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ON THE PERIOD OF AN OPERATOR, DEFINED ON ANTICHAINS

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ABSTRACT

In this report we will study the period of a certain operator defined on antichains in a partially ordered set.

0. INTRODUCTION

If $(X,\leq)$ is a partially ordered set, then define for each $A \subset X$:

$$A^{\uparrow} = \{x \in X \mid \exists a \in A: a \leq x\},$$
$$A^{c} = \{x \in X \mid x \notin A\},$$
$$A^{max} = \{a \in A \mid \forall b \in A: b \geq a \rightarrow b = a\},$$

Set $A(X)$ = the set of antichains in X (order-free subsets of X), and for each $A \in A(X)$: $F(A) = A^{\uparrow c max}$.

Then one can easily see: F is a bijection of $A(X)$ onto $A(X)$, and for each $A \in A(X)$ there exists a $k > 0$ so that $F^k(A) = A$.

Motivated by an abundance of examples we conjectured that, if X is a Boolean algebra with 2^n elements, for each antichain $A \in A(X)$ the relation $F^{n+2}(A) = A$ would be valid. This conjecture turned out to be wrong. However, the following can be proved:

1. If $n \leq 4$ and X is a Boolean-algebra with 2^n elements, then for each $A \in A(X)$: $F^{n+2}(A) = A$.
2. For each $n \in \mathbb{N}$ the following propositions are equivalent:
   a. If X is a Boolean algebra with 2^n elements and $A \in A(X)$, then $F^{n+2}(A) = A$;
   b. If $\ell_1+\ell_2+\ldots+\ell_p = n$, $X = \{0,\ldots,\ell_1\} \times\ldots\times \{0,\ldots,\ell_p\}$ (the cardinal product of p chains) and $A \in A(X)$, then $F^{n+2}(A) = A$.
3. If X is the cardinal product of the 2 chains $\{0,\ldots,\ell\}$ and $\{0,\ldots,m\}$ and $A \in A(X)$, then $F^{\ell+m+2}(A) = A$.

1. BASIC DEFINITION AND PROPERTIES

In the sequel all sets (except $\mathbb{N}$ and $\mathbb{Z}$) are supposed to be finite.

If $(X_1,\leq_1),\ldots,(X_n,\leq_n)$ are p.o.sets (partially ordered sets) then the cardinal product $(\Pi_{i=1}^n X_i,\leq)$ is the p.o.set $\Pi_{i=1}^n X_i$ with order:

$$(x_1,\ldots,x_n) \leq (y_1,\ldots,y_n) \leftrightarrow x_1 \leq_1 y_1,\ldots,x_n \leq_n y_n.$$

If $n \in \mathbb{N}$ then $\bar{n}$ is the totally ordered set $\{0,\ldots,n\}$ with $0 \leq 1 \leq\ldots\leq n$.

A *lattice* L of *dimension* k is the product of k totally ordered sets. If $L = \bar{n}_1\times\ldots\times\bar{n}_k$ then $n_1+\ldots+n_k$ is called the *length* of L.

If $n \in \mathbb{N}$, we identify the following p.o.sets:

1. the power-set-algebra $P(X)$, where X is any set with $|X| = n$ (ordered by inclusion),
2. the Boolean algebra with 2^n elements (as a lattice),
3. the n-cube $\{0,1\}^n$ (with $(x_1,\ldots,x_n) \leq (y_1,\ldots,y_n) \Leftrightarrow x_1\leq y_1,\ldots,x_n\leq y_n$).

Each of these p.o.sets will be denoted by 2^n.

In particular: 2^n is a lattice with dimension n and length n.

If $(X,\leq)$ is a p.o.set and $A \subset X$, define

$A^{\uparrow} = \{x \in X \mid \exists a \in A: a \leq x\}$,
$A^{\downarrow} = \{x \in X \mid \exists a \in A: x \leq a\}$,
$A^{max} = \{x \in A \mid \forall y \in A: y \geq x \rightarrow y = x\}$,
$A^{min} = \{x \in A \mid \forall y \in A: y \leq x \rightarrow y = x\}$,
$A^{c} = \{x \in X \mid x \notin A\}$,
A is an *antichain* if $\forall x,y \in A: x \leq y \rightarrow x = y$,
$A(X)$ is the set of antichains in X.

