Bart Jansen

Vertex Cover Kernelization Revisited:

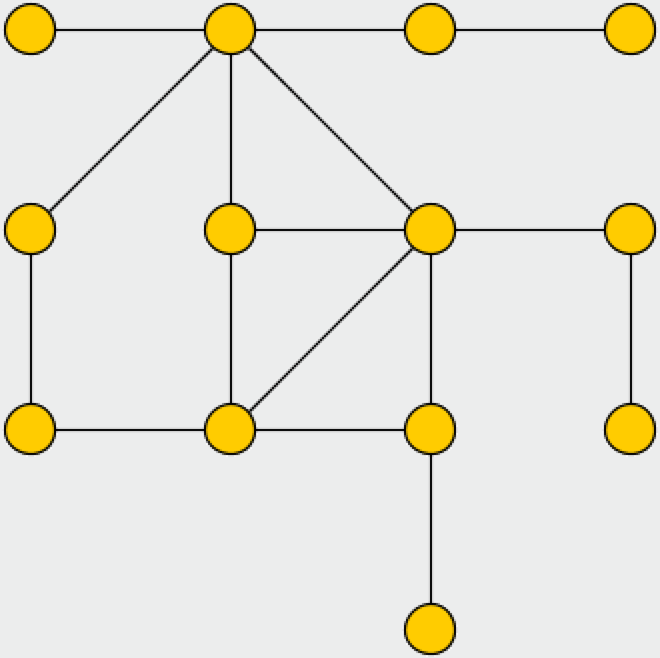
Upper and Lower Bounds for a Refined Parameter

Joint work with Hans Bodlaender

STACS 2011, Dortmund
March 10th, 2011

Vertex Cover

- Input: Graph G, integer k
- Question: Does G have a vertex cover of size ≤ k?



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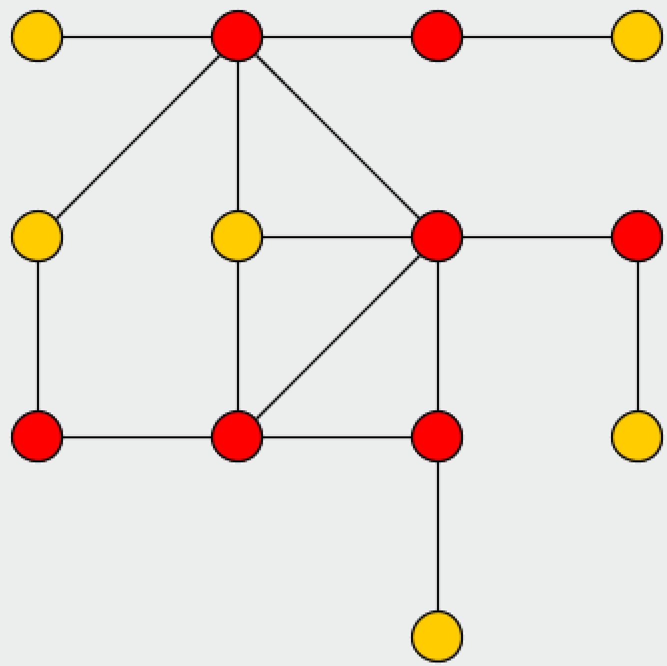

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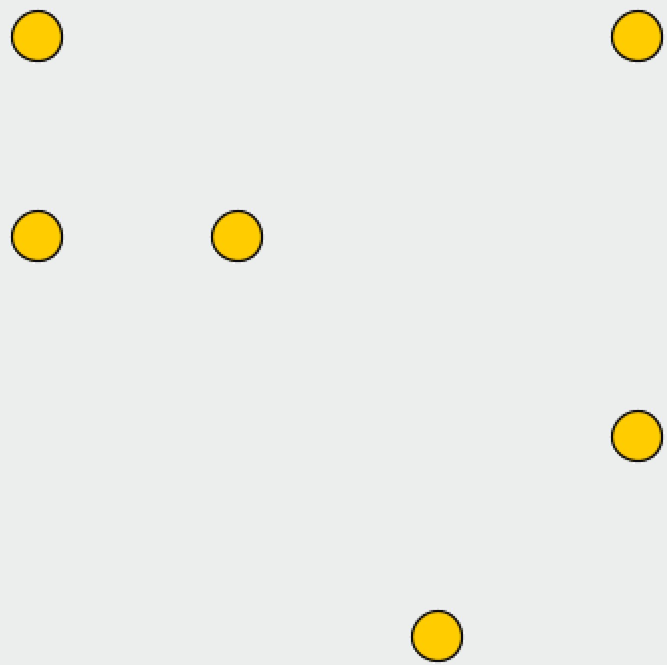

