Regular symmetric Hadamard matrices with constant diagonal

A. E. Brouwer

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Today Aart Blokhuis becomes 60. Happy birthday!

Abstract

We survey constructions of regular symmetric Hadamard matrices with constant diagonal, and point out some errors in the literature.

1 Introduction

A *Hadamard matrix* is a matrix H of order n with entries ± 1 such that $HH^\top = nI$. It is called *symmetric* when $H = H^\top$. It is called *regular* when all row sums are equal. If J denotes the all-1 matrix of order n, then all row sums are a iff $HJ = aJ$. (It follows that $JH = aJ$ and $a^2 = n$.) The matrix $H = (h_{ij})$ has *constant diagonal* when $h_{ii} = e$ for all i and some fixed $e \in \{\pm 1\}$. Abbreviate the phrase 'regular symmetric Hadamard matrix with constant diagonal' with RSHCD.

Let H be a RSHCD with parameters n, a, e. Then $a^2 = n$ so that $a = \pm\sqrt{n}$. The matrix $-H$ is a RSHCD with parameters $n, -a, -e$, so that there are the two essentially distinct cases $ae > 0$ and $ae < 0$. Put $ae = \varepsilon\sqrt{n}$ with $\varepsilon \in \{\pm 1\}$, and call H *of type* ε. If $n > 1$, then $4|n$, so $2|a$, say $a = 2t$. Then $A = \frac{1}{2}(J - eH)$ is the adjacency matrix of a strongly regular graph (degenerate for $(n, \varepsilon) = (4, -1)$) with parameters

$$v = 4t^2,\ k = 2t^2 - \varepsilon t,\ \lambda = \mu = t^2 - \varepsilon t,$$

$$r = t,\ s = -t,\ f = 2t^2 - t - (1-\varepsilon)/2,\ g = 2t^2 + t - (1+\varepsilon)/2.$$

And $J - I - A = \frac{1}{2}(J + eH - 2I)$ is the adjacency matrix of the complementary strongly regular graph with parameters

$$v = 4t^2,\ k = 2t^2 + \varepsilon t - 1,\ \lambda = t^2 + \varepsilon t - 2,\ \mu = t^2 + \varepsilon t,$$

$$r = t - 1,\ s = -t - 1,\ f = 2t^2 + t - (1+\varepsilon)/2,\ g = 2t^2 - t - (1-\varepsilon)/2.$$

Conversely, graphs with these parameters yield RSHCDs.

We see that A is the incidence matrix of a square $(4t^2, 2t^2 \pm t, t^2 \pm t)$-design (and moreover is symmetric with zero diagonal). Designs with these parameters are known as *Menon designs*.

In [9] it is shown that the sum of the absolute values of the eigenvalues (the 'energy') of a graph on n vertices is at most $n(\sqrt{n}+1)/2$, with equality precisely in the case of a graph corresponding to a RSHCD of negative type. See also [5].

2 Survey

Constructions for RSHCDs were discussed in [13] and [1]. However, not all details given there are correct, so we resurvey this area.

Let R be the set of pairs (n, ε) for which an RSHCD of order n and type ε exists.

Section 8D of Brouwer & van Lint [1] is about RSHCDs. It is the first place that kept track of the sign ε involved. It contains the recursive construction

$$(m, d), (n, e) \in R \;\Rightarrow\; (mn, de) \in R$$

and six direct constructions:

(i) $(4, \pm 1), (36, \pm 1) \in R$,

(ii) If there exists a Hadamard matrix of order m, then $(m^2, 1) \in R$ ([4], Theorem 4.4),

(iii) If both $a - 1$ and $a + 1$ are odd prime powers, and $4|a$, then $(a^2, 1) \in R$ ([4], Theorem 4.3),

(iv) If $a + 1$ is a prime power, and there exists a symmetric conference matrix of order a, then $(a^2, 1) \in R$ ([14], Corollary 17),

(v) If there is a set of $t-2$ mutually orthogonal latin squares of order $2t$, then $(4t^2, 1) \in R$,

