.

4 Proof of the theorem

We use $\sim$ for adjacency.

4.1 Bipartite or not

The *bipartite double* $\tilde{\Gamma}$ of a graph Γ with vertices v is the graph with vertices v^+ and v^-, where if $v \sim w$ then $v^+ \sim w^-$ and $v^- \sim w^+$ (cf. [1], 1.11.1). If Γ has spectrum Θ then $\tilde{\Gamma}$ has spectrum $\Theta \cup -\Theta$. Since $\Gamma_{J,J}$ is the bipartite double of Γ_J, it follows that the claims for Γ_J are equivalent to those for $\Gamma_{J,J}$.

4.2 Reduction to the case $J = \emptyset$.

There is a natural map ϕ_J from the set of objects of cotype $\emptyset$ (that is, the set of chambers) to the set of objects of cotype J, given by $gB \mapsto gP_J$. This map is a homomorphism from $\Gamma_\emptyset$ onto Γ_J, and the pair of maps (ϕ_J, ϕ_K) provides a homomorphism from $\Gamma_{\emptyset,\emptyset}$ onto $\Gamma_{J,K}$: the neighbours of gB in $\Gamma_\emptyset$ are the cosets xB contained in gBw_0B, and if $gB \sim hB$ in $\Gamma_\emptyset$, then $gP_J \sim hP_K$ in $\Gamma_{J,K}$.

By Corollary 3.2, the number of cosets xB contained in $gBw_0B \cap hP_K$ is q^b. It follows that if z is an eigenvector with eigenvalue θ of $\Gamma_{J,K}$, viewed as a map $z : V\Gamma_{J,K} \to \mathbb{R}$, then its composition with (ϕ_J, ϕ_K) is an eigenvector with eigenvalue θq^b of $\Gamma_{\emptyset,\emptyset}$. (Note that since conjugation by w_0 interchanges J and K, the value b remains the same when J and K are interchanged.)

This reduces us to the case of $\Gamma_{\emptyset,\emptyset}$, and by the previous subsection to the case $\Gamma_\emptyset$.

4.3 Eigenvector constant on Bruhat cells

Let $\Gamma := \Gamma_\emptyset$ so that the vertices of Γ are the cosets gB. Let z be an eigenvector of Γ with eigenvalue θ. We may assume that $z(B) \neq 0$. Put $\bar{z}(x) = \frac{1}{|B|} \sum_{b \in B} z(bx)$. Then $\bar{z}$ is an eigenvector of Γ with eigenvalue θ that is constant on Bruhat cells BwB/B.

4.4 Reduction to the Iwahori-Hecke algebra

We saw that each eigenvalue has an eigenvector z that can be described by $z(gB) = f(w)$ when $gB \subseteq BwB$. It remains to find the condition on f, and the resulting eigenvalue θ. Let us change notation and work with right cosets Bg instead of left cosets gB, in order to get $T_u(v)$ rather than $T_{u^{-1}}(v^{-1})$ below.

For any ring R, let RW be the ring of linear combinations of elements of W with coefficients in R. For each $u \in W$ define an R-linear operator T_u on RW by letting $T_u(v)$ (for $v \in W$) be the element of $\mathbb{Z}W$ describing the multiset of Bruhat cells reached from any point Bg in BvB by going a distance u. That is, let $T_u(v) = \sum n_w w$ when there are n_w cosets Bh in $BuBv \cap BwB$. Then

$$T_{uv} = T_u T_v \quad \text{if } l(uv) = l(u) + l(v)$$

and, for $s \in S$,

$$T_s(v) = \begin{cases} q.sv & \text{if } l(sv) > l(v) \\ (q-1).v + sv & \text{otherwise} \end{cases}$$

Now $T_s^2 = (q-1)T_s + qT_1$, so that $(T_s - q)(T_s + 1) = 0$. Let $\mathcal{H}$ be the ring $R\{T_w \mid w \in W\}$ of linear combinations of the T_w with coefficients in R. The actions of the R-linear operators T_u on RW and on $\mathcal{H}$ (by left multiplication) are isomorphic via the map $q^{l(w)}.w \mapsto T_w$. The ring $\mathcal{H}$ (algebra when R is a field) is known as the Iwahori-Hecke ring (algebra). (Cf. [5], §7.4, [3], §8.4.)

The additive function $f : \mathbb{Z}W \to \mathbb{R}$ defines an eigenvector with eigenvalue θ when $f(T_{w_0}(v)) = \theta f(v)$ for all $v \in W$. But that means that the eigenvalues of Γ are precisely the eigenvalues of T_{w_0}.