S is a Vertex Cover of G ⇔ Graph G – S is edgeless

Preprocessing for Vertex Cover

- Preprocess by computing a small, equivalent instance in polynomial time
- Reduce (G,k) to an equivalent instance on f(k) vertices

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- Evidence that factor 2 is optimal under UGC

Relevant values of k

- Consider instance (G,k) of Vertex Cover
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- So for relevant instances **k is Θ(VC(G))**

Alternative parameterizations

- Existing results guarantee that the size of instance (G,k) of Vertex Cover can be reduced to O(VC(G)) vertices
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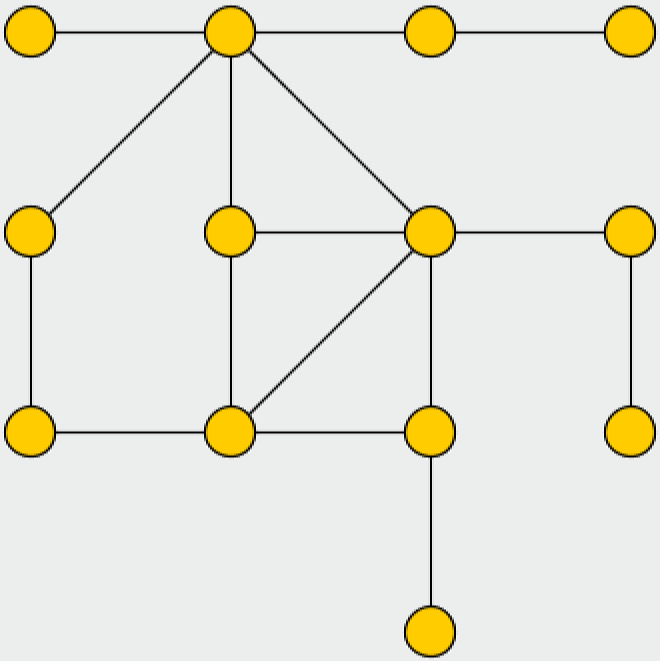
- Stronger data reduction if we can ensure $\Pi(G) \leq VC(G)$

Graph parameters

- **Vertex Cover Number VC(G)**
- Size of a smallest set S such that G – S is edgeless



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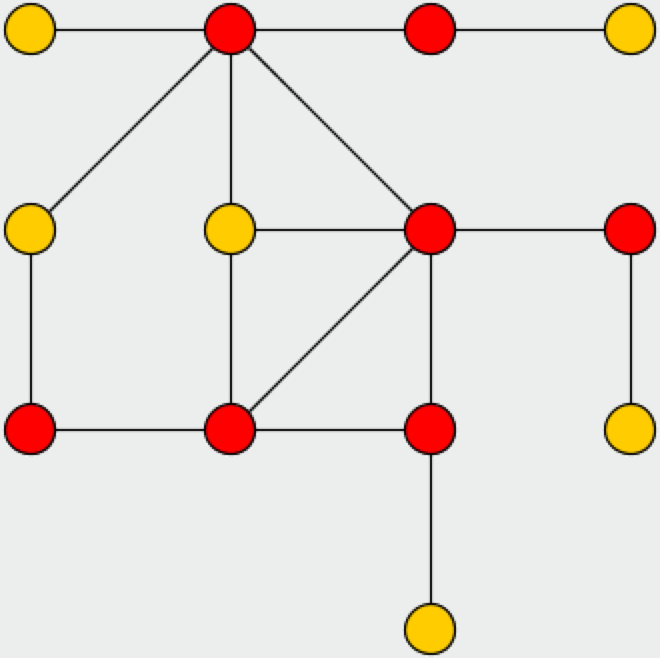

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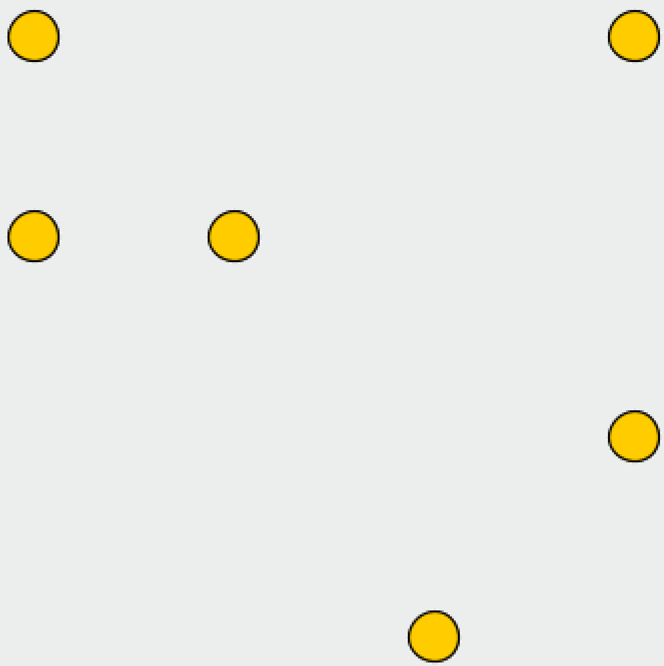

