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ON THE EDGE-COLOURING PROPERTY FOR THE HEREDITARY CLOSURE OF A COMPLETE UNIFORM HYPERGRAPH

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On the edge-colouring property for the hereditary closure of a complete uniform hypergraph

by

A.E. Brouwer

ABSTRACT

We show that the collection of subsets of cardinality at most h of a fixed set of cardinality n does not possess a parallelism (1-factorization) whenever n $\neq$ 0 or -1 (mod h) and n > 2h. Also some positive results are given.

KEYWORDS & PHRASES: *parallelism, 1-factorization, edge colouring property*

0. INTRODUCTION

Let x be some fixed set of n elements. Let for $H \subset \{0,1,\dots,n\}$ K_n^H denote the hypergraph (x,E) with vertex set x and collection of edges $E = \{y \subset x \mid |y| \in H\}$.

(Since we never need symbols for the vertices of a hypergraph, but do use collections of collections of edges, we denote sets of vertices by lower case symbols, sets of edges by capitals and collections of sets of edges with upper case script letters.)

h will always denote the largest element of H (if any). When $H = \{h\}$ we write K_n^h instead of $K_n^{\{h\}}$, the complete h-uniform hypergraph on n vertices. When $H = \{0,1,\dots,h\}$ then, following BERGE & JOHNSON, we write $\hat{K}_n^h$ for K_n^H, the hereditary closure of the complete h-uniform hypergraph on n vertices.

BARANYAI [1] proved that K_n^h has a 1-factorization iff $h|n$. BERMOND, BERGE and JOHNSON then considered the case of $\hat{K}_n^h$. Their results are as follows ([2],[3]):

Write $n = kh + \ell$ $(0 \le \ell \le h-1)$, then

(i) if $h \le 2$ $\hat{K}_n^h$ has a 1-factorization,

(ii) $\hat{K}_n^3$ has a 1-factorization iff $n \not\equiv 1 \pmod 3$ or $n = 4$,

(iii) $\hat{K}_n^4$ has a 1-factorization iff $n \equiv 0$ or $3 \pmod 4$ or $n \le 6$,

(iv) $\hat{K}_n^h$ has a 1-factorization if $\ell = 0$ and $k \ge h-1$,

(v) $\hat{K}_n^h$ does not have a 1-factorization when $h \ge 3$ and :

a) $n = 10$, $h = 5$

b) $k \ge 3$, $\ell = h-2$

c) $k = 2$, $1 \le \ell \le h-3$

d) k sufficiently large, $1 \le \ell \le h-3$.

Here we prove:

(vi) Let $k = 1$. Then $\hat{K}_{h+\ell}^h$ has a 1-factorization iff $\hat{K}_{h+\ell}^{\ell-1}$ has.

(vii) Let $k \ge 2$, $\ell = 0$. Then $\hat{K}_n^h$ has a 1-factorization iff $k \ge h-2$.

(viii) Let $k \ge 2$, $1 \le \ell \le h-2$. Then $\hat{K}_n^h$ does not have a 1-factorization.

(ix) Let $k \ge 2$, $\ell = h-1$.

a) If $\hat{K}^h_n$ has a 1-factorization then $n \geq \frac{1}{2}h(h-2)-1$, i.e., $k \geq \frac{1}{2}h-2$.

b) If $k \geq h-2$ or if ($k > \frac{1}{2}h-2$ and h is even)

then $\hat{K}^h_n$ has a 1-factorization.

Many results are given for more general H.

All the existence results follow from BARANYAI's theorem [1] which roughly says that a 1-factorization exists iff the numbers fit. E.g. $\hat{K}^3_8$ is 1-factorizable since $\binom{8}{1} = 8$, $\binom{8}{2} = 28$, $\binom{8}{3} = 56$ and one can imagine a 1-factorization consisting of

1 1-factor containing 8 singletons

and 28 1-factors each containing 2 triples and 1 pair.

On the other hand, as was remarked by R.M. WILSON, $\hat{K}^3_7$ is not 1-factorizable since $\binom{7}{1} = 7$, $\binom{7}{2} = 21$, $\binom{7}{3} = 35$, and any 1-factor not containing a singleton must contain 2 pairs and a triple so that there are at most 10 triples in a 1-factor without singleton and at most 14 triples in a 1-factor with singleton, which leaves 11 triples not in any 1-factor.