It remains to find the eigenvalues of T_{w_0} acting on $\mathcal{H}$.

4.5 The center of the Hecke algebra

If w_0 is central in W, then T_{w_0} is central in $\mathcal{H}$, and in all cases, $T_{w_0}^2$ is central in $\mathcal{H}$. (Indeed, suppose $r^{w_0} = s$ where $r, s \in S$. Then $rw_0 = w_0 s$ and $T_r T_{w_0} = T_r^2 T_{rw_0} = ((q-1)T_r + q)T_{rw_0} = T_{w_0 s}((q-1)T_s + q) = T_{w_0 s} T_s^2 = T_{w_0} T_s$. If w_0 is central in W, this shows that T_{w_0} is central in $\mathcal{H}$. In all cases $T_r T_{w_0}^2 = T_{w_0} T_s T_{w_0} = T_{w_0}^2 r$ so that $T_{w_0}^2$ is central in $\mathcal{H}$.)

4.6 The eigenvalues of $T_{w_0}^2$

Look at the action of the Hecke algebra on itself. By Schur's Lemma, $T_{w_0}^2$ acts as a multiple of the identity on each irreducible part of $\mathcal{H}$. If it is cI on a part of dimension d, then the determinant there is c^d. Look at an irreducible part of $\mathcal{H}$ of dimension d with character χ. All eigenvalues of T_r are q or -1. If T_r has a eigenvalues q and b eigenvalues -1, then $a + b = d = \chi(T_1)$, and $qa - b = \chi(T_r)$. By [3] (8.1.7), the characters of $\mathcal{H}$ become the characters of W for $q = 1$. Consequently, $|\det T_r| = q^a = q^{(\chi(1)+\chi(r))/2}$. If the Coxeter diagram has single bonds only, so that all generating involutions are conjugate, it follows that $c^d = \det T_{w_0}^2 = q^{N(\chi(1)+\chi(r))}$ so that $c = q^e$ with $e = N(1 + \chi(r)/\chi(1))$. (The d-th root of unity expected here is 1, e.g. because being opposite is a symmetric relation so that T_{w_0} has real eigenvalues, and $T_{w_0}^2$ has positive real eigenvalues.) If not all generators are conjugate, but r, s are representatives for the conjugacy classes, and there are N_r, N_s generators conjugate to r, s in an expression for w_0, then $c = q_1^e q_2^f$ with $e = N_r(1 + \chi(r)/\chi(1))$ and $f = N_s(1 + \chi(s)/\chi(1))$, if T_r has eigenvalue q_1 and T_s eigenvalue q_2. The exponents e or e, f here are integral. This computation is due to Springer, cf. [3], (9.2.2).

This completes the proof of the theorem. □

4.7 The eigenvalues of T_{w_0}

The above describes the eigenvalues of $T_{w_0}^2$. But we wanted the eigenvalues of T_{w_0}, so there is a sign to be determined. Consider an irreducible part of dimension d with character χ. If $T_{w_0}^2$ has eigenvalue θ^2 and the trace of T_{w_0} is $d\theta$ or $-d\theta$ then T_{w_0} has only eigenvalue θ or $-\theta$ there. Otherwise T_{w_0} has both eigenvalues $\pm\theta$. The trace of T_{w_0} is found from a result by Broué & Michel (see [3], (9.2.8), (9.2.9a)): $\chi(T_{w_0}) = \chi(w_0)\sqrt{\theta^2}$. Thus, if $\chi(w_0) = \pm\chi(1)$ then there is a single sign and only $(\chi(w_0)/\chi(1))\theta$ occurs. Otherwise we see $\pm\theta$.

In the particular case where w_0 is central in W we are always in the situation with a single sign.

4.8 Examples

4.8.1 A_1

Here $W = \mathrm{Sym}(2)$ and $N = 1$. Character table, $e = N(1 + \chi(r)/\chi(1))$ and θ:

	1	2	$\chi(1)$	$\chi(r)$	$\chi(w_0)$	e	θ
χ_1	1	1	1	1	1	2	q
χ_2	1	-1	1	-1	-1	0	1

4.8.2 A_2

Here $W = \mathrm{Sym}(3)$. The number of positive roots is $N = 3$, and we find the character table, and computation of $e = N(1 + \chi(r)/\chi(1))$ and θ:

	1	2	3	$\chi(1)$	$\chi(r)$	$\chi(w_0)$	e	θ
χ_1	1	1	1	1	1	1	6	q^3
χ_2	1	-1	1	1	-1	-1	0	-1
χ_3	2	0	-1	2	0	0	3	$\pm q\sqrt{q}$

