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A.E. BROUWER

A CONNECTED METRIC SPACE WITHOUT NONTRIVIAL OPEN CONNECTED SUBSPACES

2e boerhaavestraat 49 amsterdam

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A connected metric space without nontrivial open connected subspaces

by

A.E. Brouwer

ABSTRACT

An example is given of an arcwise connected metric space (which is not separable) without proper non-empty open connected subspaces.

KEY WORDS & PHRASES: *connected metric space*

0. INTRODUCTION

While trying to generalize known results for separable metric spaces to general topological spaces it was found that outside of the class of separable metric spaces requirements on all open connected subsets of a connected set are much weaker than within that class. This note shows the reason for this: while each nondegenerated connected separable metric space contains proper nonempty open connected subspaces, this is not true in general.

<u>PROPOSITION</u> *Let* X *be a connected topological space,* C *a closed connected subspace,* $\phi \neq C \neq X$. *If* X *does not contain nontrivial open connected subspaces then* $X \backslash C$ *decomposes into infinitely many components, and if* S *is such a component then* $C \cup S$ *is connected.*

<u>COROLLARY</u> *If, in addition,* X *is* T_1 *then each point of* X *is a ramification point. In particular* X *is not separable metric:*

<u>PROPOSITION</u> *Let* X *be a connected* T_1 *space in which each point is a ramification point.*

Equivalent are: (i) (*assuming* SH) X *has the Suslin property*
(ii) X *is separable*
(iii) X *is countable*

While a connected T_1 space without open connected subsets cannot be separable metric, it can be separable (hence countable) and it can be metric as we will show by examples.

1. PROOF OF THE PROPOSITIONS

(i) Let X be a connected space without nontrivial open subspaces. Let C be a closed connected subspace, $\phi \neq C \neq X$. Let S be a component of $X \backslash C$. Since S in not open while $X \backslash C$ is, $X \backslash C$ decomposes into infinitely many components. Also $X \backslash S$ is connected and not open, so S is not closed. But S is closed in $X \backslash C$ so that $C \cap \bar{S} \neq \phi$.
Therefore $C \cup S$ is connected. □

(iia) Let X be a connected T_1 space, and Y its collection of ramification points (that is, Y is the set of all $y \in X$ such that $X\backslash y$ has at least three components).

Now consider the map κ: $(x_1,x_2,x_3) \mapsto y$ if $X\backslash y = \underset{x_1}{A} + \underset{x_2}{B} + \underset{x_3}{C}$.

If D is a dense subset of X, and $M \subset X^3$ is the domain of κ, then it follows that $|Y| \le |D^3 \cap M| \le |D|^3 = |D|$ if D is infinite.

In particular if X is separable and Y = X then Y is countable.

(iib) Now let X have the Suslin property (i.e. any collection of disjoint open nonempty subsets of X is countable). We shall show that if Y is not countable then it is a Suslin tree (i.e. an uncountable partial order which does not contain the Boolean algebra 2, and in which all chains and anti-chains are countable.) It is well-known that the non-existence of a Suslin tree is equivalent to the Suslin Hypothesis (SH) [E.W. Miller, A note on Souslin's problem, Amer. J. Math. 65 (1943) 673-678]. To this end define a partial order on X as follows. Choose $x_0 \in X$ and set $x > x_0$ for each $x \in X\backslash x_0$. Set $x_1 < x_2$ for $x_1, x_2 \in X\backslash x_0$ iff x_1 seperates x_2 from x_0. Let S be an anti-chain in Y. If we choose for each $s \in S\backslash x_0$ a clopen subset U_s of $X\backslash s$ not containing x_0 then $\{U_s\}$ is a disjoint collection of open sets, hence countable. Next, let C be a chain in Y. Let for $c \in C$ $X\backslash c = \underset{x_0}{P(c)} + Q(c) + R(c)$ be a decomposition of $X\backslash c$ such that $x_0 \in P(c)$ and $C \subset P(c) \cup Q(c) \cup \{c\}$ and $R(c) \neq \phi$. Then $\{R(c) \mid c \in C\}$ is a disjoint collection of open sets, hence countable. Therefore $(Y,<)$ is a Suslin tree. $\square$

2. THE EXAMPLE

Let $Y = \{(x,y) \in I^2 \mid x = qy,\ 1 < q \in \mathbb{Q}\}$ with Euclidean topology. Y is separable metric arcwise connected space. Moreover any nonempty open connected subset of Y must contain the point $0 = (0,0) \in Y$. Also Y has no cut points different from 0. Starting from this space we construct our example X as a direct limit:

Let $Y^* = Y\backslash\{(0,0)\}$. Let $X_0 = (Y^*)^\circ = \{\lambda\}$ where λ is the empty sequence over Y^*. Let for $n > 0$ $X_n = (X_{n-1}\times\{0\}) \cup (Y^*)^n$, the set of all sequences of length n over Y such that the nonzero entries from an initial subsequence. We will identify X_{n-1} with $X_{n-1}\times\{0\} \subset X_n$. Now we may write $X := \bigcup_{n=0}^{\infty} X_n$.

If $X = (x_1,\ldots,x_n) \in X_n$ and $n \geq 1$ then define $x' := (x_1,\ldots,x_{n-1})$.