For precise statements of BARANYAI's theorem see [1],[2] and [3].

All the non-existence results are proved with the following argument: Let F be a 1-factorization of K^H_n. For each 1-factor $P \in F$ let $n(g) = n_P(g) = |\{a \in P \mid |a| = g\}|$. Suppose that $q \in H$ and that for each $P \in F$ we have $\sum \alpha_i\, n(i) \geq n(q)$. Then it follows that $\binom{n}{q} \leq \sum_i \alpha(i) \binom{n}{i}$. Using inequalities on binomial coefficients we then derive a contradiction.

1. NEGATIVE RESULTS

In this section we prove that $\hat{K}^h_n$ does not have the edge colouring property

if $n = kh + \ell$, $1 \leq \ell \leq h-2$, $k \geq 2$

or if $n = kh - 1$, $3 \leq k \leq \frac{1}{2}(h-3)$

or if $n = kh$, $3 \leq k \leq h-3$.

First two lemmata:

<u>LEMMA 1</u>. *Let* $n \geq kh-1$ *and* $g \leq h$. *Then* $\binom{n}{g} \leq (k-1)^{g-h} \binom{n}{h}$.

<u>PROOF</u>. Use induction on h-g. If g = h then the lemma is true. Next

$$\binom{n}{g-1} = \frac{g}{n-g+1}\binom{n}{g} \leq \frac{h}{n-h+1}\binom{n}{g} \leq \frac{1}{k-1}\binom{n}{g}$$

provides the induction step. □

LEMMA 2. *Let* $n \geq 2h$. *Then*

$$\sum_{i=0}^{h-1}\binom{n}{i} < \frac{h}{n-2h+1}\binom{n}{h}.$$

PROOF. As before we find $\binom{n}{h-j} \leq \left(\frac{h}{n-h+1}\right)^j\binom{n}{h}$ so that

$$\sum_{i=0}^{h-1}\binom{n}{i} < \sum_{j=1}^{\infty}\left(\frac{h}{n-h+1}\right)^j\binom{n}{h} = \frac{h}{n-2h+1}\binom{n}{h}. \quad \square$$

THEOREM 1. *Let* $n = kh+\ell$, $k \geq 3$, $1 \leq \ell \leq h-2$. *Then* K_n^H *does not have the edge colouring property.*

PROOF. Suppose K_n^H has the e.c.p., i.e., there is a collection of partitions of an n-set containing all the g-subsets of this n-set for all $g \in H$. (Note that we use the words 'partition' and '1-factor' as synonyms.) Each partition contains at least one ℓ-set or at least two other small sets, i.e.,

$$\binom{n}{h} \leq \tfrac{1}{2}k\sum_{i=1}^{h-1}\binom{n}{i} + \tfrac{1}{2}k\binom{n}{\ell}.$$

Using lemma's 1 and 2 we find

$$1 < \tfrac{1}{2}k\left(\frac{1}{k-2} + \frac{1}{(k-1)^2}\right).$$

so that $k \leq 4$.

Now suppose $\ell = h-2$, $n = (k+1)h-2$. In the same way we find

$$1 \leq \tfrac{1}{2}k\left(\frac{1}{k-1-(1/h)} + \frac{1}{(k-1)^2}\right),$$

which implies $k = 3$ since $h \geq 3$.

On the other hand, if $\ell \leq h-3$ then we argue that each partition either contains at least one g-set for some $g \leq h-2$ or at least $(h-\ell)$ $(h-1)$-sets, and it follows that

$$\binom{n}{h} \leq k\sum_{i=1}^{h-2}\binom{n}{i} + \max\left(0, \frac{n-(h-\ell)(h-1)}{h(h-\ell)}\binom{n}{h-1}\right),$$

hence

$$\frac{n-h+1}{h} < k- \frac{h-1}{n-2h+3} + \max(0, \frac{k+1}{h-\ell} -1),$$

$$k - 1 < \frac{k}{k-2} + \frac{k-1}{3} ,$$

which again implies k = 3.

Now let $n = 3h+\ell$. If $\ell \le h-5$ then a partition cannot contain only h-sets and (h-1)-sets, i.e. we have

$$\binom{n}{h} \le \frac{3}{2}\sum_{i=1}^{k-2} \binom{n}{i} + \frac{3}{2}\binom{n}{\ell} + \frac{1}{2}\binom{n}{\ell+1}.$$

[Note that if a partition contains 3 h-sets then it contains either an ℓ-set or at least two smaller sets; if a partition contains 2 h-sets then it contains at least two g-sets with $g \le h-2$ unless it is $n = h+h+(h-1) + (\ell+1)$.]

