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A NOTE ON $\Gamma\Delta$ -REGULAR GRAPHS

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A note on $\Gamma\Delta$ -regular graphs

by

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dedicated to M. Voorhoeve & R. Tijdeman

ABSTRACT

A $\Gamma\Delta$ -regular graph is a nonregular graph such that for each vertex x the graphs induced on its neighbours and on its nonneighbours are both regular. We show that if G is $\Gamma\Delta$ -regular, G and \bar{G} are connected, and $\text{diam } G = 3$ then G is one of two graphs on 4 resp. 8 vertices.

GODSIL & MCKAY [1] introduced the concept of a $\Gamma\Delta$ -regular graph (although they called it differently - we adopt Van Lint's terminology) - see the abstract. For the case we are interested in: G and \bar{G} are connected, they proved the following.

Let n be the number of vertices, λ the valency of $\Gamma(x)$ in G and $\bar{\lambda}$ the valency of $\bar{\Gamma}(x)$ in \bar{G} .

- (1) In G there occur exactly two valencies, k_1 and k_2 , where $k_1 < k_2$.
- (2) $k_1 + k_2 = \frac{1}{2}n + 2\lambda + 1$
- (3) $\lambda + \bar{\lambda} = \frac{1}{2}n - 2$

Let $M_i = \{x \mid x \text{ has valency } k_i\}$ ($i = 1, 2$). Write $m_i := |M_i|$.

- (4) Each M_i (viewed as induced subgraph of G) is regular with valency α_i ($i = 1, 2$), $\alpha_1 + \alpha_2 = \frac{1}{2}n - 1$, $(2\alpha_1 - m_1 + 1)(k_1 - k_2) = (\lambda + 1)(n - 1) - k_1 k_2$.
- (5) Let $x_1 \neq x_2$. Then $|\Gamma(x_1) \cap \Gamma(x_2)| = \lambda + 1 + \epsilon(k_1 - k_2)$, where

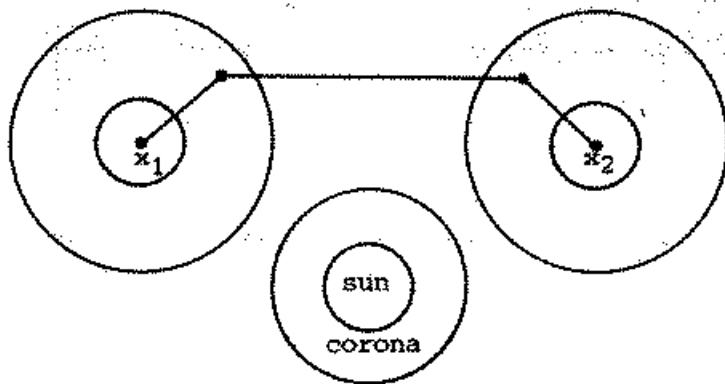
$$\epsilon = 0 \text{ if } x_1 \in M_1, x_2 \in M_2$$

$$\epsilon = 1 \text{ if } x_1, x_2 \in M_1,$$

$$\epsilon = -1 \text{ if } x_1, x_2 \in M_2.$$

- (6) $\text{diam } G \leq 3$.

Now suppose G has diameter 3, and let $\text{dist}(x_1, x_2) = 3$. Then $\Gamma(x_1) \cap \Gamma(x_2) = \emptyset$ so that by (5) $x_1, x_2 \in M_1$ and $k_2 - k_1 = \lambda + 1$. Again by (5) points in M_1 do not have distance two, so that M_1 is a disjoint union of cliques ('sun's'). Also, no point of M_2 is adjacent to points of different suns but each point of M_2 is adjacent to some point in M_1 (in fact to $k_2 - \alpha_2$ such points; $k_2 - \alpha_2 > 0$ since G is connected), so that the partition of M_1 into suns induces a partition of M_2 into 'corona's. From (2) and $k_2 - k_1 = \lambda + 1$ we find $n = 4k_1 - 2\lambda$. On the other hand, choosing one vertex in each sun we find $n \geq (k_1 + 1) \cdot \#$ of suns. Consequently the number of suns $\#$ is less than four (and larger than one since $\text{diam } G = 3$), i.e. two or three.



