Kernel Bounds for Structural Parameterizations of Pathwidth

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Joint work with
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Universiteit Utrecht

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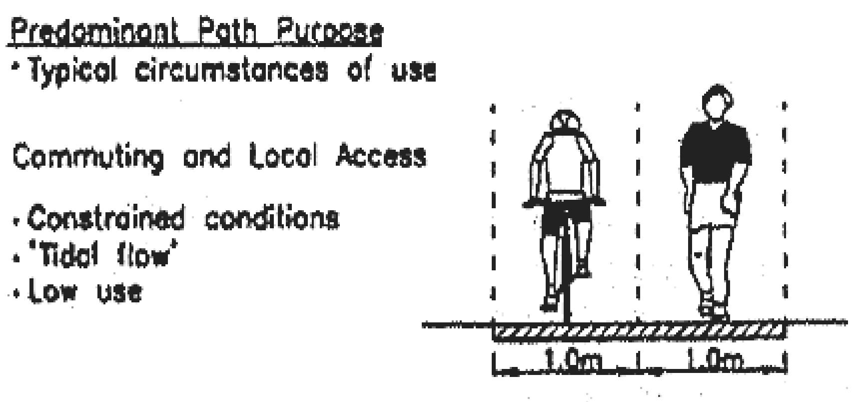

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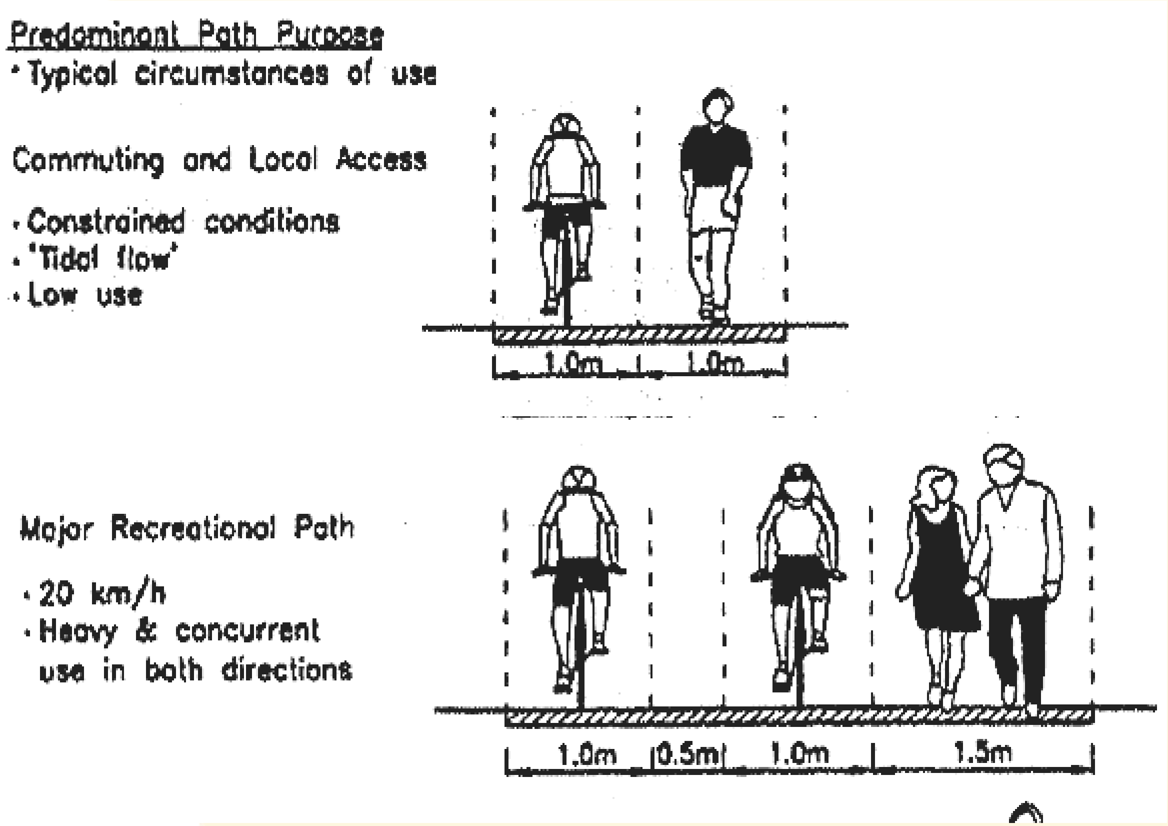

What is pathwidth?

<u>Predominant Path Purpose</u>
• Typical circumstances of us

Commuting and Local Acces

• Constrained conditions
• 'Tidal flow'
• Low use

Major Recreational Path

• 20 km/h
• Heavy & concurrent use in both directions

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- Both pathwidth and treewidth have been introduced many times under different names
  - (vertex separation number, node search number, partial k-tree, etc …)
- Play crucial roles in Robertson & Seymour's proof of the Graph Minor Theorem

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- When comparing pathwidth to treewidth:
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  - Dynamic programming algorithms for path decompositions are simpler and use less memory
- Important to find **low-width** path and tree decompositions **efficiently**

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  - Pathwidth is even NP-complete on planar graphs
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- Runtime ☹(n) for every fixed k

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- Cannot guarantee output is smaller than input (else P=NP)
- So given a graph G of “difficulty” k, shrink G to poly(k)
  - Afterwards we can shrink no more

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- Pick a measure for graph difficulty that is larger than pw(G)
- Can we shrink to size polynomial in the larger measure?
  - For example: size of a minimum vertex cover
  - (Vertex set that covers all edges)

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- This work: reduction rules, analysis & lower bounds

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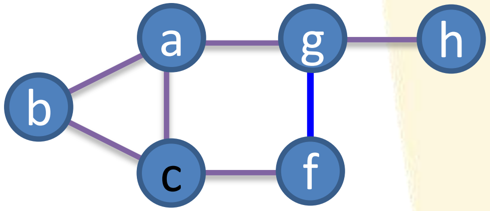

