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THE NONEXISTENCE OF A REGULAR NEAR HEXAGON ON 1408 POINTS

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The nonexistence of a regular near hexagon on 1408 points *)

by

A.E. Brouwer

ABSTRACT

There are three parameter sets of sporadic regular near hexagons with lines of size 4. The smallest has 1408 vertices and $(s,t,t_2) = (3,9,1)$. Here we show that there is no regular near hexagon with these parameters.

KEY WORDS & PHRASES: *near hexagon*

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*) This report will be submitted for publication elsewhere.

INTRODUCTION

A *near hexagon* is a partial linear space (X,L) such that

a) For any point $p \in X$ and line $\ell \in L$ there is a unique point on ℓ nearest p.

b) Every point is on at least one line.

c) The distance between any two points is at most three.

(The distances are measured in the point graph: $d(p,q) = 1$ iff p and q are collinear.)

A *regular* near hexagon with parameters (s,t,t_2) is a near hexagon such that each line contains 1+s points, each point is in 1+t lines, and a point at distance 2 from a fixed point x_0 is in $1+t_2$ lines containing a neighbour of x_0. A *sporadic* regular near hexagon is one that is not a generalized quadrangle, a generalized hexagon, or a dual polar space.

There are exactly 12 regular near hexagons with lines of size 3, two of which are sporadic (they are connected to the ternary Golay code and the Witt design S(5,8,24), respectively).

Now looking at sporadic regular near hexagons with s=3 we find (applying the conditions that the multiplicities of the associated association scheme are integral, and a few trivial divisibility conditions and inequalities) exactly three feasible parameter sets, namely

(i) $v = 1408$, $s = 3$, $t = 9$, $t_2 = 1$,

(ii) $v = 20020$, $s = 3$, $t = 48$, $t_2 = 3$,

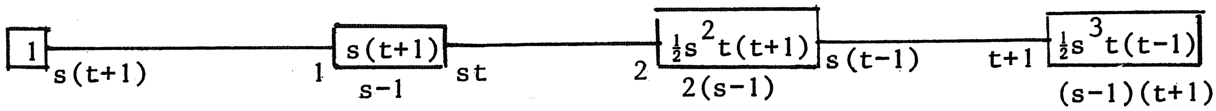
(iii) $v = 20608$, $s = 3$, $t = 34$, $t_2 = 1$.

E. SHULT (oral communication) excluded possibility (ii). Here we kill case (i).

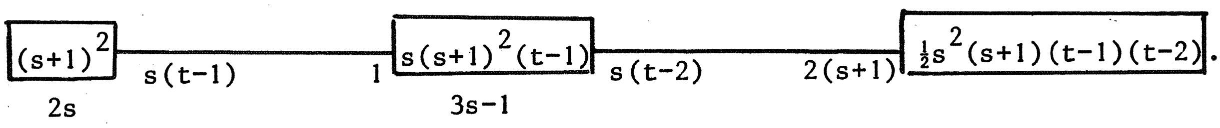
1. COUNTING QUADS

For the definition of a quad and the basic properties of quads see SHULT & YANUSHKA [1]. If H is sporadic, then quads exist, and there exist points of ovoid type w.r.t. a quad. Now let $t_2=1$. Then a quad is a generalized quadrangle GQ(s,1), i.e., a $(s+1)\times(s+1)$ square lattice.