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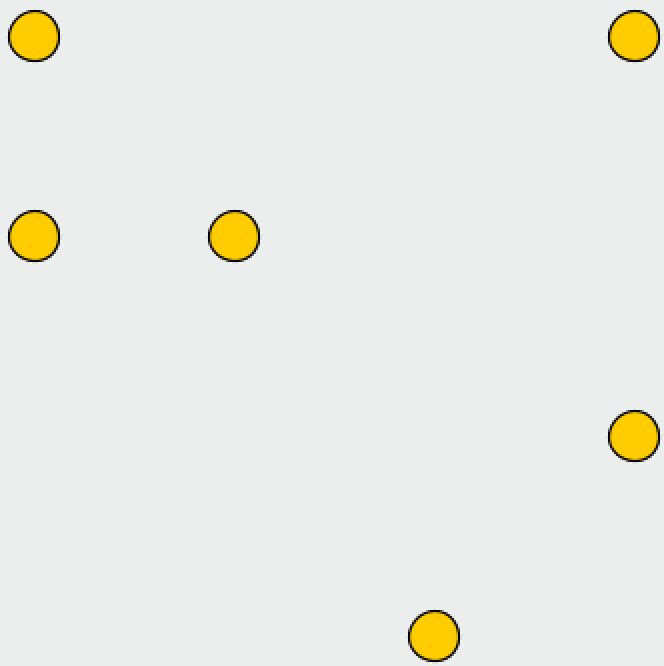

Vertex Cover
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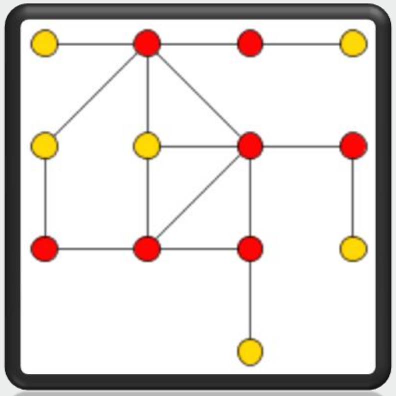

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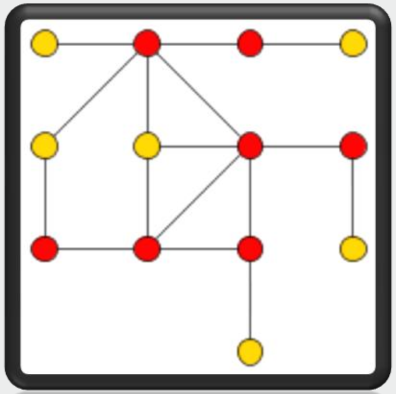

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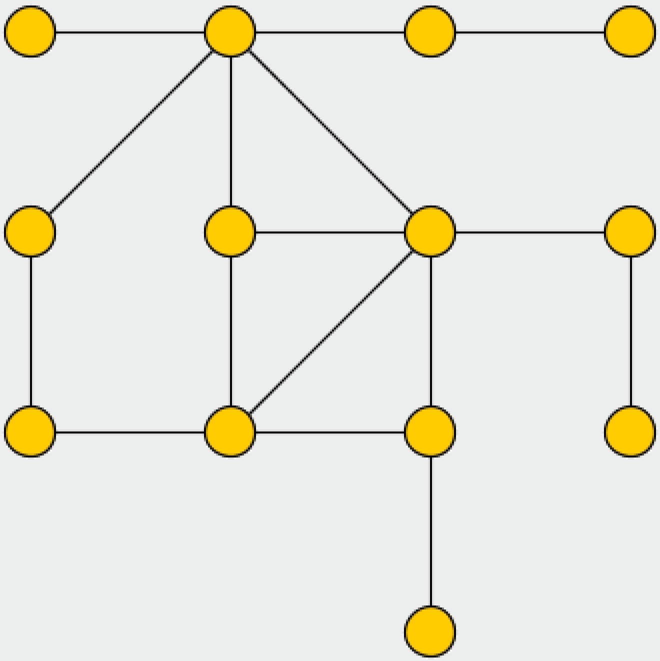