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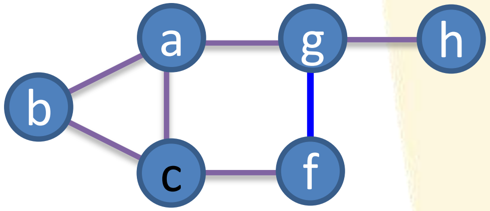

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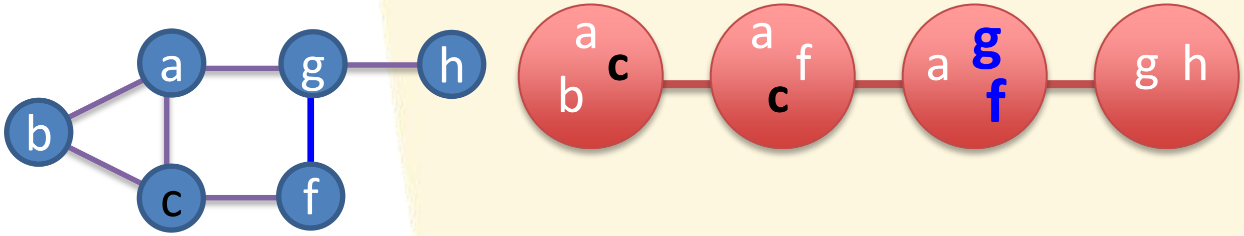

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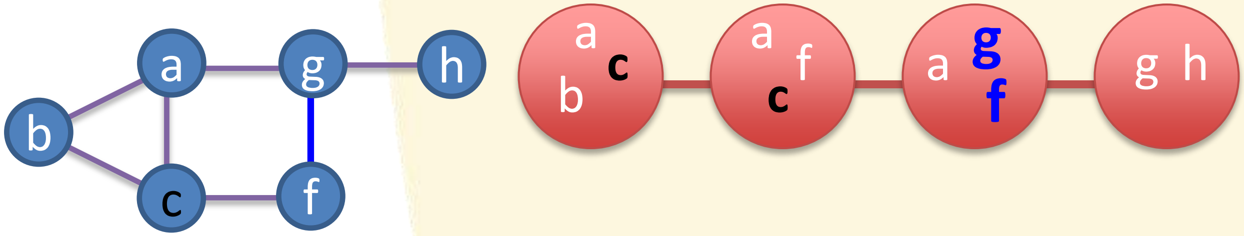

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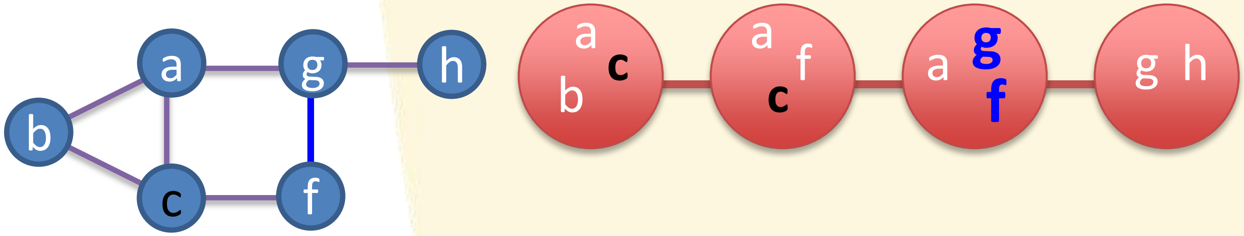

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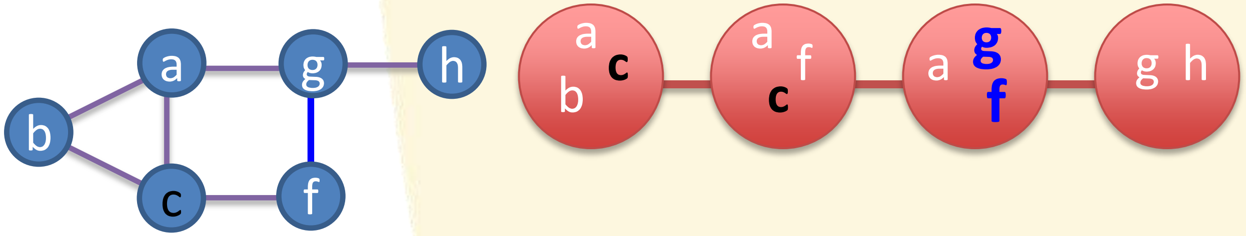

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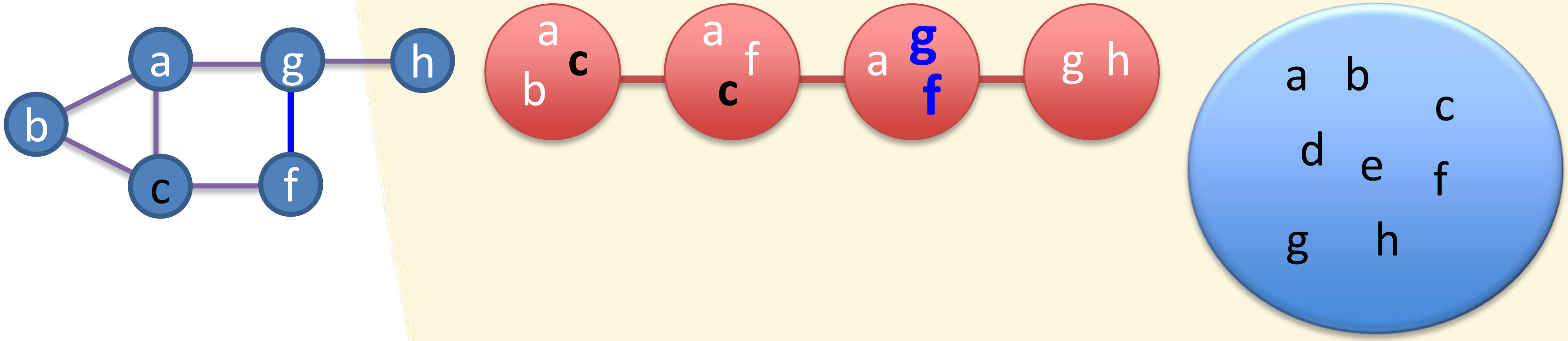

