(0,2)-graphs and root systems

Andries E. Brouwer & Leonard Chastkofsky

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Abstract

We construct (0,2)-graphs from root systems with simply laced diagram and study their properties.

1 Introduction

In the study of the mod p cohomology of the Lie algebra of the unipotent radical U of groups of Lie type with simply laced diagram, it was found that the connected components of the Hasse diagram of the Koszul complex are (0,2)-graphs. This note is the result of an attempt to understand these (0,2)-graphs.

2 (0,2)-graphs

A *(0,2)-graph* is a connected graph with the property that any two vertices have either 0 or 2 common neighbours. The first thing one shows (cf. [10]) is that two adjacent vertices have the same number of neighbours, so that a (0,2)-graph is regular of some valency k (finite or infinite). For a classification of the (0,2)-graphs of valency at most 8, see [2, 5].

A (0,2)-graph without triangles is known as a *rectagraph*. Rectagraphs play a role in diagram geometry, cf., e.g., [11].

A *semibiplane* is a connected incidence structure with points and blocks, where any two points are together in 0 or 2 blocks, and any two blocks meet in 0 or 2 points. Thus, the incidence graph of a semibiplane is a bipartite (0,2)-graph, and conversely any bipartite (0,2)-graph defines a semibiplane, up to duality (that is, up to the choice which part of the bipartition is the set of points and which part is the set of blocks). Semibiplanes were first introduced in order to study projective planes with involution, see [8].

Given a non-bipartite (0,2)-graph, its *bipartite double* (the unique bipartite 2-cover, cf. [4]) is a bipartite (0,2)-graph.

A (0,2)-graph of finite valency k has at most 2^k vertices, and the k-cube is the unique (0,2)-graph for which equality holds (see [11]).

A (0,2)-graph is called *signable* if it is possible to label its edges with ± 1 in such a way that the product of the signs of the four edges of a quadrangle is always -1. Clearly, a (0,2)-graph with more than one edge is signable if and only if it has a 2-cover without quadrangles. It is known ([7]) that hypercubes are signable, and ([4], p. 372) that the Gewirtz graph is not.

3 (0,2)-graphs from root systems

Let Φ be a finite root system with simply laced diagram, and let Φ^+ be the collection of positive roots (for some choice of fundamental roots). For any vector u (the *target vector*) in the span of Φ we define the graph $\Gamma = \Gamma(u) = \Gamma(\Phi, u)$ as follows: The vertices of Γ are the subsets A of Φ^+ such that $\sum A (= \sum_{a \in A} a) = u$. Two vertices A and B are adjacent when their symmetric difference $A \Delta B$ has size 3. (Smaller is impossible: if the symmetric difference has size 0, then $A = B$; it cannot have size 1 since $\sum A = \sum B = u$; it cannot have size 2 since A, B are sets of positive roots.)

Theorem 3.1 *If $\Gamma(u)$ has a nonempty vertex set, it is a bipartite (0,2)-graph.*

Proof: That Γ is bipartite follows since adjacent vertices are sets of roots of which the sizes have different parity.

We show that Γ is connected. Since Φ has simply laced diagram (so that all connected components of the diagram are of type A_n, D_n or E_m, $m = 6, 7, 8$), we may assume that all roots have the same length $\sqrt{2}$. Now the roots are precisely the vectors of squared norm 2 in the (integral) lattice spanned by Φ and for $r, s \in \Phi$ we have $r+s \in \Phi$ when $(r, s) = -1$ and $r-s \in \Phi$ when $(r, s) = 1$. Let A and B be two vertices of Γ. We want to join them by a path. Use induction on the size of the symmetric difference. Let $v = \sum A \setminus B = \sum B \setminus A$. Suppose $r \in A \setminus B$ and $s \in B \setminus A$ with $(r, s) = 1$. Now $r - s$ is a root, and either $r - s$ or $s - r$ is positive. Say $t = r - s \in \Phi^+$. If $t \notin A$, then $C = A \setminus \{r\} \cup \{s, t\}$ is a vertex adjacent to A and we are done by induction. If $t \in B$, then $D = B \setminus \{s, t\} \cup \{r\}$ is a vertex adjacent to B and we are done by induction. This means that in the remaining case whenever $r \in A \setminus B$ and $s \in B \setminus A$ with $(r, s) = 1$ we have either $r - s \in A \setminus B$ or $s - r \in B \setminus A$. Since $(r - s, s) = -1$ or $(r, s - r) = -1$, and the pair r, s can be retrieved from the pair $r - s, s$ or $r, s - r$, there is a contribution of -1 for each contribution of 1 in the expanded inner product $(v, v) = (\sum A \setminus B, \sum B \setminus A)$, so that $(v, v) \le 0$ and hence $(v, v) = 0$, so that $A = B$. This proves connectedness.