It follows that

$$\frac{n-h+1}{h} < \frac{3}{2}\,\frac{h-1}{n-2h+3} + \frac{3}{2}(k-1)^{-4} + \frac{1}{2}(k-1)^{-3},$$

$$2 < \frac{3}{2} + \frac{3}{32} + \frac{1}{16} ,$$

a contradiction.

Finally, if $n = 3h+\ell$ and $h-4 \le \ell \le h-2$ then from

$$\binom{n}{h} \le \frac{3}{2}\sum_{i=1}^{h-2} \binom{n}{i} + \frac{3}{2}\binom{n}{\ell} + \frac{1}{2}\binom{n}{\ell+1} + (\frac{k+1}{h-\ell} - 1)\binom{n}{h-1}$$

we find

$$\frac{n-h+1}{h} \le \frac{3}{2}\,\frac{h-1}{n-2h+3} + \frac{3}{2}(k-1)^{-(h-\ell-1)} + \frac{1}{2}(k-1)^{-(h-\ell-2)} + (\frac{4}{h-\ell} - 1)$$

so that if $\ell' = h-\ell$:

$$3- \frac{\ell'-1}{h} \le \frac{3}{4} + \frac{3}{2^{\ell'}} + \frac{3}{2^{\ell'-1}} + \frac{4}{\ell'} - 1,$$

a contradiction for $\ell' \ge 3$. The case $\ell' = 2$ is settled by observing that in this case the term $\frac{1}{2}\binom{n}{\ell+1}$ can be dropped, i.e.

$$\frac{8}{3} \le 3- \frac{\ell'-1}{h} \le \frac{3}{4} + \frac{3}{4} + \frac{4}{2} - 1 = \frac{5}{2} ,$$

a contradiction. □

<u>THEOREM 2</u>. *Let* $n = kh$, $3 \le k \le h-3$. *Then if* $(h-1) \in H$ K_n^H *does not have the edge colouring property.*

<u>PROOF</u>. A partition containing $(h-1)$-sets contains either a g-set with $g \le k$ or at least two g-sets with $k+1 \le g \le h-2$ (for: $n = kh = k(h-1) + k$). Hence

$$\binom{n}{h-1} \le \frac{k}{2} \sum_{i=1}^{k} \binom{n}{i} + \frac{k}{2} \sum_{i=1}^{h-2} \binom{n}{i},$$

so

$$1 \le \frac{k}{2} \cdot \frac{k+1}{n-2k+1} \cdot (k-1)^{-(h-k-2)} + \frac{k}{2} \cdot \frac{h-1}{n-2h+3} \quad \ldots\ldots \tag{1}$$

hence, since $h-2 \ge k+1$:

$$1 \le \frac{k}{2} \cdot \frac{k+1}{(k+1)k-1} \cdot \frac{1}{k-1} + \frac{1}{2} \cdot \frac{k}{k-2},$$

a contradiction for $k \ge 5$. Therefore $k \le 4$. Returning to (1) we find for $k = 4$:

$$1 \le \frac{10}{n-9} \cdot \frac{1}{3} + 2 \cdot \frac{h-1}{2h+3},$$

$$\frac{1}{2h+3} \le \frac{2}{3(n-9)},$$

$$12h-27 \le 4h+6,$$

$$h \le 4,$$

a contradiction.

For $k = 3$ we find

$$\binom{n}{h-1} \le \sum_{i=1}^{h-2} \binom{n}{i} + \binom{n}{2} + 2\binom{n}{3} \le \frac{h-1}{n-2h+3} \binom{n}{h-1} + 3\binom{n}{3},$$

hence, since $h \ge 6$,

$$\frac{4}{h+3}\binom{n}{5} \le \frac{4}{h+3}\binom{n}{h-1} \le 3\binom{n}{3} = 3 \cdot \frac{4}{n-3} \cdot \frac{5}{n-4}\binom{n}{5},$$

$$(h-1)(3h-4) \le 5(h+3),$$

$$h < 5.$$

a contradiction. □

THEOREM 3. *Let* $n = kh-1$, $3 \le k \le \frac{1}{2}(h-3)$. *Then if* $h-2 \in H$ K_n^H *does not have the edge-colouring property.*