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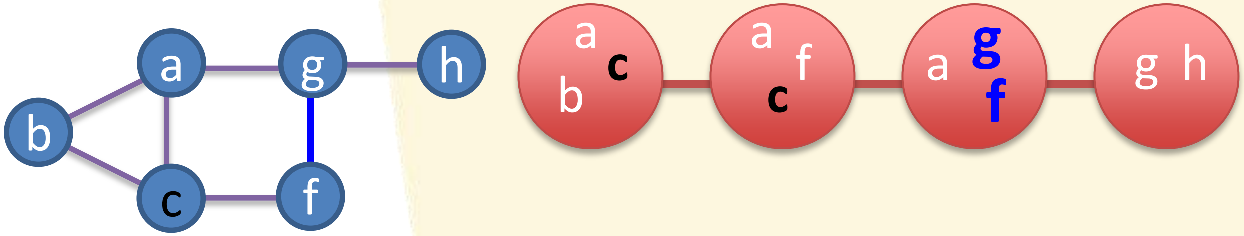

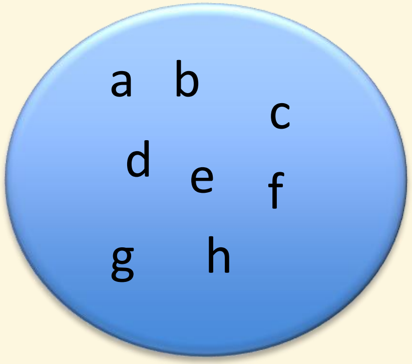

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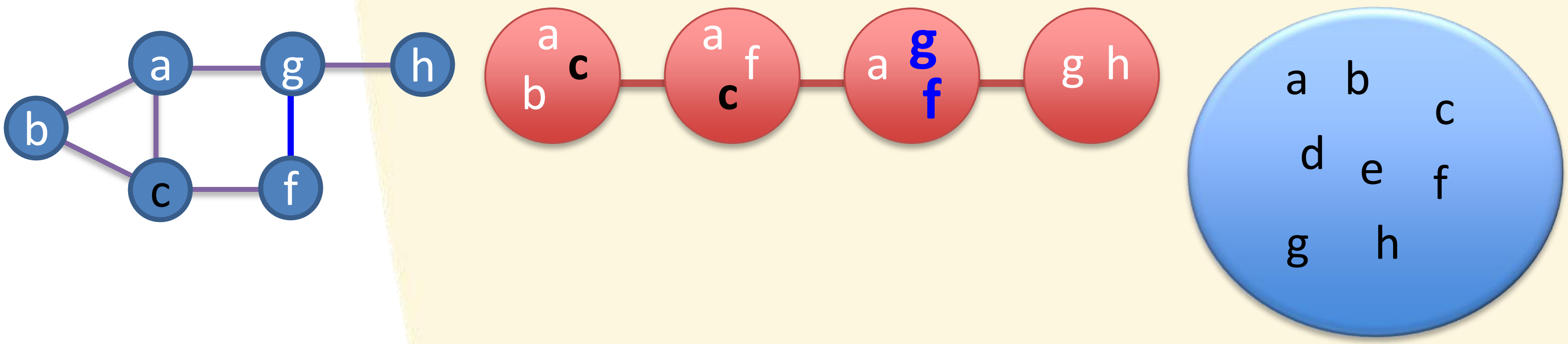

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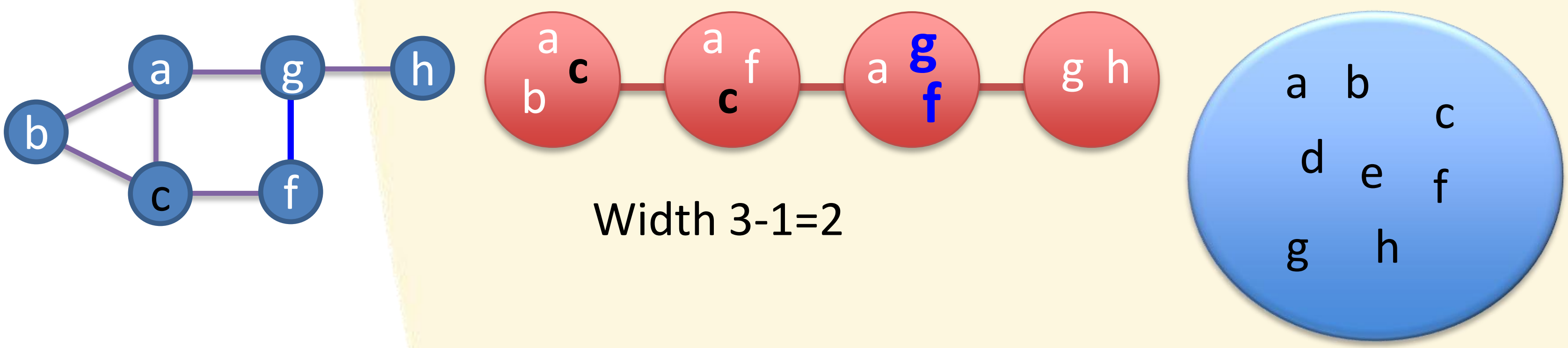

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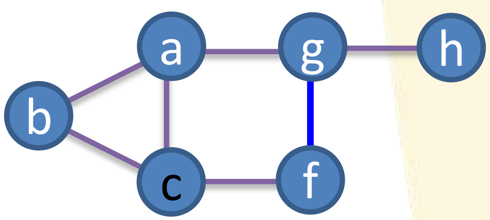
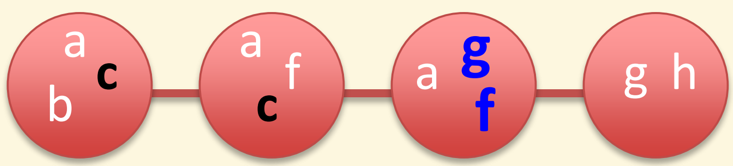
- The *pathwidth* of a graph G is the minimum width of a path decomposition of G