Fix a point $x_0 \in M_1$ and count edges between $\Gamma(x_0)$ and $\Delta(x_0)$. One finds

$$\alpha_1(k_1 - \lambda - 1) + (k_1 - \alpha_1)(k_2 - \lambda - 1) = (n - k_1 - 1 - (N-1)(\alpha_1 + 1))(k_2 - k_1)$$

(for: $|\Gamma(x_0)| = k_1$, $|\Delta(x_0)| = n - k_1 - 1$, $|\text{sun}| = \alpha_1 + 1$, etc.), i.e.,

$$k_1^2 - \alpha_1(\lambda + 1) = (3k_1 - 2\lambda - 1 - (N-1)(\alpha_1 + 1))(\lambda + 1),$$

or

$$k_1^2 - 3(\lambda + 1)k_1 + (\lambda + 1)(2\lambda + 2 + (N-2)(\alpha_1 + 1)) = 0.$$

Distinguish cases:

A. If $N = 2$ this factors as $(k_1 - (\lambda + 1))(k_1 - 2(\lambda + 1)) = 0$.

A1. $N = 2$ and $k_1 = \lambda + 1$.

Now $k_2 = 2\lambda + 2$, $n = 2\lambda + 4$, i.e. $|\text{sun}| = 1$, $|\text{corona}| = \lambda + 1$.

Considering two adjacent points in different coronas we find that they have 2λ common neighbours. Hence $\lambda = 2\lambda$, i.e. $\lambda = 0$, $n = 4$ and G looks like \square .

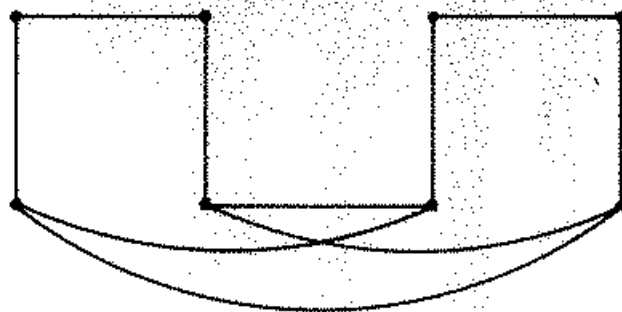
A2. $N = 2$ and $k_1 = 2(\lambda + 1)$.

Now $k_2 = 3\lambda + 3$, $n = 6\lambda + 8$, $|\text{sun}| = \alpha_1 + 1$, $|\text{corona}| = 3\lambda + 3 - \alpha_1$.

Count edges between sun and corona: $(\alpha_1 + 1)(k_1 - \alpha_1) = (3\lambda + 3 - \alpha_1)(k_2 - \alpha_2)$,

but $k_2 - \alpha_2 = 2\lambda + 2 - k_1 + \alpha_1 = \alpha_1$ (using (2) and (4)), so that

$\alpha_1 = 1 + \frac{\lambda}{\lambda+2}$. Since α_1 is integral this implies $\lambda = 0$, $n = 8$ and G looks like



B. If $N = 3$ then $|\text{sun} + \text{corona}| = \frac{4}{3}k_1 - \frac{2}{3}\lambda = 1 + k_1 + \frac{1}{3}(k_1 - 2\lambda - 3)$.

As before it follows that $k_1 \geq 2\lambda + 3$, contradicting the equation

$$k_1^2 - 3(\lambda+1)k_1 + (\lambda+1)(2\lambda+\alpha_1+3) = 0.$$

This ends the proof.

Egeldonk, 80 09 25

REFERENCE

- [1] GODSIL, C.D. & B.D. MCKAY, *Graphs with regular neighbourhoods*, to appear in: proceedings of Australian combinatorial conference, (Newcastle, 1979).