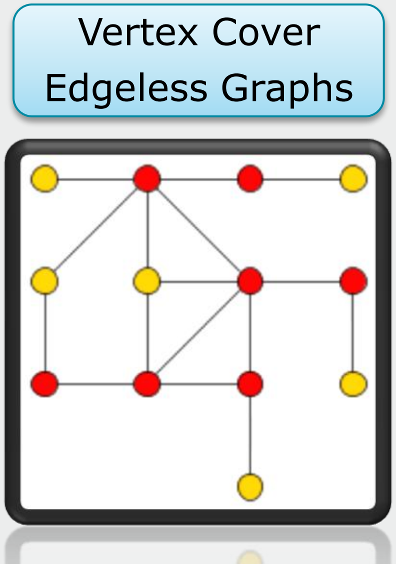

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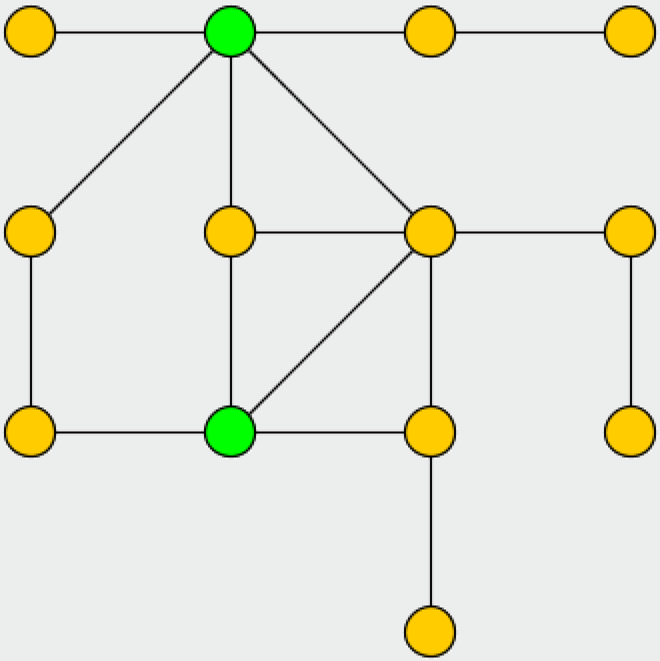

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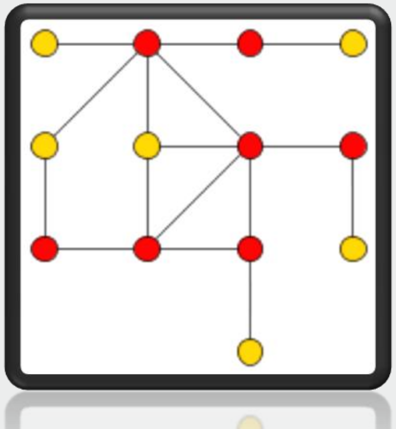

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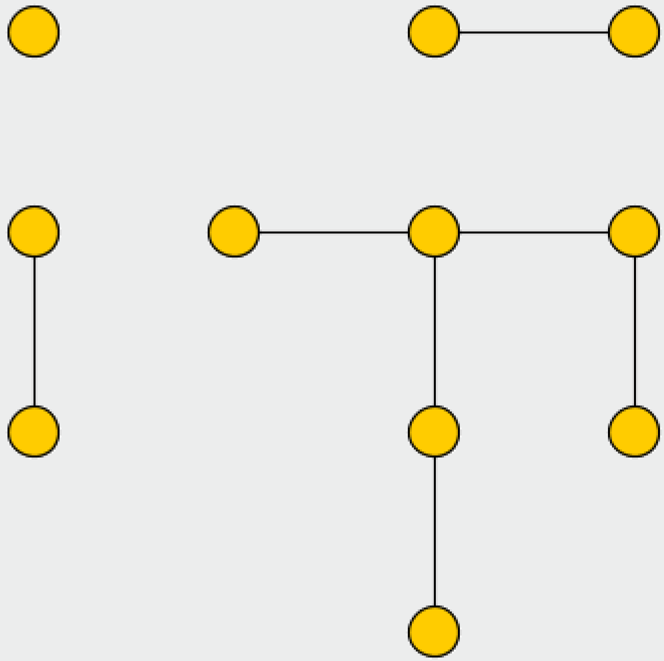

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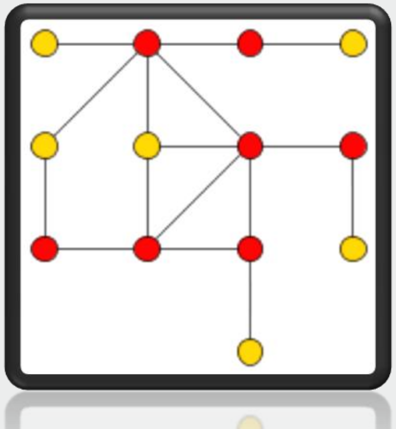