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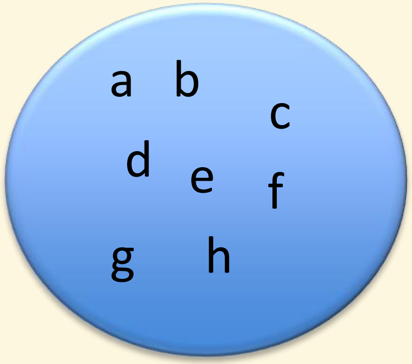

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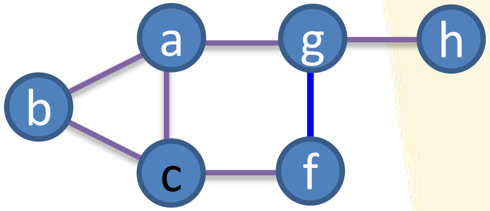

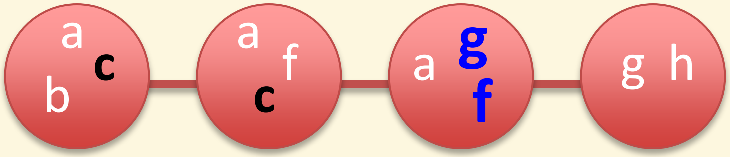

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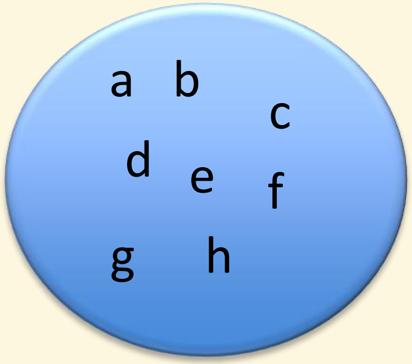

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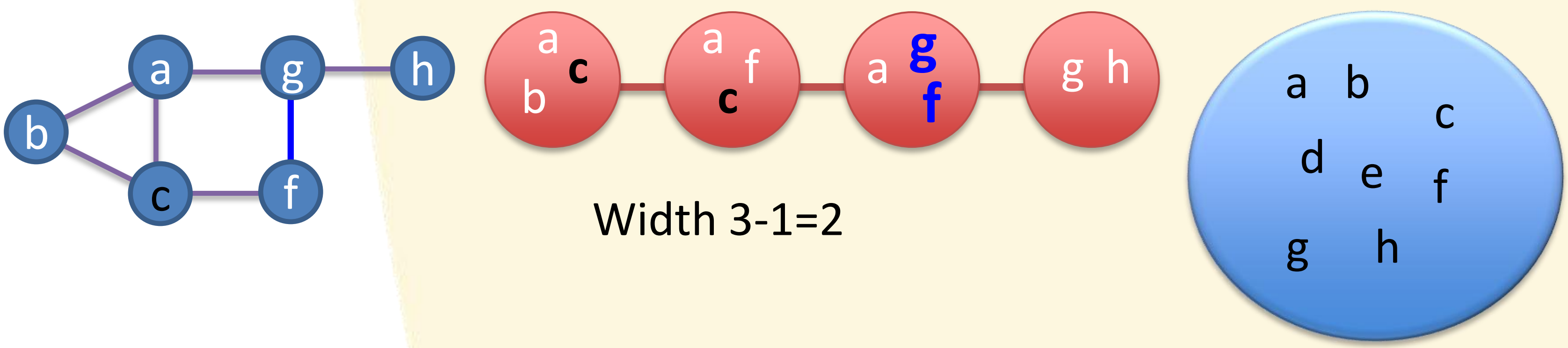

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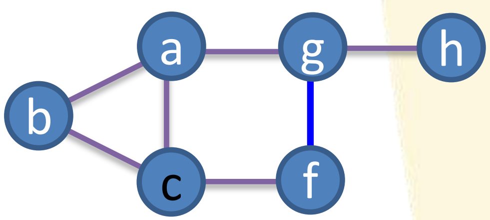

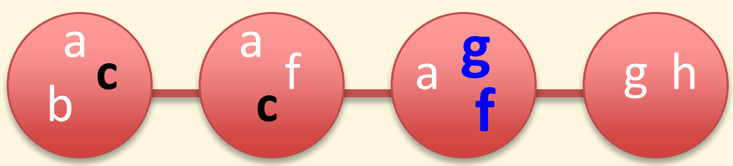

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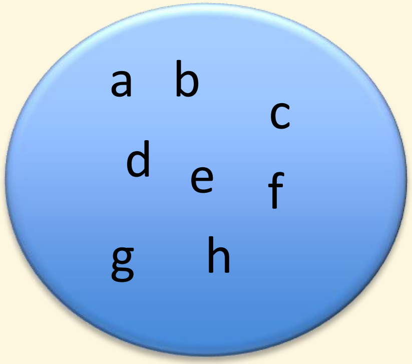

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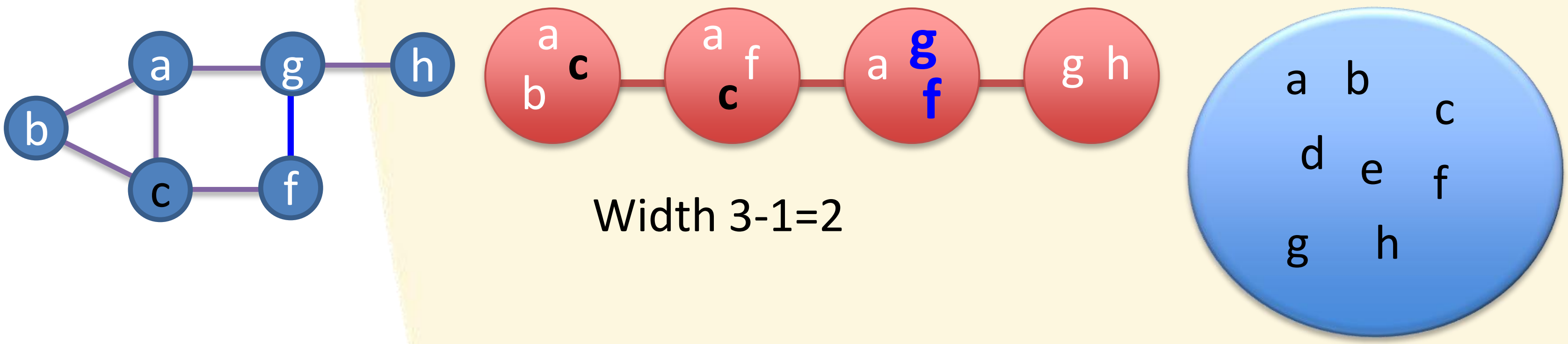
- Trees have treewidth 1, but may have pathwidth $\Theta(\log n)$