Proposition 1.1. *If* $(X,\leq)$ *is a p.o.set and* $A \subset X$, *then*

1. $A^{\uparrow\uparrow} = A^{\uparrow}$, $A^{\downarrow\downarrow} = A^{\downarrow}$,
2. $A^{max.max} = A^{max}$, $A^{min.min} = A^{min}$,
3. A *is antichain* $\leftrightarrow A^{max} = A \leftrightarrow A^{min} = A$,
4. $A^{cc} = A$,
5. $A^{\uparrow min} = A^{min}$, $A^{\downarrow max} = A^{max}$,
6. $A^{min\uparrow} = A^{\uparrow}$, $A^{max\downarrow} = A^{\downarrow}$,
7. $A^{\uparrow c\downarrow} = A^{\uparrow c}$, $A^{\downarrow c\uparrow} = A^{\downarrow c}$.

Proof. obvious. □

If $(X,\leq)$ is a p.o.set, define: $F\colon \mathcal{A}(X) \to \mathcal{A}(X)$
by $F(A) = A^{\uparrow c \max}$ for each $A \in \mathcal{A}(X)$;
and $G\colon \mathcal{A}(X) \to \mathcal{A}(X)$
by $G(A) = A^{\downarrow c \min}$ for each $A \in \mathcal{A}(X)$.

Propositions 1.2. *F and G are bijections from $\mathcal{A}(X)$ onto $\mathcal{A}(X)$ and $F^{-1} = G$.*

Proof. For each $A \in \mathcal{A}(X)$, $F(G(A)) = A^{\downarrow c \min \uparrow c \max} = A^{\downarrow c \uparrow c \max} = A^{\downarrow c c \max} =$ $= A^{\downarrow \max} = A^{\max} = A$, and similarly $G(F(A)) = A$. □

If $A \subset 2^n$, define $A^{\underline{c}} = \{x \in 2^n \mid x' \in A\}$.

Proposition 1.3. *If $A \subset 2^n$ then*

1. $A^{\underline{c}\underline{c}} = A$,
2. $A^{c\underline{c}} = A^{\underline{c}c}$,
3. $A^{\uparrow\underline{c}} = A^{\underline{c}\downarrow}$, $A^{\downarrow\underline{c}} = A^{\underline{c}\uparrow}$,
4. $A^{\min\underline{c}} = A^{\underline{c}\max}$, $A^{\max\underline{c}} = A^{\underline{c}\min}$.

Proof. obvious. □

Define $H\colon \mathcal{A}(2^n) \to \mathcal{A}(2^n)$ by $H(A) = A^{\uparrow c \max \underline{c}}$ for each $A \in \mathcal{A}(2^n)$.

Proposition 1.4.

1. *For each $A \in \mathcal{A}(2^n)$, $H(A) = F(A)^{\underline{c}} = G(A^{\underline{c}})$,*
2. *H is a bijection from $\mathcal{A}(2^n)$ onto $\mathcal{A}(2^n)$ and $H^{-1} = H$.*

Proof.

1. $F(A)^{\underline{c}} = A^{\uparrow c \max \underline{c}} = H(A)$,
   $G(A^{\underline{c}}) = A^{\underline{c}\downarrow c \min} = A^{\uparrow \underline{c} c \min} = A^{\uparrow c \underline{c} \min} = A^{\uparrow c \max \underline{c}} = H(A)$.
2. $H(H(A)) = H(F(A)^{\underline{c}}) = G(F(A)^{\underline{c}\underline{c}}) = G(F(A)) = A$. □

2. EXAMPLES

(For definitions and notations of graph- and matroid-theory see R.J. Wilson, *Introduction to graph theory*, Oliver & Boyd, Edinburgh, 1972).