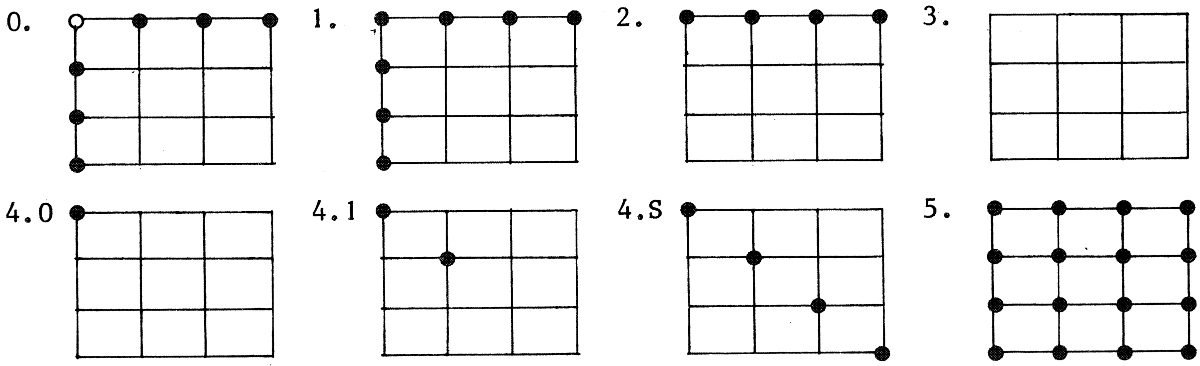
The diagram of H w.r.t. a point is



The diagram of H w.r.t. a quad is



Let Q_0 be a fixed quad. Any quad containing a point at distance two from Q_0 or contained in $\Gamma_1 Q_0$ is of one of the following types - a black dot denotes a point at distance one from Q_0, an open circle a point in Q_0.



(Type 4.i has i+1 black dots along a transversal, $0 \le i \le s$.)

Let there be N_α quads of type α.

Clearly

(1) $$N_0 = (s+1)^2 . \binom{t-1}{2}.$$

Counting pairs $(x, \{\ell, \ell'\})$ with $x \in \Gamma_1 Q_0$, $\ell, \ell' \subset \Gamma_1 Q_0$, x on ℓ and ℓ' we find

(2) $$N_1 + (s+1)^2 N_5 = s(s+1)^2(t-1).$$

Counting quads containing a line in $\Gamma_1 Q_0$ we find

(3) $$2N_1 + N_2 + 2(s+1)N_5 = 2s(s+1)(t-1)^2$$

so that also

(3A) $$N_2 - 2s(s+1)N_5 = 2s(s+1)(t-1)(t-s-2)$$

and

(3B) $$2sN_1 + (s+1)N_2 = 2s(s+1)^2(t-1)(t-2).$$

Counting vertices in $\Gamma_2 Q_0$ and pairs of incident lines both meeting $\Gamma_1 Q_0$ we see

$$s^2N_0 + s^2N_1 + \sum_{i=1}^{s} i(i+1)N_{4.i} = \tfrac{1}{2}s^2(s+1)(t-1)(t-2)\cdot(s+1)(2s+1)$$

so that

(4) $$s^2N_1 + \sum_{i=1}^{s} i(i+1)N_{4.i} = s^3(s+1)^2(t-1)(t-2).$$

Similarly, if only one of the two lines meets $\Gamma_1 Q_0$ we find

(5) $$s(s+1)N_2 + \sum_{i=0}^{s-1} 2(i+1)(s-i)N_{4.i} = s^2(s+1)^2(t-1)(t-2)(t-2s-1)$$

and if both lines are contained within $\Gamma_2 Q_0$ we get

(6) $$(s+1)^2N_3 + \sum_{i=0}^{s-1} (s-i)^2N_{4.i} = \tfrac{1}{2}s^2(s+1)(t-1)(t-2)\binom{t-2s-1}{2}.$$

From (3B), (4) and (5) it follows that

(7) $$\sum_{i=0}^{s} (i+1)N_{4,i} = \tfrac{1}{2}s(s+1)^2(t-1)(t-2)(t-3).$$

It seems that all other relations between the N_α obtained by counting various things are a consequence of the above equations.
Put $N_1 = s(s+1)^2\gamma$, for some rational γ. (Below we shall see that in fact γ is an integer.) Then

$$N_2 = 2s(s+1)((t-1)(t-2)-s\gamma)$$

$$N_5 = s(t-1-\gamma)$$

so that $0 \leq \gamma \leq t-1$.