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- In practical settings:
  - Guess k, or work with upper- and lower bounds

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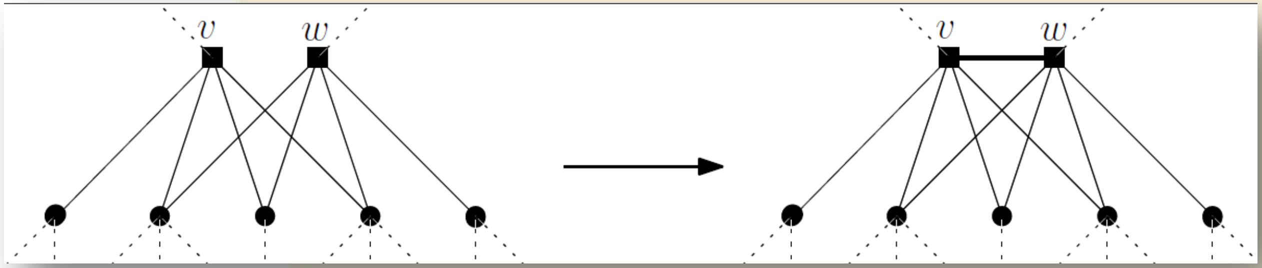

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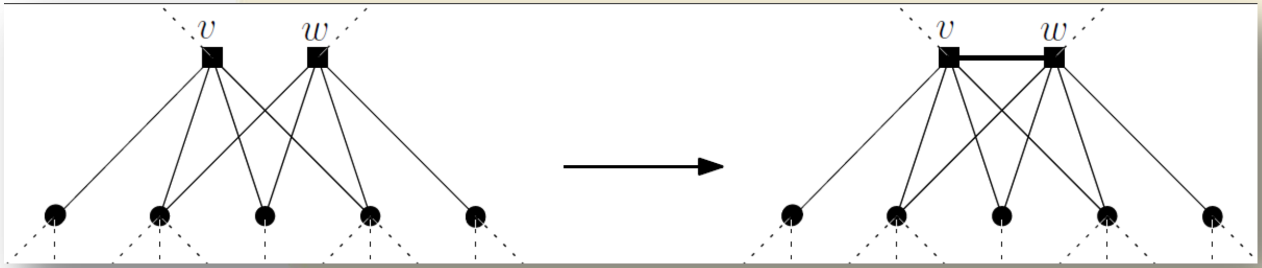

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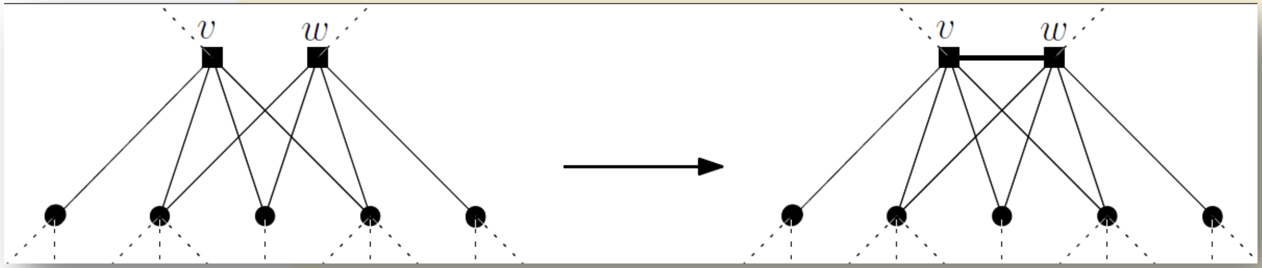

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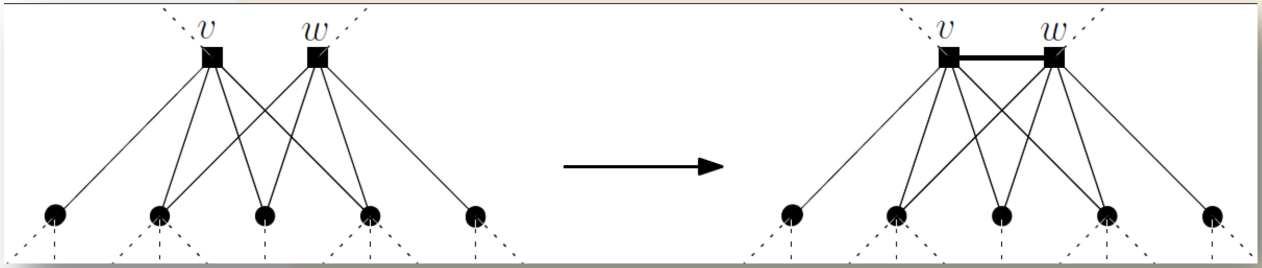

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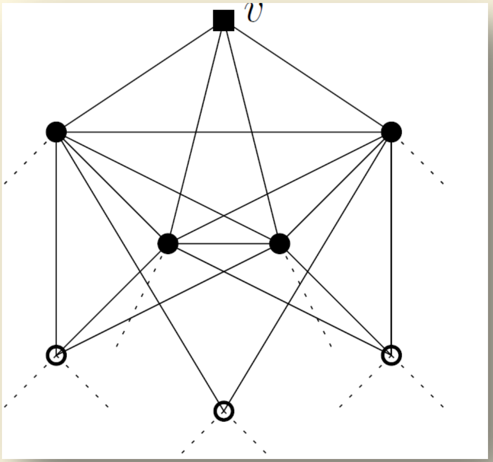

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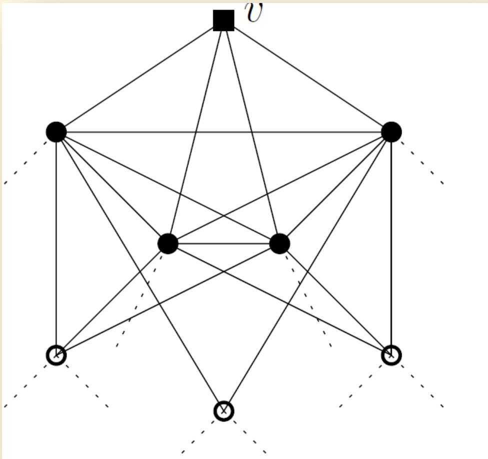

Treewidth Simplicial Vertex Rule

Let v be a simplicial vertex in G.