PROOF. Consider the partitions containing $(h-2)$-sets. Since $n = k(h-2)+(2k-1)$ and $2k-1 \le h-4$ each such partition contains at most k $(h-2)$-sets. Moreover, such a partition cannot contain only h, $(h-1)$ or $(h-2)$-sets since

$$n = ah + b(h-1) + c(h-2)$$

implies

$$b + 2c \equiv 1 \pmod h,$$
$$b + c \le k,$$
$$b + 2c \le 2k < h,$$
$$b + 2c = 1,$$
$$c = 0.$$

If a partition contains only one small set, say of size s, then

$$n = ah + b(h-1) + c(h-2) + s$$
$$b + 2c \equiv 1 + s \pmod h$$

and again it follows from $2k+1 < h$ that

$$b + 2c = 1 + s,$$
$$s \le 2k-1 \le h-4.$$

Hence

$$\binom{n}{h-2} \le \frac{k}{2} \sum_{i \le 2k-1} \binom{n}{i} + \frac{k}{2} \sum_{i \le h-3} \binom{n}{i}$$
$$\le \frac{k}{2} \cdot \frac{2k}{n-4k+1} \binom{n}{2k} + \frac{k}{2} \cdot \frac{h-2}{n-2h+5} \binom{n}{h-2} \quad \dots\dots\dots\dots\dots\dots \quad (2)$$

so that

$$\left(1 - \frac{k}{2} \cdot \frac{1}{k-2}\right) \le \frac{k}{2} \cdot \frac{2}{h-4} \cdot (k-1)^{-(h-2-2k)} \le \frac{1}{2} \cdot \frac{1}{(k-1)},$$

i.e.,

$$k \le 4.$$

Returning to (2) we find for $k = 4$ that $\frac{k}{2} \cdot \frac{h-2}{n-2h+5} = \frac{2h-4}{2h+4} = 1 - \frac{4}{h+2}$ so that $\frac{4}{h+2} \le \frac{4}{h-4} \cdot 3^{-(h-10)}$, a contradiction since $h \ge 2k+3 = 11$.

Finally, let $k = 3$, $n = 3h-1$, $h \geq 9$. By the usual arguments we find

$$\binom{n}{h-2} \leq \binom{n}{1} + \binom{n}{2} + \binom{n}{5} + \sum_{i \leq \frac{h+2}{2}} \binom{n}{i} + \sum_{i \leq \frac{2h+1}{3}} \binom{n}{i} \tag{3}$$

[as follows: If a partition contains 3 $(h-2)$-sets then by $n = 3(h-2)+5$ it also contains a g-set for $g = 1,2$ or 5. If a partition contains 2 $(h-2)$-sets then by $n = 2(h-2) + (h+3)$ it also contains a g-set for some $g \leq \frac{h+3}{2}$, and if it does not contain a g-set for some $g < \frac{h+3}{2}$ then it contains 2 $(\frac{h+3}{2})$-sets. Finally if it contains only one $(h-2)$-set and it does not contain a g-set for $g \leq \frac{h+3}{2}$ then it contains exactly three other sets, and at least one of them has size $\leq \frac{2h+1}{3}$.]
Estimating roughly we find for $g = \lfloor \frac{2h+4}{3} \rfloor$

$$\binom{n}{h-2} \leq 3 \sum_{i \leq \lfloor \frac{2h+1}{3} \rfloor}{}' \binom{n}{i} \leq 3 . \frac{g}{n-2g+1} \binom{n}{g} < \binom{n}{g+1}$$

so that

$$h-2 < \left\lfloor \frac{2h+4}{3} \right\rfloor + 1,$$

$$h \leq 10.$$

But for $h = 9,10$ it can be verified directly that (3) is contradictory. □

<u>THEOREM 4</u>. *Let* $n = 2h+\ell$, $1 \leq \ell \leq h-2$. *Then* K_n^H *does not have the edge-colouring property.*

