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TWO INFINITE SEQUENCES OF NEAR POLYGONS

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Two infinite sequences of near polygons*)

by

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ABSTRACT

We construct near polygons (non-regular, but 'with quads') belonging to the groups Sym(2n) and $O^{+}(2n,2)$.

KEY WORDS & PHRASES: *near polygon; Zara graph*

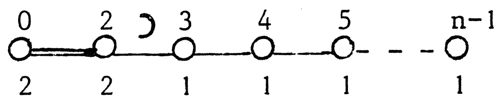
*) This report will be submitted for publication elsewhere.

A *near polygon* is a connected partial linear space (X,L) such that for any piont $p \in X$ and line $\ell \in L$ there is a unique point on ℓ nearest p. (These objects were introduced in SHULT & YANUSHKA [8]; a structure theory is developed in BROUWER & WILBRINK [3]; the connection with dual polar spaces stems from CAMERON [6]. Other interesting examples can be found in ASCHBACHER [1] and COHEN [7].) A subset $Y \subset X$ is called *geodetically closed* if for any two points $y_1, y_2 \in Y$ all shortest paths between y_1 and y_2 are contained in Y. A *quad* is a geodetically closed subspace of X of diameter two which is nondegenerate (i.e., not all of its points are adjacent to one fixed point); it follows that a quad is a generalized quadrangle.

[Here distances are measured in the collinearity graph on the points.]

1. Sym(2n)

Our first sequence of examples has Buekenhout-Tits diagram



(n-1 nodes)

and group Sym(2n) and is defined as follows:
Consider the complement of the triangular graph $\overline{T(2n)}$ (i.e., the graph with point set $\binom{2n}{2}$, the unordered pairs from a 2n-set, and two pairs being adjacent when they are disjoint;
i.e., the commuting graph on the transpositions of Sym(2n);
i.e., the Fischer space Sym(2n) together with its 2-lines).
The i-objects of our geometries are (n-i)-cliques in this graph ($i = 0,2,3,\dots,n-1$). We call 0-objects *points* and 2-objects *lines*. It is obvious that this geometry has the stated group and diagram. We shall show that the points and lines are the points and lines of a near polygon.

LEMMA. *Let* x *and* y *be two points. The distance* d(x,y) *of* x *and* y *in the collinearity graph on the points is* n-m, *where* m *is the number of connected components of* $x \cup y$ *regarded as a 2-factor of the complete graph* K_{2n}.

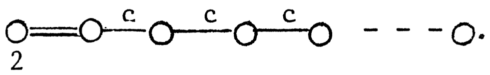
PROOF. Points are n-cliques in the graph $\overline{T(2n)}$, i.e., are partitions of a 2n-set in n pairs, i.e., are complete matchings of the complete graph K_{2n}. The union of two such matchings is a regular graph of valency two on 2n vertices and hence a union of 2k-circuits (here k=1 is allowed). The claim of the Lemma is that each such component contributes k-1 to the distance d(x,y). The proof is by induction on d(x,y): if d(x,y) = 0 then x = y and all components of $x \cup y$ are 2-circuits (doubled edges); if d(x,y) > 0 then let z be a neighbour of y such that d(x,z) = d(x,y)-1. By induction we know that $x \cup z$ has precisely n-d(x,z) components, and since y and z are collinear, they have n-2 pairs in common, so that y is obtained from z by replacing two pairs by two other pairs, covering the same 4 points. Clearby this can change the number of components by at most one so that $x \cup y$ has at least n-d(x,y) components. But conversely, we have $d(x,y) \leq n-m$ since given a 2k-circuit -p-q-r-s- ... with pq and rs in y and qr in x, we can find a neighbour z of y containing ps and qr, so that $x \cup z$ has one more component than $x \cup y$. □

PROPOSITION. (X,L) := ({0-objects}, {2-objects}) *is a near* 2(n-1)-gon, *i.e., a near polygon of diameter* n-1.