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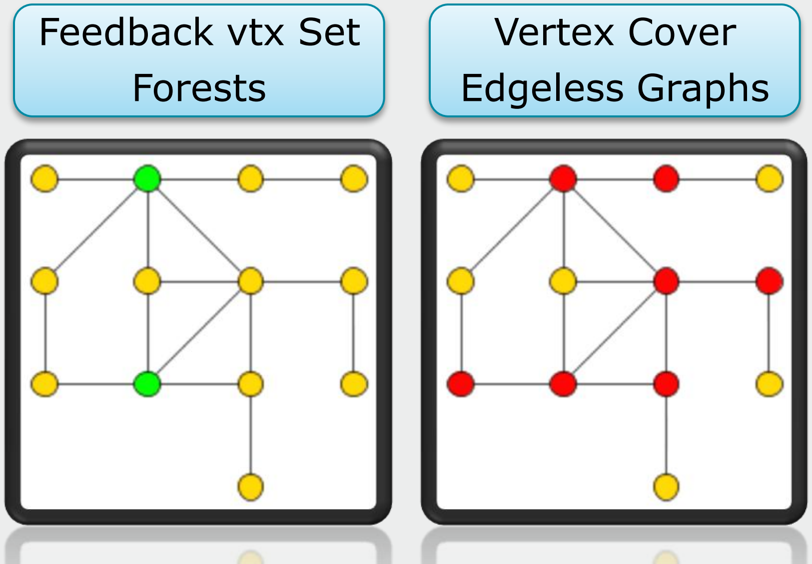
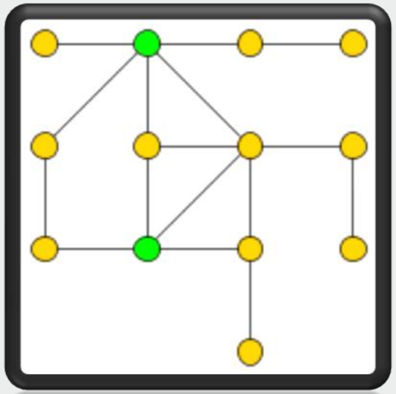


A refined parameter

Feedback vtx Set
Forests



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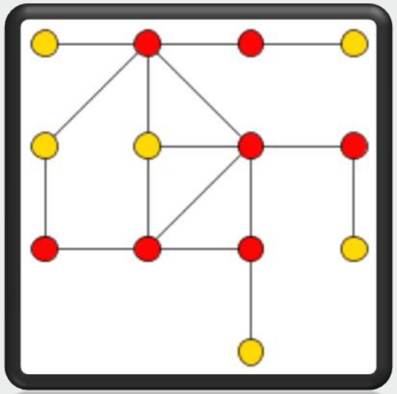

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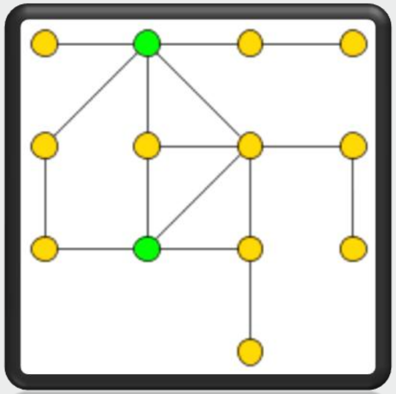

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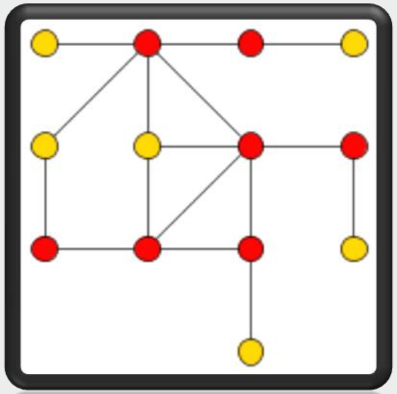

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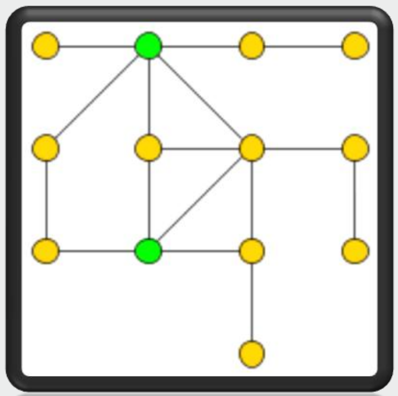