<u>PROOF</u>. The case $\ell \neq h-2$ of this theorem is due to JOHNSON [3]. We give however a shorter direct proof. For $\ell \leq h-3$ ws find as usual:

$$\binom{n}{h} \leq \sum_{i \leq \frac{h+\ell}{2}} \binom{n}{i} + \binom{n}{\ell}$$

$$< \frac{\frac{h+\ell}{2}+1}{n-(h+\ell)+1} \binom{n}{\lfloor \frac{1}{2}(h+\ell+2) \rfloor}$$

$$< \binom{n}{\lfloor \frac{1}{2}(h+\ell+2) \rfloor} + \binom{n}{\ell} \leq \binom{n}{h-1} + \binom{n}{\ell}$$

$$\left(\frac{n-h+1}{h} - 1\right)\binom{n}{h-1} \le \binom{n}{\ell},$$

$$2\binom{n}{h-1} = \frac{n-\ell}{h}\binom{n}{h-1} \le \frac{n-\ell}{\ell+1}\binom{n}{\ell} = \binom{n}{\ell+1},$$

$$h-1 < \ell+1,$$

$$\ell > h-2,$$

a contradiction.

For $\ell = h-2$, $n = 3h-2$ the slightly refined estimate

$$\binom{n}{h} \le \sum_{i<\frac{h+\ell}{2}} \binom{n}{i} + \frac{1}{2}\binom{n}{\frac{1}{2}(h+\ell)} + \binom{n}{\ell}$$

$$= \sum_{i\le h-2} \binom{n}{i} + \frac{1}{2}\binom{n}{h-1} + \binom{n}{h-2}$$

works:

$$\frac{n-h+1}{h}\binom{n}{h-1} \le \left(\frac{h-1}{n-2h+3} + \frac{1}{2} + \frac{h-1}{n-h+2}\right)\binom{n}{h-1},$$

$$\frac{2h-1}{h} \le \frac{h-1}{h+1} + \frac{1}{2} + \frac{h-1}{2h},$$

$$2 - \frac{1}{h} \le 1 - \frac{2}{h+1} + \frac{1}{2} + \frac{1}{2} - \frac{1}{2h},$$

$$\frac{2}{k+1} \le \frac{1}{2h},$$

$$4h \le h+1,$$

a contradiction. $\square$

2. THE CASE k = 1

THEOREM 5. *Let* $0 < \ell \le h$. *Then* $\hat{K}^{h}_{h+\ell}$ *has a 1-factorization if and only if* $\hat{K}^{\ell-1}_{h+\ell}$ *has one.*

PROOF. Given a 1-factorization of $\hat{K}^{\ell-1}_{h+\ell}$, add the 1-factors $\{a, x\backslash a\}$ for all $a \subset x$, $\ell \le |a| \le h$ to obtain a 1-factorization of $\hat{K}^{h}_{h+\ell}$. Conversely, suppose that F is a 1-factorization of $\hat{K}^{h}_{h+\ell}$ with maximum number of 1-factors of the form $\{a,x\backslash a\}$. Let $a \subset x$, $\ell \le |a| \le h$. If the 1-factor P_1 containing a is not $\{a,x\backslash a\}$, but, say, $\{a,a_1,\ldots,a_r\}$ and the 1-factor P_2 containing $x\backslash a$ is $\{x\backslash a,b_1,\ldots,b_s\}$, then

$$(F\backslash\{P_1,P_2\}) \cup \{\{a,x\backslash a\},\{a_1,\ldots,a_r,b_1,\ldots,b_s\}\}$$

is a 1-factorization containing one more pair of complementary sets, a contradiction. Hence F contains all complementary pairs $\{a,x\backslash a\}$ for $a \subset x$, $\ell \le |a| \le h$ and removing these yields a 1-factorization of $K_{h+\ell}^{\wedge\ell-1}$. □

REMARK. Of course $K_h^{\wedge h}$ is 1-factorizable. This theorem settles the case k = 1, and provides some information for the case k = 2, ℓ = 0:

COROLLARY 1. $K_{2h}^{\wedge h}$ *is* 1-*factorizable iff* $h \le 4$.

PROOF. By (i),(ii),(iii) of the introduction it suffices to show that $K_{2h}^{\wedge h}$ is not 1-factorizable for $h \ge 5$. By the above theorem this is equivalent to showing that $K_{2h+2}^{\wedge h}$ is not 1-factorizable for $h \ge 4$, which follows from theorem 4. □

COROLLARY 2. $K_{2h-1}^{\wedge h}$ *is* 1-*factorizable iff* $h \le 6$.