PROOF. (i) By the Lemma, d(x,y) is maximal (and equals n-1) when $x \cup y$ is connected, a 2n-circuit.
(ii) Given a line L (i.e., a partial matching consisting of n-2 edges) and a point x (i.e., a complete matching), consider $x \cup L$. This is a graph consisting of a number of closed circuits and two paths beginning and ending in a point not covered by L. There is a unique way of completing these two paths to two circuits, so there is a unique point on L closest to x. □

The near polygon (let us call it H_{n-1}) has 3 points on each line, $\binom{n}{2}$ lines on each point, and the i-objects are geodetically closed sub near 2(i-1)-gons (i = 2,3,...,n-1). In particular, the 3-objects are classical GQ(2,2) generalized quadrangles (on 15 points, with group Sp(4,2)).
Not any two lines on a point determine a unique 3-object, but they do determine a unique quad: they determine a GQ(2,2) quad whenever they have n-3 pairs in common, and a GQ(2,1) quad (the direct product of two lines) otherwise.
More generally, every two points x,y at distance j determine a unique geodetically closed sub 2j-gon H(x,y); if $x \cup y$ is the union $\cup C_{2k}^{(i)}$ of 2k-circuits then H(x,y) is the direct product $\Pi H_{k-1}^{(i)}$ of sub 2(k-1)-gons $H_{k-1}^{(i)}$ which are k-objects of the geometry (where 1-objects are identified with 0-objects).

The full geometry (with geodetically closed sub 2j-gons as objects ($j = 0,1,\ldots,n-2$)) has Buekenhout-Tits diagram



The derived geometry at a point is the geometry of partitions of an n-set.

2. $O^+(2n,2)$

Let Q be a nondegenerate quadric in PG(2n,2), and H a hyperbolic hyperplane, i.e., a hyperplane such that $H \cap Q$ has (maximal) Witt index n. Let X be the set of all maximal totally isotropic subspaces not contained in H and L the set of all totally isotropic subspaces of dimension n-1 not contained in H.

PROPOSITION. (X,L) *(with reverse inclusion as incidence) is a near polygon.*

PROOF. First remark that $d(x,y) = \mathrm{codim}_x(x\cap y)$ for $x,y \in X$. Indeed, this holds in the full polar space and one easily checks that our removal of H has not enlarged distances. Next, that our lines are full lines of the dual polar space so that (X,L) inherits the near polygon property. □

This near polygon (let us call it P_n) has diameter n, 3 points on each line and 2^n-2 lines on each point. For $i \geq 2$, the geodetically closed sub near 2i-gons are the totally isotropic (n-i)-spaces. They come in two kinds: those contained in H and those not contained in H. In particular, the quads are either GQ(2,1) or GQ(2,2) quads.

3. A SPORADIC ISOMORPHISM

PROPOSITION. $H_3 \cong P_3$.

PROOF. Suspicion that this might be the case is aroused by the coinciding parameters $v = 105$, $s = 2$, $t = 5$ and the coinciding groups $A_8 \cong P\Omega^+(6,2)$. Now, exploiting $A_8 \cong PSL(4,2)$, we see that the lines of PG(3,2) can be labelled with triples from a 7-set such that intersecting lines correspond to triples having one element in common. Under this labelling, points and planes correspond to Steiner triple systems on 7 points, and in this way we find all (30) such systems.

(Conversely, starting with the set of all 30 STS(7)'s on a fixed 7-set one finds that "meeting in one triple" is an equivalence relation splitting the 30 into 15+15. Now with "meeting in three triples" as adjacency we find the point-plane incidence graph of PG(3,2).)

Now the hyperplane H used in the construction of P_3 carries the structure of a geometry of type D_3 and hence is isomorphic to the geometry of points, lines and planes in PG(3,2). By the above remarks this means that the 30 maximal totally isotropic subspaces (planes) of H can be labelled with STS(7)'s such that two planes meet in a line iff the corresponding STS(7)'s have three triples in common.

Finally, given a point x in the near heaxagon P_3, i.e., a totally isotropic plane on Q we find a line $x \cap H$ carrying two totally isotropic planes x_1 and x_2 contained in H. Viewed as Steiner triple systems, these meet in three triples abc, ade, afg, and we label x with (∞a) (bc)(de)(fg) where ∞ is a new symbol. This defines a map i: $P_3 \to H_3$. If x and x' are adjacent points in P_3 then $x \cap x' \cap H$ is a point (of the D_3 geometry, corresponding with a line in the A_3 geometry PG(3,2)) labelled with a triple (one of abc, ade, afg), say abc. It follows that x' is labelled w.l.o.g. either (∞a) (bc)(df)(eg) (and $i(x) \sim i(x')$) or (∞b)(ac)(--)(--). But since the union of the set of 3 triples belonging to x and the set of 3 triples belonging to x' cannot be contained in an STS(7) it follows that the latter set is not abc, bde, bfg but abc, bdf, beg and consequently x' is called (∞b)(ac)(df)(eg). Thus $i(x) \sim i(x')$ in all cases, and i is an isomorphism. □

4. A CHARACTERIZATION OF H_3

In BROUWER, COHEN & WILBRINK [4] a geodetically closed proper sub near polygon Q of a near polygon H is called *big* when each point of H is collinear with a point of Q, and all near hexagons with quads and with lines of size 3 containing a big 27-point quad are determined.
Here we determine all near hexagons with quads and with lines of size 3

containing a big quad.