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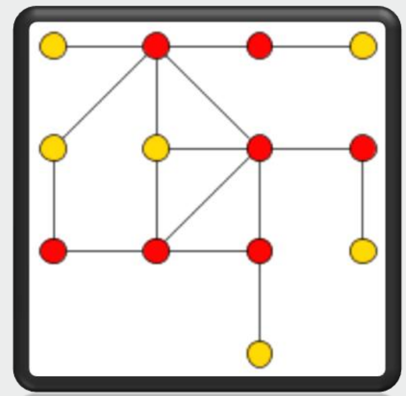

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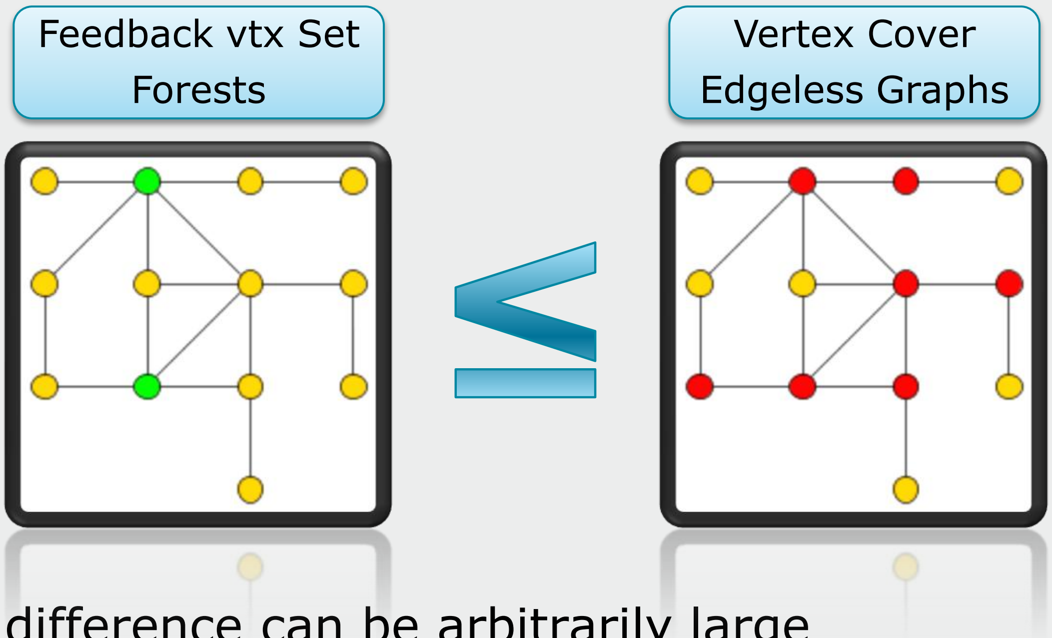

- The difference can be arbitrarily large
- The feedback vertex number is a **refined** parameter

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- In the language of parameterized complexity:
  - Vertex Cover parameterized by the size of a feedback vertex set admits a cubic-vertex kernel

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    - (unless the polynomial hierarchy collapses)
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*** This strenghtens the result as given in the paper**

Sketch of the reduction rules

THE UPPER BOUNDS

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3) Use the structure of X within G to apply reduction rules
   - When no rules apply, the instance is provably small

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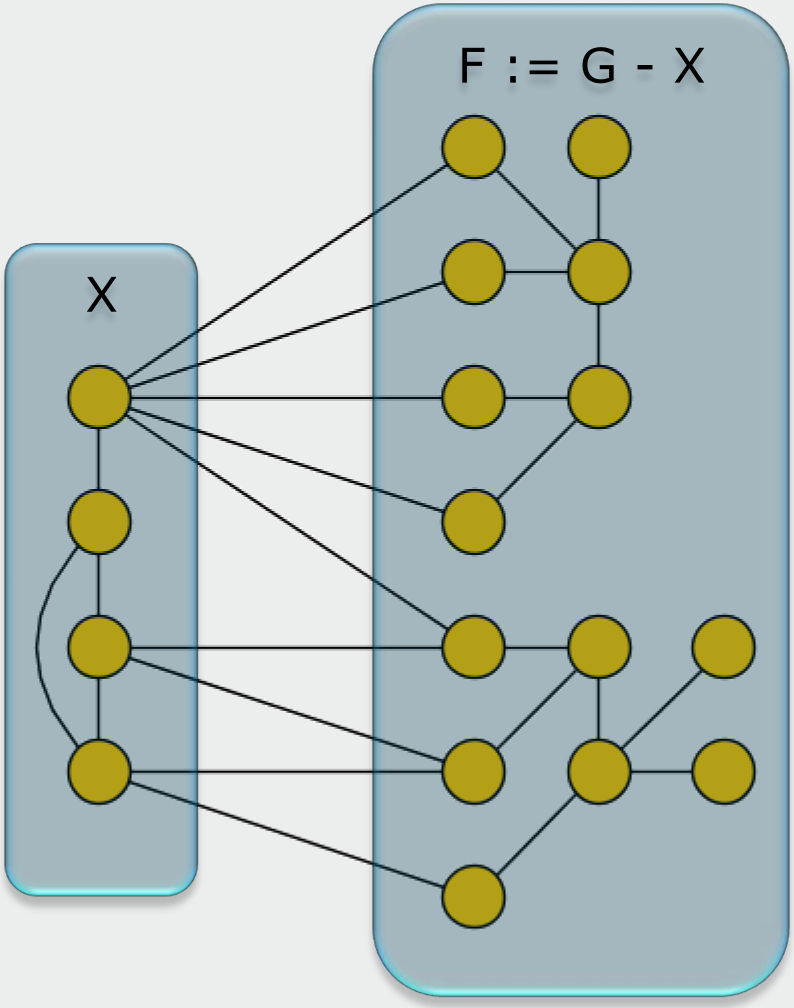
- Instance (G,k) of Vertex Cover is equivalent to asking "Does G have an Independent Set of size n – k?"
- Reduction rules are easier to formulate in Independent Set perspective
  - Interpret (G,k) as an instance (G, n – k) of Independent Set
  - Apply reduction rules to obtain a small instance of Independent Set (G', n' – k')
  - Equivalent to the small Vertex Cover instance (G', k')