a. If $G = (V,E)$ is an undirected graph (V is vertex-set and E is edge-set), $|E| = n$, $\mathcal{P}(E) = 2^n$ and $A = \{C \subset E \mid C$ a circuit in G and C contains no other circuit$\}$,

then: $A \in \mathcal{A}(\mathcal{P}(E))$,

$F(A) = \{F \subset E \mid F$ is a spanning forest of $G\} \in \mathcal{A}(\mathcal{P}(E))$.

and $H(F(A)) = F^2(A)^{\underline{c}} = \{C \subset E \mid C$ is a cutset in $G\} \in \mathcal{A}(\mathcal{P}(E))$.

b. If X is a k-dimensional linear space over a finite field, $|X| = n$, $\mathcal{P}(X) = 2^n$, and $A = \{L \subset X \mid L$ is linear dependent in X and L contains no other linear dependent set$\}$,

then: $A \in \mathcal{A}(\mathcal{P}(X))$,

$F(A) = \{B \subset X \mid B$ is basis of $X\}$,

and $F^2(A) = \{Y \subset X \mid Y$ is a $(k-1)$-dimensional subspace of $X\}$.

c. If $G = (V,E)$ is an undirected graph, $|V| = n$, $\mathcal{P}(V) = 2^n$, and $A = \{\{x,y\} \mid (x,y) \in E\}$,

then: $A \in \mathcal{A}(\mathcal{P}(V))$,

$F(A) = \{I \subset V \mid I$ is a maximal edge-independent set of vertices$\}$,

and $H(A) = \{C \subset V \mid C$ is a minimal edge-covering set of vertices$\}$.

d. If $(X,\leq)$ is a p.o.set, $|X| = n$, $\mathcal{P}(X) = 2^n$, and $A = \{\{x,y\} \mid x \nleq y$ and $y \nleq x\}$,

then: $A \in \mathcal{A}(\mathcal{P}(X))$,

and $F(A) = \{A \subset X \mid A$ is a maximal chain in $X\}$.

e. If $M = (X,\mathcal{B})$ is a matroid, $|X| = n$, $\mathcal{P}(X) = 2^n$, and $\mathcal{B}$ is the collection of basis of M,

then: $\mathcal{B} \in \mathcal{A}(\mathcal{P}(X))$,

$G(\mathcal{B})$ is the collection of circuits of M,

and $H(\mathcal{B})$ is the collection of co-circuits of M.

Our principal interest will be in the orbit of an antichain under repeated application of F. In the remainder of this section we show these orbits in a few cases, both in lattices and in general p.o.sets.

f. If $X = \{a,b,c,d\}$

and $A = \{\{a,b\},\{a,c\},\{b,c,d\}\}$,

then $A \in \mathcal{A}(\mathcal{P}(X))$,

$A^{\uparrow} = \{\{a,b\},\{a,c\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\ \{a,b,c,d\}\}$,

$A^{\uparrow c} = \{\emptyset,\{a\},\{b\},\{c\},\{d\},\{a,d\},\{b,c\},\{b,d\},\{c,d\}\}$,

$A^{\uparrow c\max} = F(A) = \{\{a,d\},\{b,c\},\{b,d\},\{c,d\}\}$.

$F^2(A) = \{\{a,b\},\{a,c\},\{d\}\}$,

$F^3(A) = \{\{a\},\{b,c\}\}$,

$F^4(A) = \{\{b,d\},\{c,d\}\}$,

$F^5(A) = \{\{a,b,c\},\{a,d\}\}$,

and $F^6(A) = \{\{a,b\},\{a,c\},\{b,c,d\}\} = A$.

g. If $X = \{a,b,c,d\}$

and $A = \{Y \subset X \mid |Y| = 2\}$,

then $F(A) = \{Y \subset X \mid |Y| = 1\}$,

$F^2(A) = \{\emptyset\}$,

$F^3(A) = \emptyset$,

$F^4(A) = \{X\}$,

$F^5(A) = \{Y \subset X \mid |Y| = 3\}$,

$F^6(A) = \{Y \subset X \mid |Y| = 2\} = A$.