We show that Γ is a (0,2)-graph. Let A, B be two vertices with at least one common neighbour, and put $Z = A \cap B$. Since neighbours have sizes that differ by 1, the sizes of A and B differ by 0 or 2. Suppose first that $|A| = |B| + 2$. If $A = Z \cup \{a, b, c, d\}$ and $B = Z \cup \{a+b, c+d\}$ then the common neighbours are $Z \cup \{a+b, c, d\}$ and $Z \cup \{a, b, c+d\}$. If $A = Z \cup \{a, b, c\}$ and $B = Z \cup \{a+b+c\}$ and $(a, b) = (b, c) = -1$, then the common neighbours are (i) $Z \cup \{a+b, c\}$ if $a + b \notin Z$ and $Z \setminus \{a+b\} \cup \{a, b, a+b+c\}$ otherwise, and (ii) $Z \cup \{a, b+c\}$ if $b + c \notin Z$ and $Z \setminus \{b+c\} \cup \{b, c, a+b+c\}$ otherwise. Note that since $a+b+c \in \Phi$, so that $(a+b+c, a+b+c) = (a, a) = (b, b) = (c, c) = 2$, precisely two of the inner products $(a, b), (a, c), (b, c)$ are -1 (and the third is 0). Now suppose that $|A| = |B|$. If the symmetric difference of A and B has size 6, so that $A = Z \cup \{a, b, c+d\}$ and $B = Z \cup \{a+b, c, d\}$, then the common neighbours are $Z \cup \{a, b, c, d\}$ and $Z \cup \{a+b, c+d\}$. If their symmetric difference has size 4, then $A = Z \cup \{a, b\}$ and $B = Z \cup \{c, d\}$, where $a+b = c+d$. If $a+b \in \Phi^+$ then one common neighbour is $Z \cup \{a+b\}$ if $a+b \notin Z$ and $Z \setminus \{a+b\} \cup \{a, b, c, d\}$ otherwise. If $\pm(c-a) \in \Phi$, say $c - a \in \Phi^+$, then one common neighbour is $Z \cup \{a, c-a, d\}$ if $c - a \notin Z$ and $Z \setminus \{c-a\} \cup \{b, c\}$ otherwise. In this way we find one common neighbour for each of $c - a = b - d$ and $c - b = a - d$ that is in Φ, i.e., for each inproduct (a,c), (a,d) that equals 1. Now $(a,c)+(a,d)=(a,a+b)=2+(a,b)$, and $(a,b),(a,c),(a,d)\in\{-1,0,1\}$. If $(a,b)=-1$, then $a+b\in\Phi^+$ and precisely one of (a,c), (a,d) is 1, and we find two common neighbours. If $(a,b)=0$, then $(a,c)+(a,d)=2$ so that $(a,c)=(a,d)=1$, again two common neighbours. Finally if $(a,b)=1$, then $(a,c)+(a,d)=3$, impossible. □

4 Isomorphisms

Different choices for u may yield isomorphic graphs $\Gamma(u)$.

The map $A\mapsto\Phi^+\setminus A$ sending each set of positive roots to its complement induces an isomorphism from the graph $\Gamma(u)$ onto the graph $\Gamma(2\rho-u)$, where 2ρ is the sum of the positive roots.

Let W be the Weyl group (generated by the reflections in the elements of Φ), let $w\in W$, and let $N=\Phi^+\setminus w^{-1}\Phi^+$ be the set of positive roots made negative by w. The graph $\Gamma(u)$ is mapped isomorphically onto the graph $\Gamma(w(u-n))$, where $n=\sum N$, by the map $A\mapsto w(A\setminus N)\cup -w(N\setminus A)$ that sends A to the union of $w(A)$ and $w(-N)$ with pairs of opposite elements removed. If we parametrize the graphs $\Gamma(u)$ using $\mu=\rho-u$ instead of u, this means that Γ_μ is mapped isomorphically onto $\Gamma_{w\mu}$. (Indeed, $w\rho=\rho+wn$, so $\rho-w(u-n)=w\mu$.) It follows that we may choose $u=\rho-\mu$ where μ is a dominant weight.