PROOF. $K_{2h-1}^{\wedge h}$ is 1-factorizable iff $K_{2h-1}^{\wedge h-2}$ is. But $2h-1 = 2(h-2)+3$ hence $K_{2h-1}^{\wedge h-2}$ is not 1-factorizable for $h \ge 7$ by theorem 4. But by (i)-(iii) of the introduction we see the 1-factorizability for $h \le 6$. □

3. POSITIVE RESULTS

Although not mentioned in all cases we sometimes assume $k \ge 3$ (which we may by theorems 4,5 and corollaries 1,2).

THEOREM 6. *Let* $n = hk$ *and* $k \ge h-1$. *Then* K_n^H *is* 1-*factorizable.*

PROOF. Use 1-factors of the form

$$n = \frac{h}{(h,g)} * g + (k - \frac{g}{(h,g)}) * h$$

for $g \in H\backslash\{h\}$. In order to accomodate all g-sets we need $\frac{(h,g)}{h}\binom{n}{g} =: N_g$ such 1-factors for each g, and

$$\frac{1}{k}\left(\binom{n}{h} - \sum_{\substack{g\in H\\ g\neq h}} \frac{(h,g)}{h} \cdot \binom{n}{g} \cdot \left(k - \frac{g}{(h,g)}\right)\right) =: N_h$$

1-factors of the type $n = k*h$ are needed for the remaining h-sets.
By Baranyai's theorem this setup will produce a solution iff

(i) N_g is integral for each $g \in H$
(ii) $N_g \geq 0$ for each $g \in H$
(iii) $k \geq \frac{g}{(h,g)}$ for each $g \in H$.

Ad (i): if $(h,g) = ah + bg$ for some integers a and b, then

$$\frac{(h,g)}{h}\binom{n}{g} = a\binom{n}{g} + b\binom{n-1}{g-1}$$

is integral. Concerning N_h, since $\binom{n}{h} = k\binom{n-1}{h-1}$ and $\frac{g}{h}\binom{n}{g} = k\binom{n-1}{g-1}$ we find that

$$N_h = \binom{n-1}{h-1} - \sum_{\substack{g\in H\\ g\neq h}} \left(N_g - \binom{n-1}{g-1}\right)$$

is also integral.

Ad (ii): Since $N_g \leq \frac{1}{2}\binom{n}{g}$ for $g < h$ it suffices to prove that

$$\binom{n}{h} \geq \sum_{g=0}^{h-1} \frac{1}{2}\binom{n}{g}(k-1).$$

But

$$\sum_{g=0}^{h-1} \binom{n}{g} \leq \frac{h}{n-2h+1}\binom{n}{h},$$

and for $k \geq 3$ we indeed have

$$\frac{1}{2}(k-1) . \frac{h}{n-2h+1} < \frac{1}{2}\,\frac{k-1}{k-2} \leq 1.$$

(For $k = 2$ one may verify directly that $N_h \geq 0$, or appeal to corollary 1.)

Ad (iii): Since $k \geq h-1$ it follows immediately that $\frac{g}{(h,g)} \leq g \leq h-1 \leq k$. □

In fact we proved the more general

THEOREM 6A. *Let* $n = hk$ *and let for each* $g \in H$ $k \geq \frac{g}{(h,g)}$. *Then* K_n^H *is* 1-*factorizable.* □

COROLLARY 3. *Let* h *be even and* $H = \{0,2,4,\ldots,h\}$, *and let* $k \geq \frac{1}{2}(h-2)$. *Then* K_n^H *is* 1-*factorizable.* □

THEOREM 7. *Let* $n = hk-1$ *and* $k \geq h-1$. *Then* $\hat{K}_n^h$ *is* 1-*factorizable.*
This theorem is an immediate consequence of theorem 6 and the following proposition:

PROPOSITION 1. *Let* $H \subset \{1,2,\ldots,n\}$ *be such that* $j \in H \longrightarrow j-1 \notin H$. *Let* $H' = \{j \mid j \in H$ *or* $(j+1) \in H\}$. *Then if* K_{n+1}^H *is* 1-*factorizable,* $K_n^{H'}$ *is* 1-*factorizable also.*

PROOF. Let $x' = x \cup \{\infty\}$ be some set of n+1 elements. Given a 1-factorization of K_{n+1}^H (with vertex set x'), remove the point ∞ from each set containing it. This yields a 1-factorization of $K_n^{H'}$. □