THEOREM. *There are, up to isomorphism, 8 distinct near hexagons with quads and with lines of size 3 containing a big quad. Their parameters are:*

	v	t+1	t_2+1	group	# of big quads	Fischer space
(i)	27	3	2	Sym(3) ~ Sym(3)	9	3 lines F_3
(ii)	45	4	2,3	Sym(3) × Sym(6)	3	1 line
(iii)	105	6	2,3	Sym(8)	28	triangular graph T(8)
(iv)	135	7	3	$PO_7(2)$	63	polar space $O_7(2) \cong Sp(6,2)$
(v)	81	6	2,5	Sym(3) × $P\Gamma U(4,2^2)$	3	1 line
(vi)	243	9	2,5	$(\mathbb{Z}_3 \times AGL(2,3) \times AGL(2,3)).2$	18	2 planes F_9
(vii)	567	15	3,5	$PO_6^-(3)$	126	$\{x \mid Q(x)=1\}$ in PG(5,3); Q quadratic form with nonmax. Witt index
(viii)	891	21	5	$P\Gamma U(6,2^2)$	693	polar space $U(6,2^2)$.

PROOF. (We assume the terminology and results of [4] known.) If a big quad Q contains q points then $v = q+sq(t-t_2)$ where Q has t_2+1 lines on each point. For a quad with lines of size 3 only q = 9,15,27 is possible. If q = 27 then the work has been done in [4] and we find cases (v)-(viii). If q = 9 then all quads are 9-point quads unless t = 3, v = 45 and also 15-point quads are big. But if all quads are 9-point quads we have case (i). Thus we may assume q = 15.

LEMMA. *Let* H *be a near polygon with quads containing a big subspace* Q *such that each point of* Q *is on a unique line not contained in* Q. *Then* H *is the direct product* Q × L *of* Q *with a line* L.

(Sketch of the easy) PROOF. 0. This is obvious when Q is (a point or) a line.

1. Define the projection π: H → Q by $x \sim \pi x \in Q$ for $x \in H\backslash Q$ and $\pi x = x$ for $x \in Q$. Then π is well defined and the projection of a line not meeting Q is again a line.
2. Given a line $\ell \subset Q$ and a point x with $\pi x \in \ell$ there is a unique line ℓ_0 on x with $\pi\ell_0 = \ell$.
3. Given a square in Q and a point x with πx on the square, there is a unique square on x projecting to the given square.

4. Given two geodetes in Q with the same endpoints one can transform one into the other by a homotopy where an elementary homotopy consists in replacing a subpath of length 2 by another (going along the other side of a square).
5. Now everything is clear. □

In our case we find for $q = 9,15,27$ the cases (i), (ii), (v) of the Theorem. Now we may assume $t+1 \geq 5$.

If Q and Q' are quads and Q is big then $Q \cap Q'$ cannot be a single point (otherwise Q' would contain points at distance 2 from Q); in the local spaces L_x this means that each line meets every 3-line.

LEMMA. *Let* τ *be a collection of triples such that any two have precisely one point in common. Then either all triples of* τ *have a common element, or* τ *can be extended to the Fano plane* PG(2,2).

PROOF. Easy verification. □

If some local space L_x has at least four 3-lines on a point u then any line not on u has size at least four, contradiction. If some local space L_x has precisely three 3-lines on a point u then all other lines are 3-lines and L_x is a Fano plane. Now it follows that L_x is a Fano plane for each x, i.e. all quads are 15-point quads and we have case (iv).

Now we may assume that in no local space L_x there are three 3-lines on a point. It follows that $5 \leq t+1 \leq 6$.

If some local space has 2-lines only then there is a local space L_x with a 3-line and with a point u that is not in a 3-line. But such a local space can have only 4 points, and t+1 would be 4, contradiction.

If $t+1 = 5$ then all local spaces L_x are unions of two 3-lines; we find $v = 75$ and 10 big quads. Each big quad intersects 5 others so that we find a Fischer space on 10 points with two 3-lines on each point, impossible.

Thus $t+1 = 6$ and $v = 105$. All local spaces L_x look like the Fischer space F_6. There are 28 big quads, and each big quad intersects 15 others.