5 The number of vertices

Let $\rho=\frac{1}{2}\sum\Phi^+$ be half the sum of the positive roots.

Proposition 5.1 *The number of vertices of the graph $\Gamma(u)$, where $u=\rho-\mu$ and μ is a dominant weight, equals the multiplicity of the weight μ in the Verma module V_ρ.*

Proof: This is Lemma 5.9 in [9]. Or, from Weyl's character formula: the formal character of V_ρ equals $A_{2\rho}/A_\rho=e(-\rho)\prod_{a\in\Phi^+}(e(a)+1)$, so that the multiplicity of μ equals the number of ways to write $\rho+\mu$ as sum of positive roots. □

Now Freudenthal's formula gives a straightforward way to compute the number of vertices for any given u.

6 The valency

Let $\Pi=\{\alpha_1,\ldots,\alpha_n\}$ be the set of fundamental roots.

Proposition 6.1 *The graph $\Gamma(u)$ has valency $k=(u,2\rho-u)/2$. For $u=\sum u_i\alpha_i$ this becomes $k=\sum_i u_i-\sum_i u_i^2+\sum_e u_iu_j$, where the last sum is over all edges $e=ij$ in the Coxeter diagram.*

Proof: Let A be a vertex, and $B=\Phi^+\setminus A$ its complement. Compute $(\sum A,\sum B)$ (which equals $(u,2\rho-u)$). If r,s are distinct positive roots with $(r,s)\neq 0$, then r and s determine a root system of type A_2 with a unique third positive root t, and w.l.o.g. $t=r+s$, so that $(r,t)=(s,t)=1$, $(r,s)=-1$.

The pairs taken from $\{r,s,t\}$ only contribute to $(\sum A, \sum B)$ when $r,s \in A$, $t \in B$ or $r,s \in B$, $t \in A$ (and then the contribution is 2). This means that we find a nonzero contribution (of 2) for each neighbour $A \setminus \{r,s\} \cup \{t\}$ or $A \setminus \{t\} \cup \{r,s\}$ of A. This proves the formula for the valency k. On the basis Π of fundamental roots, inner products are given by the Cartan matrix, and $(u,u)/2 = \sum u_i^2 - \sum_e u_i u_j$, and $(u,\rho) = \sum u_i$. □

Lemma 6.2 *Let $u = \sum u_i \alpha_i = \rho - \mu$ where μ is a dominant weight. Then the graph $\Gamma(u)$ has valency $k \geq \frac{1}{2}\sum u_i$.*

Proof: Let C be the Cartan matrix. A vector is converted from root coordinates to weight coordinates by multiplication by C. In weight coordinates $\rho = \mathbf{1}$ and $\mu \geq 0$, so that $Cu \leq \mathbf{1}$, and $k = \sum u_i - \frac{1}{2}u^\top Cu \geq \frac{1}{2}\sum u_i$. □

7 Direct products

Let us write things like $\Gamma(E_6, 112321)$ for the graph $\Gamma(\Phi, u)$ where Φ is a root system of type E_6 and $u = (u_i)_i = (1,1,2,3,2,1)$ on the root basis, where fundamental roots are numbered as in Bourbaki [1].

If the Coxeter diagram is disconnected (or the support of u is), then clearly the corresponding graph is the direct product of the graphs for the components. For example, $\Gamma(D_8, 11102322)$ is the direct product $\Gamma(A_3, 111) \times \Gamma(D_4, 2322)$.

If the target vector u contains a 1 on a nonterminal position i, then the graph is the direct product of the two or three graphs that arise by splitting the Coxeter diagram (and target vector) into components at i, preserving a copy of i in each component. For example, $\Gamma(A_7, 1221221)$ is the direct product $\Gamma(A_4, 1221) \times \Gamma(A_4, 1221)$. And $\Gamma(D_5, 12111)$ is the direct product $\Gamma(A_3, 121) \times \Gamma(A_2, 11) \times \Gamma(A_2, 11)$. (Proof: Since $u_i = 1$ each vertex A contains a unique root r involving α_i. Let the projection πA on one of the components consist of the roots in A with support in that component, together with the projection of r on that component. This establishes an isomorphism.)

In particular, $\Gamma(A_{n+1}, 11...1)$ is the n-cube.