h. If $X = \{a,b,c\}$,

and $A = \{\{a,b\},\{a,c\}\}$,

then $F(A) = \{\{a\},\{b,c\}\}$,

$F^2(A) = \{\{b\},\{c\}\}$,

$F^3(A) = \{\{a\}\}$,

$F^4(A) = \{\{b,c\}\}$,

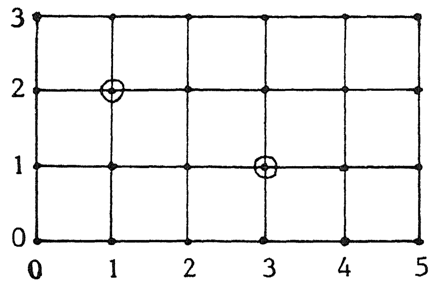
$F^5(A) = \{\{a,b\},\{a,c\}\} = A$.

i. If $X = \{a_1,\ldots,a_n\}$ and $A = \{\{a_1,\ldots,a_n\}\} = \{X\}$,
then $F(A) = \{\text{all } (n-1)\text{-subsets of } X\}$,

$\cdot$
$\cdot$
$\cdot$
$\cdot$

$F^{n-1}(A) = \{\text{all singletons in } X\}$,
$F^{n}(A) = \{\emptyset\}$,
$F^{n+1}(A) = \emptyset$,
$F^{n+2}(A) = \{X\} = A$.

j. If $X = \bar{5} \times \bar{3}$
and $A = \{(1,2),(3,1)\}$,
then $F(A) = \{(0,3),(2,1),(5,0)\}$,
$F^{2}(A) = \{(1,2),(4,0)\}$,
$F^{3}(A) = \{(0,3),(3,1)\}$,
$F^{4}(A) = \{(2,2),(5,0)\}$,
$F^{5}(A) = \{(1,3),(4,1)\}$,
$F^{6}(A) = \{(0,3),(3,2),(5,0)\}$,
$F^{7}(A) = \{(2,2),(4,1)\}$,
$F^{8}(A) = \{(1,3),(3,1),(5,0)\}$,
$F^{9}(A) = \{(0,3),(2,2),(4,0)\}$,
$F^{10}(A) = \{(1,2),(3,1)\} = A$.



k. If $L = \bar{n}_1 \times \bar{n}_2 \times \ldots \times \bar{n}_k$, $n_1 + \ldots + n_k = N$,
and $A = \{(n_1,\ldots,n_k)\}$,
then $F(A) = \{(x_1,\ldots,x_k) \mid \sum_{i=1}^{k} x_i = N - 1\}$,

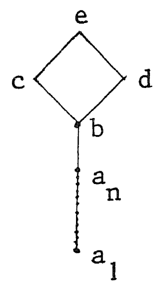
$\cdot$
$\cdot$
$\cdot$

$F^{N-1}(A) = \{(x_1,\ldots,x_k) \mid \sum_{i=1}^{k} x_i = 1\}$,
$F^{N}(A) = \{(0,\ldots,0)\}$,
$F^{N+1}(A) = \emptyset$,
$F^{N+2}(A) = \{(n_1,\ldots,n_k)\} = A$;
this orbit is called the *principal orbit* of the lattice.

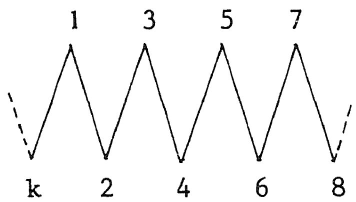
In general, the principal orbit of a p.o.set is the orbit (under action of F) of the empty set. It is easily seen, that the principal orbit of a p.o.set, on which a height function can be defined, consists of those sets, that contain all elements of a given height, and has length H+2 if H is the height of the p.o.set.