Define a metric d on X as follows: Let d_0 be the Euclidean metric of I^2 (and Y). If $x \in X_n\backslash X_{n-1} = (Y^*)^n$ and $y \in (Y^*)^m$ then define

$$d_1(x,y) = d_1(y,x) := \begin{cases} d_0(x_n,y_n) & \text{if } m = n \text{ and } x' = y' \\ d_0(x_n,(0,0)) & \text{if } (m=n-1 \text{ and}) \ y = x' \\ d_0(x_n,(1,1)) & \text{if } (m=n-2 \text{ and}) \ y = x'' \\ \infty & \text{otherwise} \end{cases}$$

Let $d(x,y) = \min\{\sum_{i=0}^{\ell} d_1(x_i,x_{i+1}) \mid x_0 = x,\ x_{\ell+1} = y\}$, then d is a metric on X: d is finite because for each x the chain $x_0 = x,\ x_1 = x',\ldots,x_n = \lambda$ shows that $d(x,\lambda) < \infty$. If x and y are points in X_n then it is not necessary to consider chains with points in $X\backslash X_n$ since for each $z \in X$ we have $d_1(z',z'') \leq \sqrt{2} \leq d(z',z) + d(z,z'')$, as soon as z'' is defined. This shows that 'min' instead of 'inf' is justified and that $d(x,y)$ is nonzero if $x \neq y$.

Now we have:

(i) Each X_n is a closed subspace of X: If $x \in X_m\backslash X_{m-1}$ and $m > n$ and $\varepsilon_x = \min(d(x,x'), d(x,x''))$ then $d(x,X_n) \geq \varepsilon_x$.

(ii) The map $j_u : Y \to X_{n+1}$ defined by
$y \mapsto (u,y)$
where $u \in (Y^*)^n$, is a homeomorphic embedding;
for $n = 0$ $j_\lambda : Y \xrightarrow{\sim} X_1$ is a homeomorphism onto X_1,
for $n \geq 1$ $\mathrm{cl}\ j_u[Y] = j_u[Y] \cup \{u'\}$.

For: Obviously j_u is $1-1$ and open since $d(j_u y_1, j_u y_2) \leq$ $\leq d_0(y_1,y_2)$. j_u is also continuous since if $x = j_u a$ then $d(x,j_u b) \leq \varepsilon_x \Rightarrow d(x,j_u b) = d_0(a,b)$ $(a\neq 0)$ and $d(u,j_u b) < \frac{1}{2}\sqrt{2} \Rightarrow d(u,j_u b) = d_0(0,b)$.

Since X_{n+1} is closed $\mathrm{cl}\ j_u[Y] \subset X_{n+1}$, and from the definition of d_0 it is easily seen that $\mathrm{cl}\ j_u[Y] = j_u[Y] \cup \{u'\}$.

(iii) Each X_n is (arcwise) connected, as is X itself. For: x is joined by an arc with x', hence by induction each point is in the arc component of λ.

(iv) Let $L_q = \{(x,y) \in Y \mid x = qy\}$ and $L_q^* = L_q\backslash\{0\}$. Then $Y\backslash\{0\}$ decomposes into the components L_q^* • $(1<q\in\mathbb{Q})$.

(v) Define the projection $\pi_m : X \to X_m$ by

$$x = (x_1,\ldots,x_m,\ldots) \mapsto (x_1,\ldots,x_m).$$

π_m is not continuous (the image of the connected $j_u[Y] \cup \{u'\}$ is the doubleton $\{u,u'\}$ if $u \in X_m\backslash X_{m-1}$), but it is continuous on $X\backslash X_{m-1}$.

(vi) Let $u \in (Y^*)^{n-1}$. Then $X\backslash u$ decomposes into the components $X\backslash\pi_{n-1}^{-1}\{u\} = X\backslash\pi_n^{-1}j_u[Y]$ and $\pi_n^{-1}j_uL_q^*$ $(1<q\in\mathbb{Q})$.
For: for each point $a \in (Y^*)^n$ $\pi_n^{-1}\{a\}$ is homeomorphic to X and hence connected. Therefore since $j_uL_q^*$ is connected each $\pi_n^{-1}j_uL_q^*$ is connected. Also $X\backslash\pi_{n-1}^{-1}\{u\}$ is connected (and has at most 2 arc components). If $M_r = \cup\{L_q^* \mid r < q \in \mathbb{Q}\}$ then M_r is clopen in Y^* and $\pi_n^{-1}j_uM_r$ is clopen in $X\backslash u$ $(r>1)$, so the candidate components are indeed maximal connected.

(vii) X is a non-separable metric arcwise connected space without proper nonempty open subspaces.
For: Let $U \subset X$ be open and connected and let $u \in U$, $u = (u_1,\ldots,u_n) \in (Y^*)^n$. If $n > 0$ and $v = u'$ then u is not an interior point of its component in $X\backslash v$. Therefore U is not contained in $X\backslash v$, that is, $v \in U$. By induction it follows that $\lambda \in U$. Next if $u \in U$ and $u = v'$ then again u is accumulation point of components of $X\backslash v$ so $v \in U$. This proves $U = X$.

This finishes the description of a metric example. The space Y was a fan of copies of the unit interval I. When I is replaced by a countable connected T_2 space and the construction is carried out as before (now defining an appropriate topology instead of a metric) then we obtain a countable connected T_2 space not containing nontrivial open connected subspace.

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