This discussion, together with Lemma 6.2, implies that one can identify all graphs $\Gamma(\Phi, u)$ with valency at most k_0 with a finite amount of work. For the results up to $k_0 = 8$, see [3].

valency	0	1	2	3	4	5	6	7	8
(0,2)-graphs	1	1	1	2	3	8	24	96	301
bipartite	1	1	1	1	2	4	13	40	104
signable	1	1	1	1	2	3	6	17	50
from rootsystem	1	1	1	1	2	3	5	7	11

8 Signability

Theorem 8.1 *The graphs $\Gamma(\Phi, u)$ are signable.*

Proof: Let $L = H \oplus \bigoplus_{r \in \Phi} \langle e_r \rangle$ be a semisimple complex Lie algebra with Cartan subalgebra H and root system Φ, where the structure constants are chosen such that $[e_r, e_s] = \pm e_{r+s}$ for $r, s, r+s \in \Phi$. (This is possible because Φ is simply laced—see, e.g., [6], Theorem 4.2.1.)

Let k be a field, and let V be the k-vectorspace with basis Φ^+. Let $<$ be an arbitrary fixed total order on Φ^+ and identify each vertex $A = \{a_1, \ldots, a_m\}$, where $a_1 < \ldots < a_m$, with the exterior product $a_1 \wedge \ldots \wedge a_m \in \bigwedge V$. Give the edge joining A and B, where $B = A \setminus \{r+s\} \cup \{r, s\}$, the sign $\varepsilon\eta$ (with $\varepsilon, \eta = \pm 1$) if $\varepsilon A = (r+s) \wedge a_1 \wedge \ldots \wedge a_m$ and $\eta B = r \wedge s \wedge a_1 \wedge \ldots \wedge a_m$, where $[e_r, e_s] = e_{r+s}$. We show that this assignment of signs has the property that the product of the signs on the four edges of a quadrangle is always -1. Note that if $ABCD$ is a quadrangle, we may choose the ordering of the roots at these four vertices independently (since each of these vertices is on two edges of the quadrangle), and invariant tails in the exterior products can be ignored.

Examine the possible shapes of a 4-gon $ABCD$, as found in the proof of the (0,2)-property. Define $f(r, s) = \pm 1$ by $[e_r, e_s] = f(r, s)e_{r+s}$.

If $A = (a+b) \wedge (c+d)$, $B = (a+b) \wedge c \wedge d$, $C = a \wedge b \wedge c \wedge d$, $D = a \wedge b \wedge (c+d)$, then the edges AB, BC, CD, DA have signs $-f(c, d)$, $f(a, b)$, $f(c, d)$ and $f(a, b)$ with product -1, as desired.

If $A = a+b+c$, $B = (a+b) \wedge c$, $C = a \wedge b \wedge c$, $D = a \wedge (b+c)$ then the signs are $f(a+b, c)$, $f(a, b)$, $-f(b, c)$, $f(a, b+c)$ with product -1, since the Jacobi identity $[[e_a, e_b], e_c] + [[e_b, e_c], e_a] + [[e_c, e_a], e_b] = 0$ reduces to $f(a, b)f(a+b, c) + f(b, c)f(b+c, a) = 0$. (Note that $a + c \notin \Phi$.)

That covers the case where $|A| = |C| \pm 2$ (or $|B| = |D| \pm 2$). Remains the case where $|A| = |C|$ and $|B| = |D|$ and, say, $|A| = |B| - 1$.

If $A = a \wedge b \wedge (a+b)$, $B = a \wedge b \wedge c \wedge d$, $C = c \wedge d \wedge (a+b)$, $D = a \wedge (c-a) \wedge d \wedge (a+b)$, with $a + b = c + d$, then the signs are $f(c, d)$, $f(a, b)$, $f(a, c-a)$, $f(d, c-a)$ with product -1 since the Jacobi identity $[[e_{c-a}, e_a], e_d] + [[e_a, e_d], e_{c-a}] + [[e_d, e_{c-a}], e_a] = 0$ reduces to $f(c-a, a)f(c, d) + f(d, c-a)f(b, a) = 0$.

If $A = a \wedge b$, $B = a \wedge (c-a) \wedge d$, $C = c \wedge d$, $D = a \wedge (d-a) \wedge c$ (or $D = (a-d) \wedge d \wedge b$), with $a + b = c + d$, then the signs are $f(d, b-d)$, $f(a, c-a)$, $-f(a, d-a)$ (or $-f(a-d, b)$), $f(c, b-c)$ (or $f(a-d, d)$) with product -1 by the Jacobi identity on e_a, e_{d-a}, e_{b-d} (or e_d, e_{a-d}, e_{c-a}).

That covers all cases. $\square$

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