Structure of an instance: a canonical solution

- Let forest F := G – X
  - Maximum Independent Set (MIS) of F is poly-time computable



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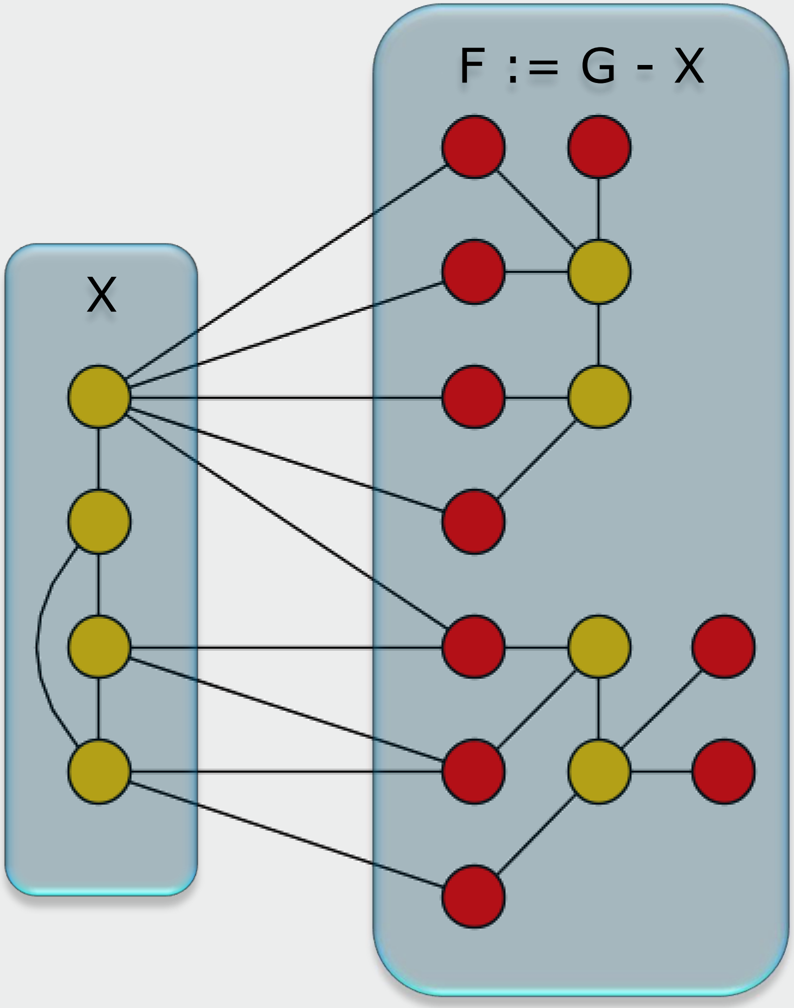