As the following examples show, the length of an orbit other than the principal one need not be correlated with the height of the p.o.set.

l. If $X = \{a_1,\dots,a_n,c,d,e\}$ with $a_1<a_2<\dots<a_n<b<c<e; b<d<e$
and $c \nless d$, $d \nless c$,
and $A = \{\{c\}\}$,
then $F(A) = \{\{d\}\}$,
$F^2(A) = \{\{c\}\} = A$.



m. If $k \in \mathbb{N}$, k even, $k \geq 8$ and $X = \{1,\dots,k\}$,
and $\ell < \ell - 1$ and $\ell < \ell + 1$ if ℓ is even,
and $k < 1$,
and $A = \{1,4\}$,
then $F(A) = \{2\} \cup \{\ell \mid \ell \text{ odd}, 7 \leq \ell \leq k - 1\}$,
$F^2(A) = \{5\} \cup \{\ell \mid \ell \text{ even}, 8 \leq \ell \leq k\}$,
$F^3(A) = \{3,6\}$,
$F^6(A) = \{5,8\}$,
$F^m(A) = \{1,4\} = A$, if $m = \frac{3}{2}k$.



(For k=6 we get, if $A=\{1,4\}$, $F^5(A) = A$ which is not a special case of the behaviour shown above).

3. THE CONJECTURE

<u>Proposition 3.1</u>. *If* $(X,\leq)$ *is a p.o.set and* $A \in \mathcal{A}(X)$, *then there exists a* $k > 0$ *with* $F^k(A) = A$.

<u>Proof</u>. F is a permutation of $\mathcal{A}(X)$. □

Examples f,g,h,i suggest the following

Conjecture. *If* $A \in A(2^n)$ *then* $F^{n+2}(A) = A$.

In fact one can prove

Theorem 3.2. *If* $n \leq 4$ *and* $A \in A(2^n)$, *then* $F^{n+2}(A) = A$.

Proof. Check each antichain. □

Theorem 3.3. *If* $n \in \mathbb{N}$ *and* $A = \{2^n\}$ *then* $F^{n+2}(A) = A$.

Proof. See example i. □

There exists a connection between the above conjecture and an analogous one for lattices:

Theorem 3.4. *If* $n \in \mathbb{N}$, *then the following assertions are equivalent:*
(i) $\forall A \in A(2^n)$: $F^{n+2}(A) = A$ (*the conjecture for* n),
(ii) *each lattice* L *with length* n *satisfies*: $\forall A \in A(L)$: $F^{n+2}(A) = A$.

Proof. (ii) → (i): 2^n is a lattice with length n.

(i) → (ii): Suppose $L = \bar{n}_1 \times \ldots \times \bar{n}_k$, with $n_1 + \ldots + n_k = n$.
Let $X_1, \ldots, X_k$ be k pairwise disjoints sets with $|X_j| = n_j$ $(1 \leq j \leq k)$, and set $X = X_1 \cup \ldots \cup X_k$.
Now $|X| = n$ and $P(X) = 2^n$.

Define $\phi: P(X) \to L$ by

$$\phi(Y) = (m_1, \ldots, m_k) \text{ if } Y \subset X \text{ and } |Y \cap X_j| = m_j \ (1 \leq j \leq k).$$

Then it is easily seen that:

(1) ϕ is a function onto,
(2) for each $A \in A(L)$ $\phi^{-1}[A] \in A(P(X))$, and
(3) for each $A \in A(L)$ $\phi^{-1}[F(A)] = F(\phi^{-1}[A])$.

Therefore, if $A \in A(L)$, for each $k \in \mathbb{Z}$, $\phi^{-1}[F^k(A)] = F^k(\phi^{-1}[A])$ and $\phi^{-1}[F^{n+2}(A)] = F^{n+2}(\phi^{-1}[A])$. But, by (i), since $\phi^{-1}[A] \in A(P(X))$, $F^{n+2}(\phi^{-1}[A]) = \phi^{-1}[A]$, and so
$F^{n+2}(A) = \phi\phi^{-1}[F^{n+2}(A)] = \phi F^{n+2}(\phi^{-1}[A]) = \phi\phi^{-1}[A] = A$. □

Corollary 3.5. *Each lattice* L *with length* $n \leq 4$ *satisfies:* $\forall A \in A(L)$: $F^{n+2}(A) = A$.