(vi) Suppose we have a Steiner system $S(2, K, V)$ with $V = K(2K - 1)$. If we form the block graph, and add an isolated point, we get a graph in the switching class of a regular two-graph. The corresponding Hadamard matrix is symmetric with constant diagonal, but not regular. If this Steiner system is invariant under a regular abelian group of automorphisms (which necessarily has orbits on the blocks of sizes V, V, and $2K - 1$), then by switching with respect to a block orbit of size V we obtain a SRG with parameters

$$v = 4K^2,\ k = K(2K - 1),\ \lambda = \mu = K(K - 1)$$

showing that $(4K^2, 1) \in R$. Steiner systems $S(2, K, K(2K-1))$ are known for $K = 3, 5, 6, 7$ or 2^t, but only for $K = 2, 3, 5, 7$ are systems known that have a regular abelian group of automorphisms. Thus we find $(196, 1) \in R$. The required switching set also exists when the design is resolvable: take the union of K parallel classes. Resolvable designs are known for $K = 3$ or 2^t.

See also Goethals & Seidel [4], Section 4, and Wallis [13], Section 5.3.

More recent constructions:

(vii) In Jørgensen & Klin [8] it is shown that $(100, -1) \in R$.

(viii) In Haemers [5] is is shown that if there exists a Hadamard matrix of order m, then $(m^2, -1) \in R$.

(ix) In Muzychuk & Xiang [10] it is shown that $(4m^4, 1) \in R$ for all m.

(x) In Haemers & Xiang [6] it is shown that $(4m^4, -1) \in R$ for all m.

3 Errata

This note was prompted by questions from Nathann Cohen who implemented a large number of constructions for strongly regular graphs, and encountered flaws in various descriptions. Now there is a report [2] describing this effort.

3.1 Ad (iii)

In [1] the condition $4|a$ was omitted from (iii) above. But it seems necessary. (Here [1] referred to [13], which gives the result without this condition in Theorem 5.11, and Corollary 5.12 and in the table on p. 454. It says 'we strengthen a theorem of Goethals and Seidel', but the proof is wrong.)

After correction, (iii) becomes a special case of (ii).

3.2 Ad (iv)

Many of the parameter sets that would be produced by (iii) without the condition $4|a$ are also produced by (iv). In this way one finds e.g. $(676, 1) \in R$ and $(900, 1) \in R$. Now in [2] the authors found that also (iv) was wrong, or at least could not be reproduced. The reference for (iv) was [13], Corollary 5.16 which uses the construction of [13], Theorem 5.15. There is a typo in that theorem: the expression given for H misses a minus-sign in front of the C in the bottom-right entry. In [14] the expression is correct. So, construction (iv) stands.[1]

3.3 Ad (vi)

In the Handbook of Combinatorial Designs the chapter on Hadamard matrices [3] contains (Theorem 1.44, p. 277) the statement

If there is a BIBD$(u(2u-1), 4u^2-1, 2u+1, u, 1)$, *then there is a regular graphical Hadamard matrix of order* u^2.

with a reference to [11]. Here 'graphical' means 'symmetric with constant diagonal'. However, that reference constructs the Hadamard matrix by observing that the block graph is strongly regular with parameters $(v, k, \lambda, \mu) = (4u^2-1, 2u^2, u^2, u^2)$ and bordering its $(-1, 1)$-adjacency matrix with a constant border, so that the resulting Hadamard matrix is not regular. In [4], Theorem 4.5 and also in [13], Theorem 5.14 this same result is shown without the 'regular'. In [13], p. 454, construction GV is mistakenly starred.

In [6] the statement $(196, \pm 1) \in R$ is attributed to [7], p. 258. As we saw, $(196, 1) \in R$ was shown in [1] as application of [4], Theorem 4.5. It is still unknown whether $(196, -1) \in R$. The proof of Theorem 8.2.26 (iii) in [7] is wrong. For [6], §5 this means that the smallest open case again is $n = 196$.

[1] The construction uses Szekeres difference sets, and if one tries to find those in the original Szekeres paper [12] one may stumble over another sign typo: in (4.2) the $-$ should be a $+$.

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