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AFDELING ZUIVERE WISKUNDE ZW 150/81 FEBRUARI
(DEPARTMENT OF PURE MATHEMATICS)

A.E. BROUWER

AN INEQUALITY IN BINARY VECTOR SPACES

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An inequality in binary vector spaces

by

A.E. Brouwer

ABSTRACT

We prove that if an n-dimensional vector space over GF(2) is the irredundant union of k subspaces, and this collection of subspaces has zero intersection, then $n < k$. This answers a question of B. Ganter.

KEY WORDS & PHRASES: *blocking set*

In [1] GANTER posed the following problem: "Let V be a vector space over GF(2) which is the irredundant union of k subspaces which have a trivial global intersection, i.e.,

$$V = \bigcup_{i=1}^{k} U_i, \quad V \neq \bigcup_{\substack{1\le i\le k\\ i\neq j}} U_i \quad (j=1,\dots,k), \quad \bigcap_{i=1}^{k} U_i = \{0\}.$$

Does this imply that dim V < k?"

Here we answer this question affirmatively. In fact, in order to make the induction work we prove the slightly stronger

THEOREM. *Let* X *be a vector space over* GF(2) *and* V, U_i $(1 \le i \le k)$ *subspaces of* X *such that for certain vectors* $a_i \in X$ *we have*

$$V \subset \bigcup_{i=1}^{k} (a_i + U_i), \quad V \not\subset \bigcup_{\substack{1\le i\le k\\ i\neq j}} (a_i + U_i) \qquad (j = 1,\dots,k).$$

Then, if $W := V \cap \bigcap_{i=1}^{k} U_i$, *we have* $k \ge \dim V - \dim W + 1$.

(Clearly, Ganter's problem is the case $V = X$, $W = \{0\}$, $a_i = 0$ $(1 \le i \le k)$.)

PROOF. Induction on k and for fixed k on decreasing $\Sigma_{i=1}^{k} \dim(U_i \cap V)$. (Note that if $(a+U) \cap V \neq \emptyset$ then $\dim((a+U) \cap V) = \dim(U \cap V)$, in fact $(a+U) \cap V = b + (U \cap V)$ for some $b \in (a+U) \cap V$.) If $k = 1$ then the statement of the theorem is obvious. Now assume $k > 1$. Let $n := \dim V$. Since the union is irredundant V meets all $a_i + U_i$ and since $k > 1$ it follows that $\dim(U_i \cap V) \le n-1$ for all i. If $\dim(U_i \cap V) = n-1$ for all i, then $W = V \cap \bigcap_{i=1}^{k} U_i$ implies $\dim W \ge \dim V - k$, and we are done unless $\dim W = \dim V - k$. But in the latter case $\dim(V \setminus \bigcup_{i=1}^{k} (a_i+U_i)) \ge \dim W \ge 0$ so that $V \setminus \bigcup_{i=1}^{k} (a_i+U_i) \neq \emptyset$, a contradiction.

Consider $W_I := V \cap \bigcap_{i\notin I} U_i$. Then $W_\emptyset = W$.

LEMMA. *If* $0 < |I| < k$ *then* $\dim W_I \le |I| + \dim W - 1$. *In particular* $W_{\{i\}} = W$.

PROOF. Induction on $|I|$. $V \setminus \bigcup_{i\notin I} (a_i+U_i)$ is a nonempty union of translates of W_I, so that for some a we have $a + W_I \subset \bigcup_{i\in I} (a_i+U_i)$. If this union is irredundant than by the theorem (applied with $|I|$ instead of k) we find

$\dim W_I \le |I| + \dim W - 1$ (note that $W_I \cap \bigcap_{i\in I} U_i = W$). On the other hand, if the union is redundant then we may choose $J \subsetneq I$ such that $a + W_I \subset \bigcup_{i\in J} (a_i+U_i)$ and this latter union is irredundant. By the theorem and the induction hypothesis we find

$$\dim W_I \le |J| + \dim W_{I\setminus J} - 1 \le |J| + |I\setminus J| + \dim W - 2$$

$$< |I| + \dim W - 1. \qquad \square$$

Returning to the proof of the theorem: we shall carry out the induction by either enlarging some U_i or reducing the number of subspaces k. We may suppose that $\dim(U_g \cap V) < n-1$ for some g $(1 \le g \le k)$. Set $U_g' = U_g \cup (a+U_g)$ and and $U_i' = U_i$ for $1 \le i \le k$, $i \ne g$ where a is chosen such that $\dim((a_g+U_g') \cap V) > \dim((a_g+U_g) \cap V)$. Now $V \subset \bigcup_{i=1}^{k} (a_i+U_i')$ and $W' := V \cap \bigcap_{i=1}^{k} U_i' = W$ (for: $W \subseteq W' \subset W_{\{g\}} = W$) so if the union is irredundant we succeeded in reducing the problem to one with larger U_g. On the other hand, if the union is redundant then we may choose I such that $g \notin I$ and $V \subset \bigcup_{i\notin I} (a_i+U_i')$ is irredundant. Since $\dim(U_g' \cap V) < n$ we have $|I| < k-1$ so that by the lemma $\dim W' = \dim(U_g' \cap W_{I\cup\{g\}}) \le \dim W_{I\cup\{g\}} \le |I| + \dim W$. By the theorem (applied with $k-|I|$ instead of k) we find

$$\dim V \le k - |I| + |I| + \dim W - 1 = k + \dim W - 1. \qquad \square$$

<u>REMARK</u>. It is natural to ask what happens for vector spaces over $GF(q)$ with $q > 2$. It is easy to see that there are examples with $k = (n-1)(q-1) + 2$ where $n = \dim V$. We have seen that $k \ge (n-1)(q-1) + 2$ for $q = 2$, and it is trivial to prove the same inequality for $n = 2$. But already for $n = 3$ smaller k occur: First rephrase the problem as a projective problem, and then dualize. Now our problem is:

> "Let V be a projective space of dimension $n+1$ over $GF(q)$ which is spanned by k subspaces U_i $(1 \le i \le k)$ such that any hyperplane contains at least one of the U_i, and where there are hyperplanes H_i such that H_i does not contain any U_j $(j \ne i,\ 1 \le i \le k)$.
> Find a lower bound for k."

In the special case $n = 3$ we get $\dim V = 2$ and ask for a minimal blocking set (with less than $2q$ elements). If q is a square then a Baer subplane will do - it provides us with an example with $q + \sqrt{q} + 1$ elements. Also when q is not a square one may have $k < 2q$. For example, if $q = 5$ one may take 4 points on a line and 5 points forming a transversal of the remaining two parallel classes. This gives $k = 9$. (See HIRSCHFELD [2], Ch. 13 for a discussion of blocking sets.)

Note that for $q = 2$, $n = 3$ we have a blocking family $\{U_i\}_i$ consisting of two points and two lines, but a blocking set consisting of points only does not exist. It is easily seen that for $q \geq 3$ we may restrict attention to blocking sets, and thus $k \geq q + \sqrt{q} + 1$, with equality precisely in case of a Baer subplane.

The case $n > 3$ remains completely open.

REFERENCES

[1] GANTER, BERNARD, letter to J.A. Thas, dated 23.6.80.

[2] HIRSCHFELD, J.W.P., *Projective geometries over finite fields*, Clarendon Press, Oxford, 1979.