THEOREM 8. *Let* $n = h(h-2)$. *Then* $\hat{K}_n^h$ *is* 1-*factorizable.*

PROOF. Observe that $\hat{K}_n^h = K_n^h \cup \hat{K}_n^{h-1}$, where K_n^h is 1-factorizable since $h|n$, and $\hat{K}_n^{h-1}$ is 1-factorizable by theorem 7 since $n = (h-1)^2-1$. □

THEOREM 9. *Let* $n = h(h-2)-1$. *Then* $\hat{K}_n^h$ *is* 1-*factorizable.*

PROOF. $\hat{K}_n^h = (K_n^h \cup K_n^{h-1}) \cup \hat{K}_n^{h-2}$ is 1-factorizable by proposition 1 and theorem 7 since $n = (h-2)h-1$. □

THEOREM 10. *Let* $n = kh - 1$, *where* h *is even and* $k \geq \frac{1}{2}h-1$. *Then* $\hat{K}_n^h$ *is* 1-*factorizable.*

PROOF. follows immediately from corollary 3 and proposition 1. □

What remains to be proved is

> Let $n = kh-1$ where h is odd, $k \geq 3$, and $\frac{1}{2}(h-1) \leq k \leq h-3$.
>
> Then $\hat{K}_n^h$ is 1-factorizable.

This is surely true, but I was unable to find a conclusive argument. Some partial results follow:

- Note that $\hat{K}_n^h$ is 1-factorizable so that it suffices to find a 1-factorization for K_n^H where $H = \{k+1, k+2, \ldots, h\}$.
- Let $h = 2k+1$ so that $n = kh-1 = 2k^2+k-1 = (k+1)(2k-1) = (k+1)(h-2)$.

A 1-factorization of K_n^H is obtained as follows:

Let $k+1 \leq g \leq h-3$. If $s := (g,h-2) > 1$ then put the g-sets into 1-factors looking like

$$n = (k+1 - \tfrac{g}{s}) * (h-2) + \frac{h-2}{s} * g.$$

Note that the coefficients are integral and positive since $\frac{g}{s} \leq \frac{h-3}{2} = k-1 < k+1$. The number N_g of 1-factors of this type is $\frac{s}{h-2} \binom{n}{g}$ which is integral also. If $(g,h-2) = 1$ we use N'_g 1-factors of the type

$$n = (h-2-g) * (h-1) + (k+2+g-h) * (h-2) + 1 * g$$

and N''_g 1-factors of the type

$$n = (h-2-g) * h + (k+1+g-h) * (h-2) + 2 * g.$$

Again all coefficients are integral and positive. We shall choose $N''_g = (k-1).N'_g$ so that $N'_g = \frac{1}{2k-1} \binom{n}{g}$, which is integral since $(g,2k-1) = 1$. The remaining (h-2)-sets are put into 1-factors of the type

$$n = (k+1) * (h-2),$$

of which we need

$$N_{h-2} = \frac{1}{k+1} \left(\binom{n}{h-2} - \sum_{\substack{g \\ (g,h-2)>1}} (h+1+\tfrac{g}{s})\, N_g - \sum_{\substack{g \\ (g,h-2)=1}} (k(k+1+g-h)+1) N'_g \right).$$

Now $\binom{n}{g} = \frac{(k+1)(h-2)}{g} \binom{n-1}{g-1}$ so that

$$N_{h-2} = \binom{n-1}{h-3} - \sum_{\substack{g \\ (g,h-2)>1}} \left(N_g - \binom{n-1}{g-1} \right) - \sum_{\substack{g \\ (g,h-2)=1}} \left((k+g-h+2) N'_g - \binom{n-1}{g-1} \right)$$

which is integral. It is also positive, for $(k+1-\frac{g}{s})\frac{s}{h-2} < \frac{1}{3}k$ and $(k(k+1+g-h)+1).\frac{1}{2k-1} < \frac{1}{2}(k-1)$ so that it suffices to show that

$$\binom{n}{h-2} \geq \tfrac{1}{2}(k-1)\sum_{g=1}^{h-3}\binom{n}{g}$$

which is true by lemma 2.