Proof. Consequence of theorems 3.2 and 3.4. □

Remark. Example ℓ shows that not each modular (general) lattice L with length $n \le 4$ satisfies $\forall A \in A(L)$, $F^{n+2}(A) = A$.

Corollary 3.5 gave a sufficient condition on the length of a lattice. The next theorem gives a sufficient condition on the dimension of a lattice.

Theorem 3.6. *Each lattice* L *with dimension* 2 *and length* n *satisfies:* $\forall A \in A(L)$, $F^{n+2}(A) = A$.

Proof. Suppose $L = \bar{\ell} \times \bar{m}$, $n = \ell + m$ and $A \in A(L)$. Define for each $t \in Z$ the sets $X(t) \subset \bar{\ell}$ and $Y(t) \subset \bar{m}$, by

$$X(t) = \{x \in \bar{\ell} \mid \exists y \in \bar{m}: (x,y) \in F^t(A)\}$$

and

$$Y(t) = \{y \in \bar{m} \mid \exists x \in \bar{\ell}: (x,y) \in F^t(A)\}.$$

The theorem will be proved by demonstrating the following facts:

I. for each $t \in \mathbb{Z}$ is $F^t(A)$ completely determined by $X(t)$ and $Y(t)$,

II. for each $t \in \mathbb{Z}$ is $X(t+n+2) = X(t)$ and $Y(t+n+2) = Y(t)$.

Then, of course, for each $t \in \mathbb{Z}$, $F^{t+n+2}(A) = F^t(A)$ holds, and thus $F^{n+2}(A) = A$.

Proof of I: $\forall t \in \mathbb{Z}$ is $|X(t)| = |Y(t)|$ (if (x_1,y) and $(x_2,y) \in F^t(A)$, then $x_1=x_2$).
Suppose $X(t) = \{x_1,\ldots,x_k\}$ with $0 \le x_1 < \ldots < x_k \le \ell$,
and $Y(t) = \{y_1,\ldots,y_k\}$ with $m \ge x_1 > \ldots > x_k \ge 0$.
Then $F^t(A) = \{(x_1,y_1),\ldots,(x_k,y_k)\}$, so $F^t(A)$ is determined by $X(t)$ and $Y(t)$.

Proof of II: We first prove:

$$\forall t \in \mathbb{Z} \quad X(t+1) = \{x \in \bar{\ell} \mid x+1 \in X(t)\} \cup \{\ell \mid 0 \notin Y(t)\}$$
$$\text{and } Y(t+1) = \{y \in \bar{m} \mid y+1 \in Y(t)\} \cup \{m \mid 0 \notin X(t)\}. \qquad (*)$$

For suppose $x \in \bar{\ell}$ and $x+1 \in X(t)$. Then, by definition of $X(t)$, for some y $(x+1,y) \in F^t(A)$, so $(x,y) \notin F^t(A)^{\uparrow}$, i.e. $(x,y) \in F^t(A)^{\uparrow c}$. Then there exists $(u,v) \in F^t(A)^{\uparrow c\max}$ so that $(x,y) \le (u,v)$. But $(x+1,y) \in F^t(A)$, so $x = u$ and

$(x,v) \in F^t(A)^{\uparrow cmax}$, hence $x \in X(t+1)$. If $0 \notin Y(t)$, then for all $x \in \bar{\ell}$ $(x,0) \notin F^t(A)$ and consequently $(\ell,0) \notin F^t(A)^{\uparrow}$, i.e. $(\ell,0) \in F^t(A)^{\uparrow c}$. So there exists $(u,v) \in F^t(A)^{\uparrow cmax}$ such that $(\ell,0) \leq (u,v)$. But then $\ell = u$ and $(\ell,v) \in F^t(A)^{\uparrow cmax}$, thus $\ell \in X(t+1)$.