4.8.3 A_3

Here $W = \mathrm{Sym}(4)$ with $N = 6$. Character table and computation of θ:

	1	2	2^2	3	4	$\chi(1)$	$\chi(r)$	$\chi(w_0)$	e	θ
χ_1	1	1	1	1	1	1	1	1	12	q^6
χ_2	1	-1	1	1	-1	1	-1	1	0	1
χ_3	2	0	2	-1	0	2	0	2	6	q^3
χ_4	3	1	-1	0	-1	3	1	-1	8	$\pm q^4$
χ_5	3	-1	-1	0	1	3	-1	-1	4	$\pm q^2$

4.8.4 BC_2 : r s

Here W is the dihedral group of order 8, and $N = 4$. The two generators r, s are not conjugate. Write $e = N_r(1 + \chi(r)/\chi(1))$ and $f = N_s(1 + \chi(s)/\chi(1))$, where $N_r = N_s = 2$. Character table and computation of eigenvalues:

	1	$2r$	$2s$	2	4	$\chi(1)$	$\chi(r)$	$\chi(s)$	$\chi(w_0)$	e	f	θ
χ_1	1	1	1	1	1	1	1	1	1	4	4	$q_1^2 q_2^2$
χ_2	1	-1	-1	1	1	1	-1	-1	1	0	0	1
χ_3	1	1	-1	1	-1	1	1	-1	1	4	0	q_1^2
χ_4	1	-1	1	1	-1	1	-1	1	1	0	4	q_2^2
χ_5	2	0	0	-2	0	2	0	0	-2	2	2	$-q_1 q_2$

4.8.5 BC_3 : r s t

Consider the eigenvalues θ, α, β, γ of the four graphs and and and , where the final double stroke is a $GQ(q, q^e)$.

Here $|W| = 48$ and $N = 9$. Of the three generators r, s, t, the first two are conjugate (being joined by a single stroke in the diagram), but t is not conjugate to r, s. The conjugacy class of r has size $N_r = 6$ (and consists of $r, s, srs, tst, tsrst, stsrsts$), the conjugacy class of t has size $N_t = 3$ (and consists of $t, sts, rstsr$). Write $f = N_r(1 + \chi(r)/\chi(1))$ and $g = N_t(1 + \chi(t)/\chi(1))$.

χ	$\chi(1)$	$\chi(r)$	$\chi(t)$	$\chi(w_0)$	f	g	θ	α	β	γ
χ_1	1	1	1	1	12	6	q^{6+3e}	q^{4+e}	q^{5+2e}	q^{3+3e}
χ_2	1	-1	-1	-1	0	0	-1			
χ_3	1	1	-1	-1	12	0	$-q^6$			$-q^3$
χ_4	1	-1	1	1	0	6	q^{3e}			
χ_5	2	0	-2	-2	6	0	$-q^3$			
χ_6	2	0	2	2	6	6	q^{3+3e}	q^{1+e}	q^{2+2e}	
χ_7	3	-1	-1	3	4	2	q^{2+e}			
χ_8	3	1	-1	3	8	2	q^{4+e}		q^3	q^{1+e}
χ_9	3	-1	1	-3	4	4	$-q^{2+2e}$		$-q^{1+e}$	
χ_{10}	3	1	1	-3	8	4	$-q^{4+2e}$	$-q^2$	$-q^{3+e}$	$-q^{1+2e}$

Here for $J = \{s,t\}$ and $\{r,t\}$ and $\{r,s\}$, the longest word of W_J is $stst$, rt, rsr, respectively, so that compared to θ the eigenvalues lose a factor q^{2+2e}, q^{1+e}, q^3, respectively. In these cases, $1_{W_J}^W$ decomposes as $\chi_1 + \chi_6 + \chi_{10}$ and $\chi_1 + \chi_6 + \chi_8 + \chi_9 + \chi_{10}$ and $\chi_1 + \chi_3 + \chi_8 + \chi_{10}$, respectively.

References

[1] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-regular graphs*, Springer, Heidelberg, 1989.

[2] R. J. Blok and A. E. Brouwer, *The geometry far from a residue*, pp. 29–38 in: Groups and Geometries, L. di Martino, W. M. Kantor, G. Lunardon, A. Pasini, M. C. Tamburini (eds.), Birkhaüser Verlag, Basel, 1998.

[3] M. Geck and G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, Clarendon Press, Oxford, 2000.

[4] J. Eisfeld, *The eigenspaces of the Bose-Mesner-algebras of the association schemes corresponding to projective spaces and polar spaces*, Designs, Codes and Cryptography **17** (1999) 129-150.

[5] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge, 1990.

[6] F. Vanhove, pers. comm., 2009.