- If deg(v) ≥ k+1 then tw(G) > k
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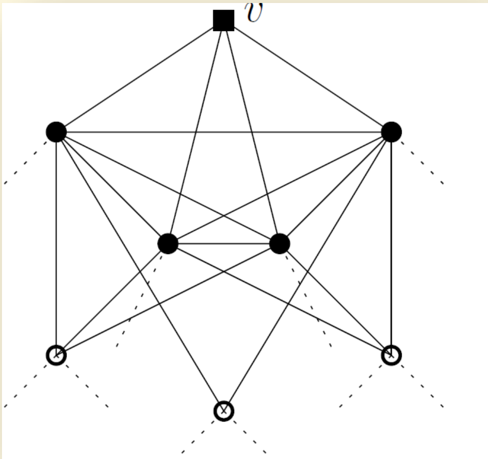

Closed neighborhood of v is a k+2 clique

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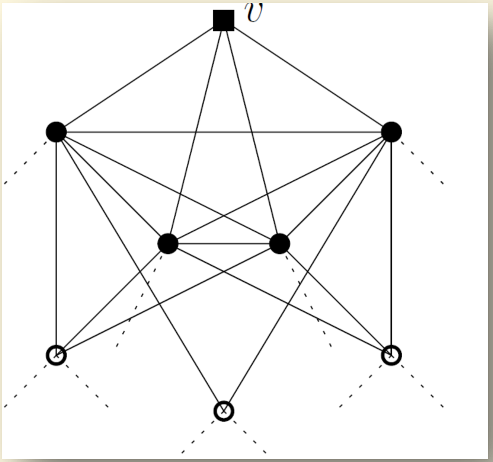

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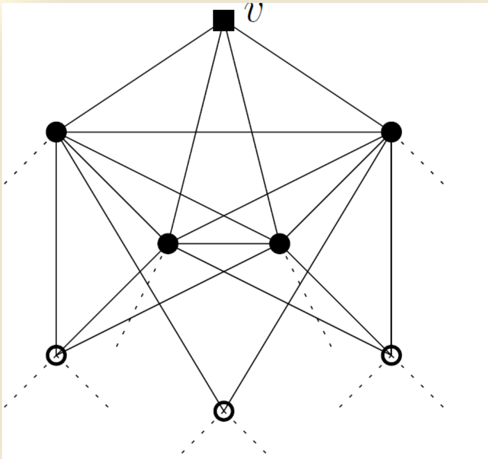

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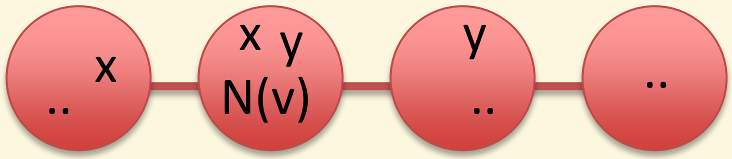

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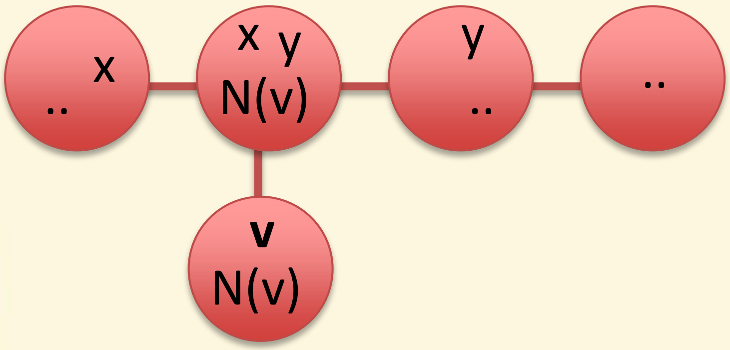

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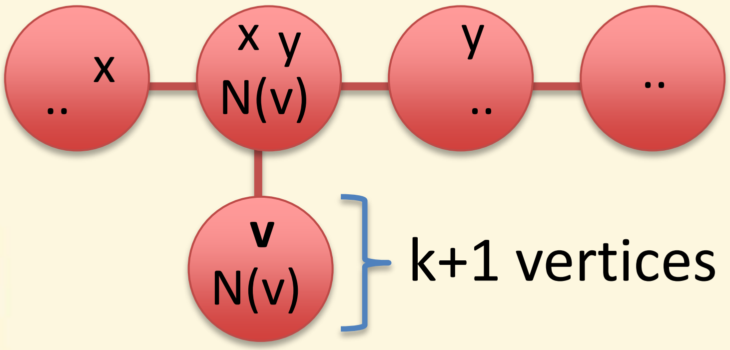


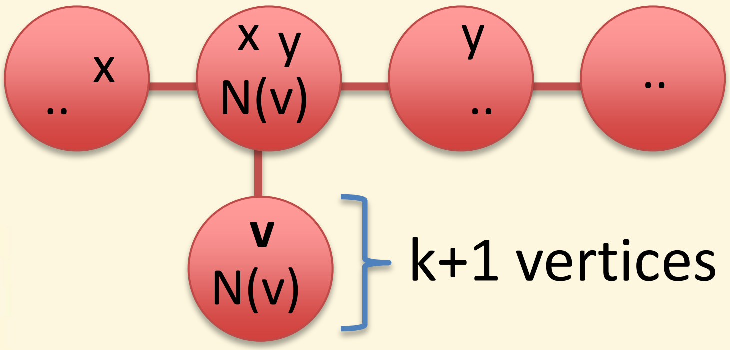
Treewidth Simplicial Vertex Ru

Let v be a simplicial vertex in G

- If deg(v) ≥ k+1 then tw(G) > k
- If deg(v) ≤ k then deleting v is safe for treewidth k

Helly property for trees ensures that there is a bag containing N(v)

Simplicial Vertices



Pathwidth Simplicial Vertex Rule?