Conversely (follows from the above by considering the reverse order on L): if $x \in X(t+1)$ and $x \neq \ell$ then there exists y so that $(x,y) \in F^{t+1}(A)$, then $(x+1,y) \notin F^{t+1}(A)^{\downarrow}$, i.e. $(x+1,y) \in F^{t+1}(A)^{\downarrow c}$. Then for some $(u,v) \in F^{t+1}(A)^{\downarrow cmin}$ $(u,v) \leq (x+1,y)$. But $(x,y) \in F^{t+1}(A)$, so $u = x + 1$ and $(x+1,v) \in F^{t+1}(A)^{\downarrow cmin}$, hence $x + 1 \in X(t)$. Finally, if $\ell \in X(t+1)$, then for some y, $(\ell,y) \in F^{t+1}(A)$, so $(\ell,0) \in F^{t+1}(A)^{\downarrow}$ and $(\ell,0) \notin F^{t+1}(A)^{\downarrow c}$. Thus $\forall x \in \bar{\ell}$ $(x,0) \notin F^{t+1}(A)^{\downarrow cmin} = GF^{t+1}(A) = F^t(A)$, i.e. $0 \notin X(t)$. This proves (*).

The proof of II is then as follows: for each $t \in Z$

$x \in X(t) \Leftrightarrow 0 \in X(t+x) \Leftrightarrow m \notin Y(t+x+1) \Leftrightarrow 0 \notin Y(t+x+1+m) \Leftrightarrow$

$\ell \in X(t+x+1+m+1) \Leftrightarrow x \in X(t+x+1+m+1+(\ell-x)) = X(t+\ell+m+2) = X(t+n+2)$.

Similarly: $y \in Y(t) \Leftrightarrow y \in Y(t+n+2)$. □

Finally we prove the following

Theorem 3.7. *The conjecture is false.*

Proof. Take $X = \{a,b,c,d,e\}$ and $A = \{\{a,b\},\{b,c\},\{c,d\},\{d,e\},\{e,a\}\}$; $n = 5$.

Then $F(A) = \{\{a,c\},\{b,d\},\{c,e\},\{d,a\},\{e,b\}\}$,

and $F^2(A) = \{\{a,b\},\{b,c\},\{c,d\},\{d,e\},\{e,a\}\} = A$,

therefore $F^7(A) = F(A) \neq A$. □

4. SOME NUMERICAL RESULTS

For $n \leq 5$ the integers which occur as the period of the F-operator on antichains in 2^n are completely known.

For $n > 5$ we only have some incidental results.

n	periods occurring in 2^n
0	2
1	3
2	2 and 4
3	5
4	2,3 and 6
5	2,3,7,16 and 27
6	among others:
	2,4,6,8,10,12,14,16,18,20,
	24,28,32,34,35,39,40,42,48,54,
	64,68,72,76,78,81,82,86,90,92,94,98,
	102,104,106,108,120,124,128,132,134,144,168,
	188,204,216,219,222,228,234,252,256,270,
	288,348,366,380,384,396,414,616,
	1026,1032

For n = 5 and n = 6 the period n + 2 is by far the most frequent one.

Below the output of a conversational program on the PDP8/I computer is reproduced. Input is the value of n and an antichain in 2^n. Output is the length of the period and if desired the entire period. Note that all numbers are in octal notation.