2. CUBES

Let ℓ_0,ℓ_1,ℓ_2 be three lines through a point x_0, and let $Q_0 = Q(\ell_1,\ell_2)$ $Q_1 = Q(\ell_0,\ell_2)$, $Q_2 = Q(\ell_0,\ell_1)$ be the three quads spanned by two of these lines. Suppose that there is a point y adjacent to each of the three quads but nonadjacent to x_0. Then the collection of all such points y together with the three quads forms a set C called a *cube* with the following properties:

$|C| = (s+1)^3$, each point of C is on three lines in C, each line in C is on two quads in C, C is closed under the formation of quads (i.e., the quad determined by two intersecting lines or two points at distance two in C is contained within C)- in particular the distance between two points is the same measured in C as it was in H.

PROOF. If z is a point adjacent to Q_i, then let $\pi_i(z)$ be its unique neighbour in Q_i $(i = 0,1,2)$. Let y be a point as in the hypothesis, and put $y_1 = \pi_1(y)$, $y_2 = \pi_2(y)$. Let m_i be the line in Q_i passing through y_i and meeting ℓ_0 $(i = 1,2)$. Then m_1 and m_2 meet ℓ_0 in the same point z (namely the point on ℓ_0 closest to y). Let $Q' = Q(m_1,m_2)$. Now $Q' \subset \Gamma_1 Q_0$: the set of points in Q' at distance ≤ 1 from Q_0 is line closed and contains y,y_1,y_2,z hence equals Q'; but Q' cannot intersect Q_0 otherwise $Q' \cap Q_0$ would be a line intersecting yy_1 or yy_2, say yy_1, - but now y_1 has two neighbours in Q_0, a contradiction.

Now let u,v,w be three points on ℓ_0,ℓ_1,ℓ_2, respectively. We shall show that there is a unique point $x \in C$ such that u,v,w are just the points on ℓ_0,ℓ_1,ℓ_2 closest to x.

First suppose u = z. Then let v' be the neighbour of v on m_2 in Q_2 and w' the neighbour of w on m_1 in Q_1. There is a unique point $x \in Q'$ adjacent to v' and w' different from z. Now $x \in C$ since $d(x,Q_0) = 1$.

Since any point of $Q'\backslash m_1,m_2$ can take the rôle of y and the rôles of the Q_i can be interchanged this proves everything. □

Let Q be a quad, and Q' a quad of type 5 w.r.t. Q (i.e. $Q' \subset \Gamma_1 Q$).

Then Q and Q' determine a cube C, and C contains s quads of type 5 w.r.t. Q so that $s|N_5$. From equation (2) we see that $s(s+1)^2|N_1$ so that γ is an integer, as promised.

<u>LEMMA</u>. $\gamma = 0$ *or* $\gamma \geq s+1$.

<u>PROOF</u>. Let Q_0 be a fixed quad. If x,y are adjacent points in Q_0, and ℓ is a line meeting Q_0 in the point x, then define $\tau_{xy}(\ell)$ to be the line in the quad $Q(\ell,xy)$ meeting Q_0 in the point y. Suppose that Q_0 and ℓ do not determine a cube.

<u>CLAIM</u>. If x,y,z,u are four points forming a square in Q_0 (i.e., $x \sim y \sim z \sim u \sim x, x \not\sim z$) then $(\tau_{ux} \circ \tau_{zu} \circ \tau_{yz} \circ \tau_{xy})(\ell) \neq \ell$.
<u>For</u>: let x' be a point on ℓ distinct from x, y' its neighbour on $\tau_{xy}(\ell)$, z' the neighbour of y' on $(\tau_{yz} \circ \tau_{xy})(\ell)$, u' the common neighbour of u and z' distinct from z and x" the common neighbour of u' and x distinct from u. If x,x',x" are collinear and $x' \neq x''$ then we have the 5-gon x'y'z'u'x" so that z' must be collinear with a point of ℓ, impossible.
If $x' = x''$ then the three lines ℓ,xu,xy and the point z' take the place of ℓ_0,ℓ_1,ℓ_2 and y in the discussion above so that ℓ and Q_0 determine a cube. □