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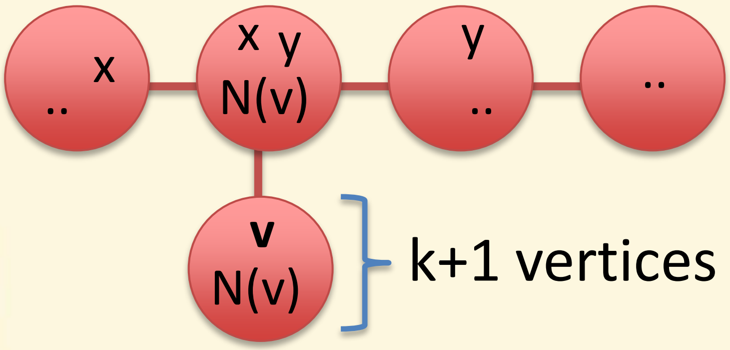

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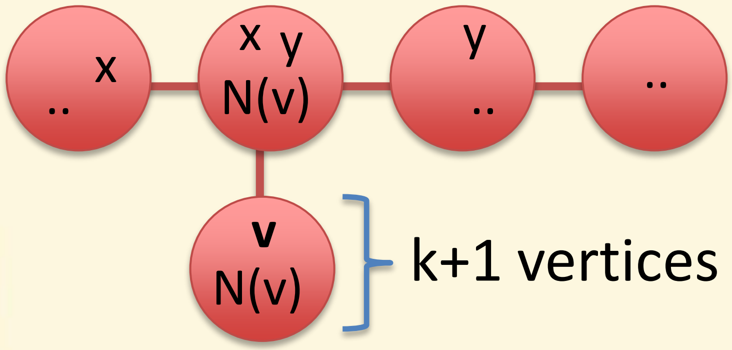

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Repeated application would eat up a tree

Degree-one vertices

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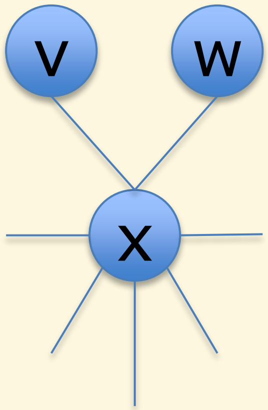

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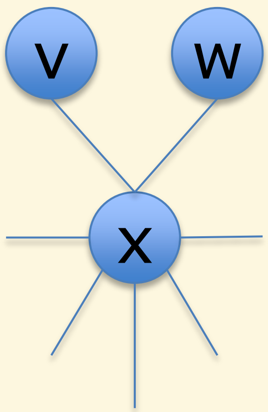

Pathwidth Degree-1 Vertex Rule
If vertex v is only adjacent to x, and
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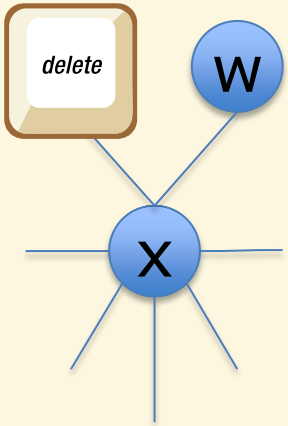

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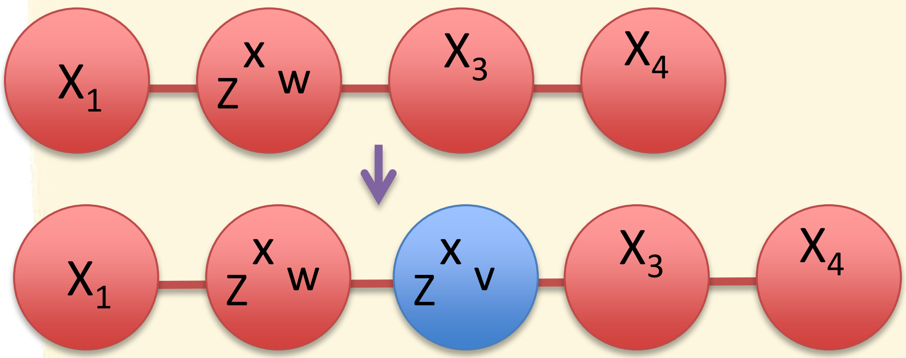

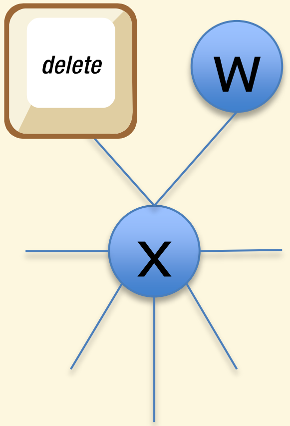

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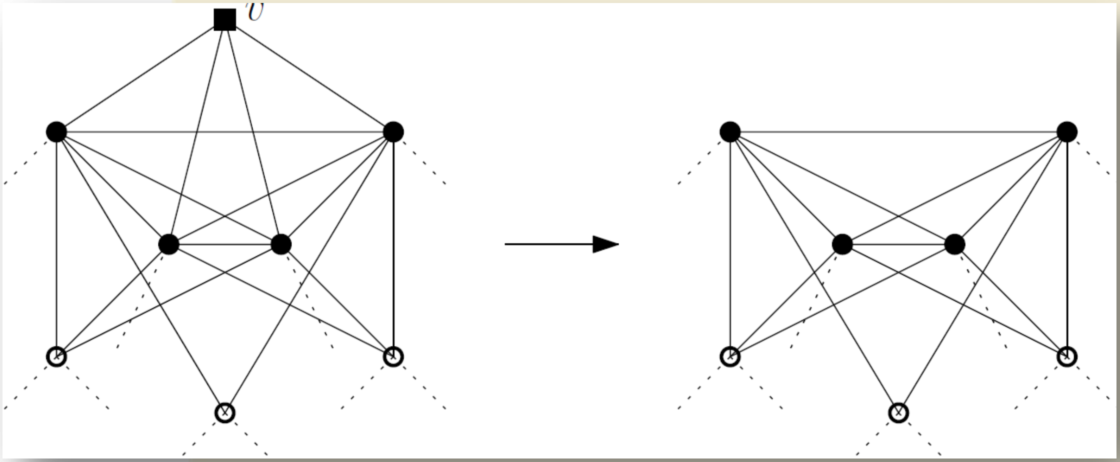