Antichain notation: {{a,b,c},{a,b,d},{c,d}} is printed ABC/ABD/CD/,
{∅} is printed /,
∅ is printed .

```
.R ANTI

N=5
INPUT ANTICHAIN: /
0007     /

PRINT THE ENTIRE CYCLE?Y
0001
0002     ABCDE/
0003     BCDE/ACDE/ABDE/ABCE/ABCD/
0004     CDE/BDE/BCE/BCD/ADE/ACE/ACD/ABE/ABD/ABC/
0005     DE/CE/CD/BE/BD/BC/AE/AD/AC/AB/
0006     E/D/C/B/A/
0007     /
```

```
N=5
INPUT ANTICHAIN: DE/CE/CD/BE/AD/AB/

0020      DE/CE/CD/BE/AD/AB/

PRINT THE ENTIRE CYCLE?Y
0001      BD/BC/AE/AC/
0002      CDE/BE/AD/AB/
0003      DE/BCD/ACE/
0004      BCE/ACD/ABE/ABD/ABC/
0005      CDE/BDE/BCD/ADE/ACE/AB/
0006      DE/BD/BCE/AE/ACD/
0007      CE/CD/BE/AD/ABC/
0010      DE/BD/BC/AE/AC/AB/
0011      CE/CD/BE/AD/
0012      DE/BD/AE/ABC/
0013      BCE/ACD/AB/
0014      CDE/BDE/BCD/ADE/ACE/
0015      DE/BCE/ACD/ABE/ABD/ABC/
0016      DE/BCD/AD/ACE/AB/
0017      CDE/BD/BC/AE/AC/
0020      DE/CE/CD/BE/AD/AB/

N=5
INPUT ANTICHAIN: DE/CE/BE/BCD/AD/
0033      DE/CE/BE/BCD/AD/

PRINT THE ENTIRE CYCLE?Y
0001      CD/BD/AE/ABC/
0002      DE/BCE/AD/AC/AB/
0003      CE/BE/BCD/AE/
0004      DE/ACD/ABD/ABC/
0005      BCE/BCD/AD/ACE/ABE/
0006      CDE/BDE/AE/ABC/
0007      DE/BCE/BCD/ACD/ABD/
0010      CD/BD/AD/ACE/ABE/ABC/
0011      DE/BCD/AE/AC/AB/
0012      CE/BE/BCD/AD/
0013      DE/CD/BD/AE/ABC/
0014      BCE/AD/AC/AB/
0015      CDE/BDE/BCD/AE/
0016      DE/BCE/ACD/ABD/ABC/
0017      BCD/AD/ACE/ABE/
0020      CDE/BDE/BCE/AE/ABC/
0021      DE/CE/BE/BCD/ACD/ABD/
0022      CD/BD/AE/AD/ABC/
0023      DE/BCE/AC/AB/
0024      CE/BE/BCD/AE/AD/
0025      DE/CD/BD/ABC/
0026      BCE/AD/ACE/ABE/
```

```
0027      CDE/BDE/BCD/AE/ABC/
0030      DE/BCE/ACD/ABD/
0031      BCD/AD/ACE/ABE/ABC/
0032      CDE/BDE/BCE/AE/AC/AB/
0033      DE/CE/BE/BCD/AD/

N=5
INPUT ANTICHAIN: DE/CE/BD/AC/AB/
0002       DE/CE/BD/AC/AB/

PRINT THE ENTIRE CYCLE?Y
0001       CD/BE/BC/AE/AD/
0002       DE/CE/BD/AC/AB/

N=5
INPUT ANTICHAIN: DE/BC/ACE/ABD/

0003       DE/BC/ACE/ABD/

PRINT THE ENTIRE CYCLE?Y
0001       CE/BD/ACD/ABE/
0002       CD/BE/ADE/ABC/
0003       DE/BC/ACE/ABD/

N=6
INPUT ANTICHAIN: F/DE/CE/BE/BCD/AD/
0332       F/DE/CE/BE/BCD/AD/

PRINT THE ENTIRE CYCLE?N
N=
```

•

ADDENDUM

Applying Ramsey's theorem M.M. Krieger has proved in [1] that for each n with $11 = N(3,4;2)+2 \le n < N(4,4;3)$ there exists an antichain $A \in A(2^n)$ with $F^2(A) = A$. By a result of Isbell [2] it is known that $N(4,4;3) \ge 13$.

Furthermore, for each even n there exists an $A \in A(2^n)$ with $F^2(A) = A$. (If $X = \{0.1,\ldots k,k+1,\ldots,2k\}$, then set $A = \{Y \subset X \mid |Y| = k \text{ and } |Y \cap \{1,\ldots,k\}| \text{ is even}\}$).

[1] M.M. Krieger, *On permutations of Antichains in Boolean lattices: An application of Ramsey's Theorem;* preprint Computer Science Dept., University of California, Los Angeles.

[2] J.R. Isbell, "N(4,4;3) ≥ 13", *J. Combinatorial Theory,* 6 (1969) 210.