Now fix x,y and consider the s distinct choices for z and u. Now the line ℓ and the s lines $(\tau_{ux} \circ \tau_{zu} \circ \tau_{yz} \circ \tau_{xy})(\ell)$ are pairwise distinct so that there are at least s+1 lines through x that do not determine a cube together with Q_0. But γ is exactly the number of lines through a fixed point of Q_0 not in a cube containing Q_0. □

<u>LEMMA</u>. *If* $\gamma = 0$ *(for all quads* Q_0*) then* $3|(t+1)$.

<u>PROOF</u>. Let x_0,x_1 be two points at distance 3. Define a graph on the t+1 lines through x_0 by calling two lines ℓ,ℓ' adjacent when $d(x_1,Q(\ell,\ell')) = 1$. This graph has valency $t_2+1 = 2$, i.e., is a union of polygons.
If ℓ,ℓ',ℓ'' are three consecutive lines in one of these polygons and ℓ,ℓ',ℓ'' determine a cube then also ℓ,ℓ'' are adjacent and we have a triangle. If $\gamma=0$ for all quads then any three lines determine a cube and the graph is a union of triangles. □

3. THE NEAR HEXAGON ON 729 POINTS

As an illustration consider the (unique) near hexagon with parameters $(s,t,t_2) = (2,11,1)$. Here one has $N_0 = 405$, $N_2 = 1080$, $N_3 = 540$, $N_{4.1} = 3240$, $N_5 = 20$; the number of points is 729; the number of quads is 5346. $N_1 = N_{4.0} = N_{4.2} = 0$, i.e., any three intersecting lines determine a cube. {As follows: from geometrical considerations one concludes that $N_{4.0} = N_{4.2} = 0$; now the equations have a unique solution. For details see [2].}

4. A NEAR HEAXAGON ON 1408 POINTS

Next consider a possible regular near hexagon with parameters $(s,t,t_2) = (3,9,1)$. Our equations are:

$$N_0 = 448$$

$$N_1 = 48\gamma$$

$$N_2 = 24(56-3\gamma)$$

$$N_5 = 3(8-\gamma)$$

(7') $$N_{4.0} + 2N_{4.1} + 3N_{4.2} + 4N_{4.3} = 8064$$

(4') $$2N_{4.1} + 6N_{4.2} + 12N_{4.3} = 24192 - 432\gamma$$

(6') $$16N_3 + 9N_{4.0} + 4N_{4.1} + N_{4.2} = 1008.$$

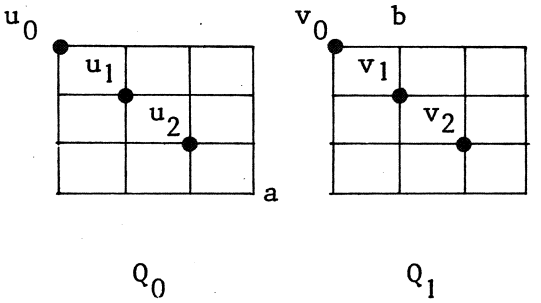
From geometric considerations we shall see that $N_{4.2} = 0$. Using this we find from (4') and (7') that

$$N_{4.0} + \frac{4}{3} N_{4.1} = 144\gamma,$$

and comparing with (6'), that $3.144\gamma \le 1008$, i.e., $\gamma \le 2$. Now the lemma in the previous section tells us that $\gamma=0$ so that any three intersecting lines determine a cube. But $3 \nmid (t+1)$, contradiction. Hence there is no hexagon with these parameters.