Pathwidth Simplicial Vertex Rule
If v is simplicial with $2 \le \deg(v) \le k$, and
$\forall$ {x,y} in N(v), $\exists$ simplicial vtx $\notin$ N[v] seeing x and y,
then deleting v is safe for pathwidth k

Effects of the reduction rules

- If G has a vertex cover X of size ℓ, and you work relative to the structure of X:
  - Easy counting arguments prove $O(\ell^3)$ vertices
  - (After some trivial rules)

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Polynomial kernels (sizes in # vertices)

- $O(\ell^3)$ when ℓ is the vertex cover number
- $O(c\ell^3 + c^2\ell^2)$ when ℓ is the size of a vertex set whose removal gives components of at most c vertices each
- $O(\ell^4)$ when ℓ is the size of a vertex set whose removal results in disjoint stars

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  - Deleting the small set yields a clique

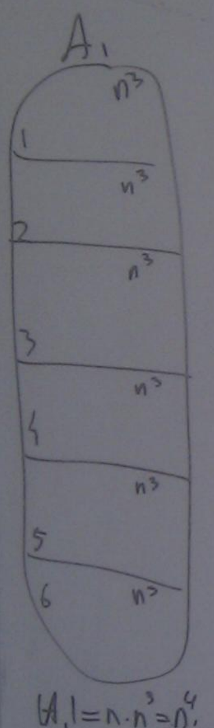
Pathwidth and Treewidth **do not** admit polynomial kernels parameterized by vertex-deletion distance to a clique (unless NP $\subseteq$ coNP/poly)

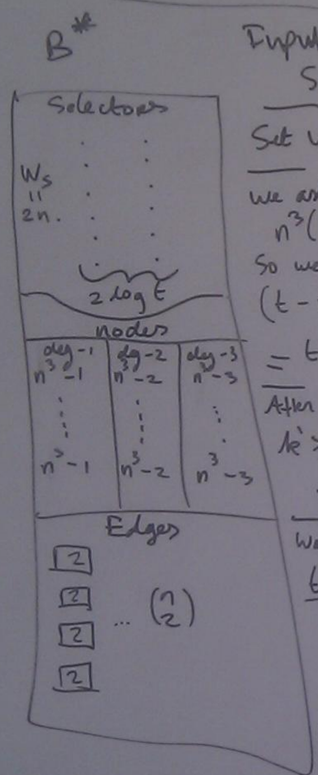
Details of the construction …

Cutwidth crosscomposes into TW by clique deletion.

known: cutwidth is NP-complete on planar max. deg ≤ 3 graphs.

Poly equ.: all input graphs have same degree sequence (characterized by n^4) and ask for same k. Sort vertices by degree. Assume $n \geq 3$.





Input instance on deg ≤ 3 graph has cutwidth $\leq |E| \leq \frac{3n}{2}$.

So $1 \leq k \leq \frac{3n}{2}$.

Set $w_s := 2n$.

we ask for (on small subinstance $(V_i \cup B^*)$):

$n^3(n+1) + k - 1$.

So we ask in the big instance for:

$(t-1)n^4 + n^3(n+1) + k - 1 + 2n \log t$

$= tn^4 + n^3 + k - 1 + 2n \log t$.

After solving instance we can eliminate the rest, provided that

$k' \geq (t-1)n^4 - 1 + 2n(2\log t) + 2\binom{n}{2} + n(n^3-1)$

$= tn^4 - 1 + 4n\log t + 2\binom{n}{2} - n$.

We need:

$\underline{tn^4} + n^3 + k - 1 + 2n\log t \geq \underline{tn^4} - 1 + 4n\log t + 2\binom{n}{2} - n$

$n^3 + k - 1 \geq 2n\log t + 2\binom{n}{2} - n - 1$

$n^3 + k \geq 2n\log t + 2\binom{n}{2} - n$.

Should work if $n \geq \log t$:

Then $2n\log t + 2\binom{n}{2} - n \leq 2n^2 + 2\binom{n}{2} - n \leq 2n^2 + \frac{2n(n-1)}{2} - n$

$\leq 2n^2 + n^2 = 3n^2$. So: $n^3 \geq 3n^2 \iff n \geq 3$.

$n^2 > c(n+1)^k$

$c(n+1)^k \leq c(2n)^k$

$\leq c \cdot 2^k \cdot n^k$.

$n^2 > c \cdot 2^k \cdot n^k$

$\iff n^{2-k} > c \cdot 2^k$

$\iff n = (c \cdot 2^k)^{\frac{1}{2-k}}$.

Zij near tweemacht. $f(n+1) = n^2$.

$f(n) = n$. $f(n+1) = 2^{2^{\lceil \lg \lg (n+1) \rceil}}$

Laat zien: $\lceil \lg \lg (n+1) \rceil = (\lg \lg n) + 1$.

We weten: $\lg \lg (n+1) > \lg \lg n$, en $\lg \lg (n) \% 1 = 0$.

Ons $\lceil \lg \lg (n+1) \rceil \geq \lg \lg n + 1$.

TW b

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- Pathwidth and treewidth do not admit polynomial kernels by deletion distance to a clique

Future directions

Experimental evaluation of the reduction rules

Lower bounds on kernel sizes

- Kernel with $O(k^{3-\varepsilon})$ bits for parameterization by vertex cover?

Pathwidth parameterized by feedback vertex set

- Polynomial kernel for treewidth by FVS
- Trees have constant treewidth but potentially large pathwidth

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THANK YOU!

Pathwidth

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