Finally we put the remaining h- and (h-1)-sets into 1-factors of the type

$$n = (h-1) * h + 1 * (h-1).$$

Indeed, since $\binom{n}{h}/\binom{n}{h-1} = \frac{n-h+1}{h} = k-1$ and $N''_g/N'_g = k-1$, these sets remain in proportion (k-1): 1. The number of such 1-factors is

$$N_{h-1} = \binom{n}{h-1} - \sum_{(g,h-2)=1}(h-2-g)N'_g$$

which is integral and positive since $\frac{h-2-g}{2k-1} < 1$ and

$$\sum_{g=1}^{h-3}\binom{n}{g} < \binom{n}{h-2} < \binom{n}{h-1}.$$

This proves

<u>THEOREM 11</u>. *Let* $n = (k+1)(2k-1) = k(2k+1)-1$. *Then* $\hat{K}_n^{2k+1}$ *is 1-factorizable.*

Having settled one extreme of the interval $\frac{1}{2}(k-1) \leq k \leq h-3$, we proceed to the other one.

- Let $h = k+3$ so that $n = kh-1 = k^2+3k-1$. Observing that $(n,h-2) = (2k-1,k+1) = (3,k+1)$ we distinguish the cases $(n,h-2) = 1$ and $(n,h-2) = 3$. In the first case we need only $h \geq k+3$, h odd. If $(n,h-2) = 1$ then use the partitions

$$\begin{aligned} n &= (k-1) * h + 1 * (h-1) \\ n &= \left(\tfrac{2k+1-h}{2}\right) * h + \left(\tfrac{h+1}{2}\right) * (h-2) \\ n &= (2k+1-h) * (h-1) + (h-k) * (h-2) \end{aligned}$$

each N_1 resp. N_2 resp. N_3 times.

Since $\binom{n}{h}/\binom{n}{h-1} = k-1$ we need $N_2/N_3 = 2(k-1)$ and it follows that in order to factorize $K_n^{\{h-2,h-1,h\}}$ we need

$$N_3 = \frac{1}{n}\binom{n}{h-2}, \; N_2 = 2(k-1)N_3, \; N_1 = \binom{n}{h-1} - (2k+1-h)N_3.$$

$N_1 \geq 0$ since $k \leq h-3$, so we proved:

PROPOSITION 2. *Let* $n = kh-1$, h *odd,* $3 \leq k \leq h-3$, $H = \{h-2,h-1,h\}$. *Then* K_n^H *is* 1*-factorizable if* $n \mid \binom{n}{h-2}$, *hence in pariteular if* $(n,h-2) = 1$.

If $(n,h-2) = 3$ we use the partitions

$$n = (k-1) * h + 1 * (h-1)$$
$$n = \left(\frac{2k+1-h}{2}\right) * h + \left(\frac{h+1}{2}\right) * (h-2)$$
$$n = \left(\frac{2k-1-h}{2}\right) * h + 2 * (h-1) + \left(\frac{h-1}{2}\right) * (h-2)$$

with frequencies $N_1 = \binom{n}{h-1} - 2N_3$, $N_2 = \frac{k-3}{2}N$, $N_3 = \frac{h-5}{6}N$ where $N = \frac{3}{n}\binom{n}{h-2}$. Note that h is odd and congruent 2 mod 3 so that $\frac{h-3}{2}$ and $\frac{h-5}{6}$ are integers. Also N is integral because if $3 = (n,h-2) = an + b(h-2)$ for some integers a and b then $N = a\binom{n}{h-2} + b\binom{n-1}{h-3}$. Next $\left(\frac{h+1}{2}\right)N_2 + \left(\frac{h-1}{2}\right)N_3 = \frac{1}{3}(h^2-3h-1)N = \binom{n}{h-2}$ and $N_1 + 2N_3 = \binom{n}{h-1}$ and $\left(\frac{h-5}{2}\right) * N_2 + \left(\frac{h-7}{2}\right) * N_3 = \frac{1}{3}(h-4)(h-5)\, N = 2(k-1)N_3$ so that the last two types of partition contain the right proportion of $(h-1)$- and h-sets. Finally $N_1 \geq 0$, which proves

THEOREM 12. *Let* $n = h(h-3)-1$, h *odd,* $h \geq 5$. *Then* $\hat{K}_n^h$ *is* 1*-factorizable.*

What remains to be proved is the 1-factorizability of $\hat{K}_n^h$ for h odd, $n = kh-1$, $k \geq 3$, $\frac{1}{2}(h+1) \leq k \leq h-4$. Continuing as above it is easy enough to do some more special cases, but I see no general way of treating all values of k for a given h at the same time. The difficulty is that just as in the case above ($k=h-3$) one has to distinguish subcases according to the divisibility properties of binomial coefficients $\binom{n}{h-i}$, $i = 0,1,2,\ldots$.

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