The Fischer space on the big quads has 28 points and 6 3-lines on each point. By BUEKENHOUT [5] this can be nothing but $\binom{8}{2}$, the Fischer space of the transpositions in Sym(8).
Having identified the big quads with pairs from an 8-set such that intersecting quads correspond to disjoint pairs we see that each point corresponds to a complete matching, and since there are as many points as matchings, each matching occurs. This identifies the near hexagon as the H_3 of the previous section. □

APPENDIX - A note on Zara graphs

F. ZARA [9] introduced the 'graphs related to polar spaces' - let us call these Zara graphs. A Zara graph is a graph (finite, undirected, without loops or multiple edges) satisfying

(Z1) all maximal cliques have the same cardinality K,

(Z2) if K is a maximal clique and $x \notin K$ then x is adjacent to precisely e points of K, where e is independent of x and K.

It is possible to show (BLOKHUIS & KLOKS [2]) that a Zara graph satisfying certain nondegeneracy conditions is strongly regular.

Given a Zara graph one may construct a geometry Γ with as Points the maximal cliques, and as Lines the e-sets that are intersections of maximal cliques. Since Zara graphs resemble collinearity graphs of polar spaces one might hope that the geometries thus constructed resemble dual polar spaces, for example, that they are near polygons.

Let us have a quick look at the known examples of Zara graphs.

A. Collinearity graphs of polar spaces.
Here Γ is a dual polar space and in particular a (classical) near polygon.

B. $\overline{T(2m)}$ - the complement of the triangular graph on 2m symbols.
Here Γ is the near polygon H_{m-1} discussed in section 1.

C. The orthogonality graph on the nonisotropic points in a vector space of dimension 2m over the field $\mathbb{F}_2$ equipped with a nondegenerate quadratic form of maximal Witt index. (With $v = 2^{m-1}(2^m-1)$, $K = 2^{m-1}$, $e = 2^{m-2}$.)
[By projecting from the nucleus into H one easily sees that this graph is isomorphic to the orthogonality graph of the isotropic points off a given hyperbolic hyperplane H in PG(2m,2) equipped with a nondegenerate quadratic form.]
Here Γ is the near polygon P_m discussed in section 2.

D. $\overline{L_2(m)}$ - the complement of the graph of an $m \times m$ lattice.
(With $v = m^2$, $K = m$, $e = m-2$.)
Here Γ is a near polygon with quads in a natural way: it has Lines of size $s+1 = 2$ and $t + 1 = \binom{m}{2}$ Lines on each Point; the point graph is bipartite (Points can be associated with even and odd permutations of m symbols) so we have a near polygon; quads are subgraphs of type $K_{2,2}$ or $K_{3,3}$.

E. The graph on the vectors of a vector space of dimension 2m over the field $\mathbb{F}_q$ equipped with a nondegenerate quadratic form Q of maximal Witt index, two vectors x,y being adjacent when $Q(y-x) = 0$.

Here again Γ is a near polygon with Lines of size 2 (the Points of Γ are translates of maximal totally isotropic subspaces. the distance between two Points is the codimension of their intersection in one of them). The subgeometry of Γ consisting of all Points containing the origin is a classical dual polar space.

F. The orthogonality graph on $\{X \mid Q(x) = 1\}$ in PG(5,3) equipped with a quadratic form Q of minimal Witt index.

(With v = 126, K = 6, e = 2.)

Here Γ is the Aschbacher near hexagon (see ASCHBACHER [1], BROUWER, COHEN & WILBRINK [4]).

G. The graph with as vertices the lines of PG(3,2), disjoint lines being adjacent.

(With v = 35, K = 5, e = 2.)

Here Γ is a geometry with Lines of size s+1 = 2 and t+1 = 10 Lines on each Point. Its point graph has V = 56 vertices, valency k = 10, is edge-regular with λ = 0, and two nonadjacent vertices have either 1 or 4 common neighbours. The graph is not bipartite, so Γ is not a near polygon.

(The complement of this graph is also a Zara graph, with parameters v = 35, k = 7, e = 3. Alternative descriptions for this graph are:

(i) it is the polar space $D_3(2)$ (i.e., $O_6^+(2)$) - so it belongs under A.

(ii) it is the graph with as vertices the $\frac{1}{2}\binom{8}{4}$ partitions of an 8-set into two 4-sets, two partitions being adjacent when they intersect as 2^4 (not $1^2 3^2$).)

H. The McLaughlin graph with parameters v = 275, K = 5, e = 2.

Here Γ is a geometry with s+1 = t+1 = 10. From the parameters one easily sees that Γ cannot be a near polygon.

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