Remains to show that $N_{4.2} = 0$.

Let Q_0 and Q_1 be two quads such that Q_1 is of type 4.2 w.r.t. Q_0.



Let $u_i \in Q_0$, $v_i \in Q_1$, $d(u_i,v_i) = 1$ $(i = 0,1,2)$.
Let a be the point in Q_0 such that $\{a,u_0,u_1,u_2\}$ is an ovoid of Q_0. Then $d(a,Q_1) = 2$ so that there are four points in Q, at distance two from a, forming an ovoid.
At least two of these points are adjacent to two of the v_i (note that $d(a,v_i) = 3$, $i = 0,1,2$) - so suppose $d(a,b) = 2$, $b \in Q_1$, $b \sim v_0,v_1$. Now $d(b,Q_0) = 2$ and b determines an ovoid in Q_0. This ovoid contains u_0,u_1 and a hence also u_2, but $d(b,u_2) = 1 + d(b,v_2) = 3$, contradiction.

REFERENCES

[1] SHULT, E.E. & A. YANUSHKA, *Near n-gons and line systems*, Geometriae Dedicata 9 (1980) 1-72.

[2] BROUWER, A.E., *The uniqueness of the near hexagon on 729 points*, to appear.

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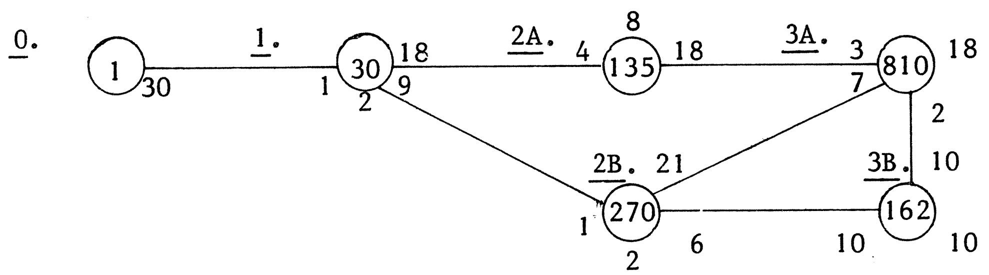
ADDED IN PROOF. A nonregular near hexagon on 1408 points.

E. Shult suggested the following description of an association scheme of which he thought that it would have the same parameters as the association scheme corresponding to our regular near hexagon on 1408 vertices. (But he couldn't find the lines of the near hexagon.)

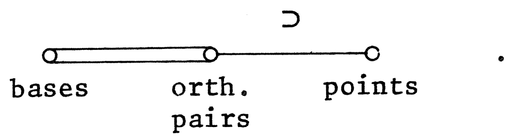
Take the 176 points not lying on a non-degenerate Hermitian quadric in $PG(4,2^2)$. In the graph on these 176 points with orthogonal pairs adjacent, there are 1408 5-cliques (orthogonal bases).

A given basis intersects itself in 5 points, 30 bases in 2 points, 135 in 1 point and is disjoint from 1242 bases.

Call two bases adjacent if they have 2 points in common. This defines a structure much resembling the association scheme looked for, but it turns out to be a 5-class association scheme instead of a 3-class one. Its diagram is



Given a pair of orthogonal points, it is contained in 4 bases. Call such quadruples of bases having a pair in common, lines. From the diagram above it is clear that this gives us a near hexagon with $s = 3$, $t = 9$ and $t_2 \in \{0,3\}$. Pairs of bases in relation 2B are joined by a unique path of length two; pairs of bases in relation 2A determine a generalized quadruple GQ(3,3) consisting of all 40 bases containing their common point. Thus the quads are in 1-1 correspondence with the 176 points. Two distinct quads are disjoint or intersect in a line; this means that all point-quad relations are classical. We find the Buekenhout diagram



(AEB, H.A. Wilbrink, A.M. Cohen - 81 07 15)