ma
the
ma
tisch

cen
trum

MC

AFDELING ZUIVERE WISKUNDE (DEPARTMENT OF PURE MATHEMATICS) ZN 78/77 NOVEMBER

A.E. BROUWER

THE UNIQUENESS OF THE TRUNCATED AFFINE PLANE OF ORDER FIVE

amsterdam 1977

stichting
mathematisch
centrum

MC

AFDELING ZUIVERE WISKUNDE ZN 78/77 NOVEMBER
(DEPARTMENT OF PURE MATHEMATICS)

A.E. BROUWER

THE UNIQUENESS OF THE TRUNCATED AFFINE PLANE
OF ORDER FIVE

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 05B05

The uniqueness of the truncated affine plane of order five

by

A.E. Brouwer

ABSTRACT

We prove that the only pairwise balanced design on 24 points with blocks of size 4 or 5 (and $\lambda = 1$) is the truncation of the affine plane of order five.

KEY WORDS & PHRASES: *Pairwise balanced design.*

Trying to construct non-isomorphic block designs with certain parameters I was led to the question whether there exist non-isomorphic PBD({4,5},1;24) designs. Unfortunately this turns out not to be the case.

Let $(X,\mathcal{B})$ be a pairwise balanced design with $|X| = 24$ and $B \in \mathcal{B} \rightarrow |B| \in \{4,5\}$. Let $\mathcal{B}_i = \mathcal{B} \cap P_i(x)$ $(i = 4,5)$. If a point x is in c_4 blocks of size 4 and in c_5 blocks of size 5 then $3c_4 + 4c_5 = 23$ so that $c_4 = 1$, $c_5 = 5$ or $c_4 = 5$, $c_5 = 2$. In the former case x is called of type I, and in latter - of type II. Let there be a points of type I and b points of type II, then we have

$$\begin{aligned} a + b &= 24 \\ 5a + 2b &= 5b_5 \\ a + 5b &= 4b_4 \end{aligned}$$

(where $b_i = |\mathcal{B}_i|$, $i = 4,5$) and we have the following five possibilities:

Case 1: $b = 0$, $a = 24$, $b_4 = 6$, $b_5 = 24$.

Here each point is in a unique block of size 4 so that the blocks of size 4 form a parallel class. Completing this parallel class by adjoining a point ∞ produces (the unique) affine plane of order five. Hence this is the case of $AG(2,5)\setminus\{p\}$.

From now on we suppose that $b > 0$.

Case 2: $b = 5$, $a = 19$, $b_4 = 11$, $b_5 = 21$.

Case 3: $b = 10$, $a = 14$, $b_4 = 16$, $b_5 = 18$.

Case 4: $b = 15$, $a = 9$, $b_4 = 21$, $b_5 = 15$.

Case 5: $b = 20$, $a = 4$, $b_4 = 26$, $b_5 = 12$.

Suppose that some block B of size 4 contains three or four points of type I. If $y \notin B$ then the blocks containing y and the points of type I in B all have size 5 so that y must be of type I. But then there can be at most one point of type II. Contradiction.

In the same way it follows that if two blocks B and B', both of size 4, each contain two points of type I, then there are no points of type II outside $B \cup B'$, so that $b \leq 4$. Contradiction.

But this shows that $a \leq b_4+1$ so that case 2 is impossible.

Let $X_1 = \{x \in X \mid x \text{ of type I}\}$ and $X_2 = X \setminus X_1$. Counting the number of pairs in X_2 covered by blocks of size 4 we find either

$$(b_4-a)\cdot\binom{4}{2} + a\cdot\binom{3}{2},$$

or

$$(b_4-a+1)\cdot\binom{4}{2} + (a-2)\cdot\binom{3}{2} + \binom{2}{2}$$

i.e.

$$6b_4 - 3a + (0 \text{ or } 1).$$

This must be less than the total number of edges $\binom{b}{2}$. In case 3 we find $54 \leq 45$. Contradiction.

Next suppose that some block B of size 4 contains two points x_1, x_2 of type I, and let B' be another block of size 4 containing a point x_3 of type I. The blocks B_i containing x_i and x_3 ($i = 1,2$) are of size 5 and we have

$$|B \cup B' \cup B_1 \cup B_2| = 13 \text{ or } 14.$$

If $y \notin B \cup B' \cup B_1 \cup B_2$ then y must be of type I, since each of the three blocks containing y and x_i ($i = 1,2,3$) is of size 5. But then $X_2 \subset B \cup B' \cup B_1 \cup B_2$ and $b = |X_2| \leq 11$. Contradiction.

Hence each block of size 4 contains at most one point of type I.

Let x_1, x_2, x_3 be three points of type I, and let B_i be the block of size 4 containing x_i ($i = 1,2,3$). Furthermore, let B_i' be the block (of size 5) containing x_j and x_k ($\{i,j,k\} = \{1,2,3\}$). As before, if $y \notin B_1 \cup B_2 \cup B_3 \cup B_1' \cup B_2' \cup B_3'$ then y must be of type I, and since

$$|B_1 \cup B_2 \cup B_3 \cup B_1' \cup B_2' \cup B_3'| \leq 21$$

it follows that

$$a \geq 3 + (24-21) = 6.$$

This disposes of case 5.

Since only case 4 is left we know that

$$|\{B \in \mathcal{B}_4 \mid B \cap X_1 = \emptyset\}| = 12$$

and

$$|\{B \in \mathcal{B}_4 \mid B \cap X_1 \neq \emptyset\}| = 9.$$

Let

$$p_i = |\{B \in \mathcal{B}_5 \mid |B \cap X_1| = i\}| \qquad (i = 0,1,2,3,4,5).$$

We have:

$$p_0 + p_1 + p_2 + p_3 + p_4 + p_5 = 15.$$

Counting edges in X_2 not covered by blocks of size 4:

$$10p_0 + 6p_1 + 3p_2 + p_3 = 6.$$

Counting edges in X_1:

$$p_2 + 3p_3 + 6p_4 + 10p_5 = \binom{9}{2} = 36.$$

It follows that

$$6p_0 + 2p_1 + 2p_4 + 6p_5 = 42 - 60 = -18.$$

Contradiction.

Hence also case 4 is impossible.