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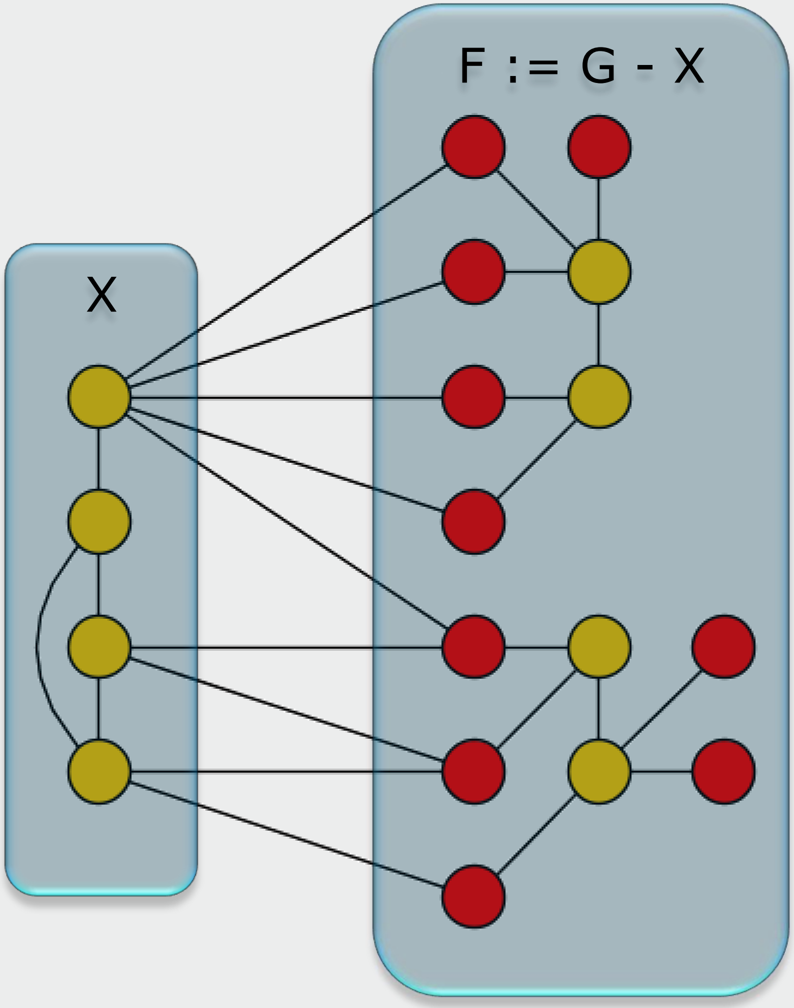

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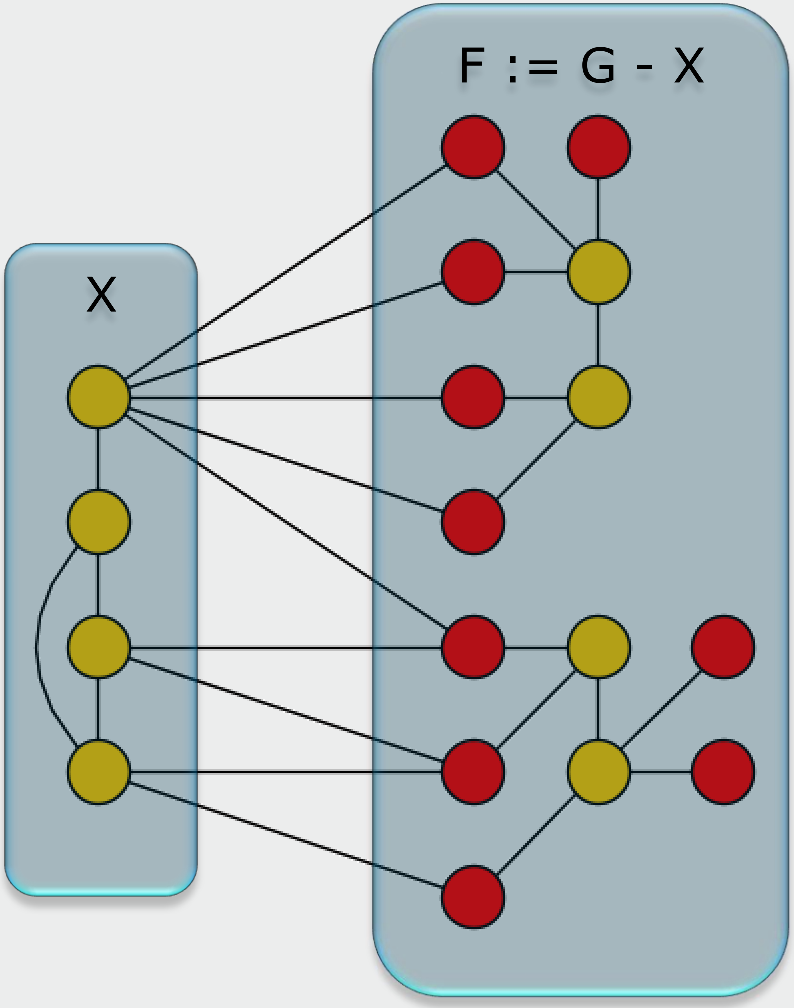
- We can test the effect of using single vertex



Using vertices from X

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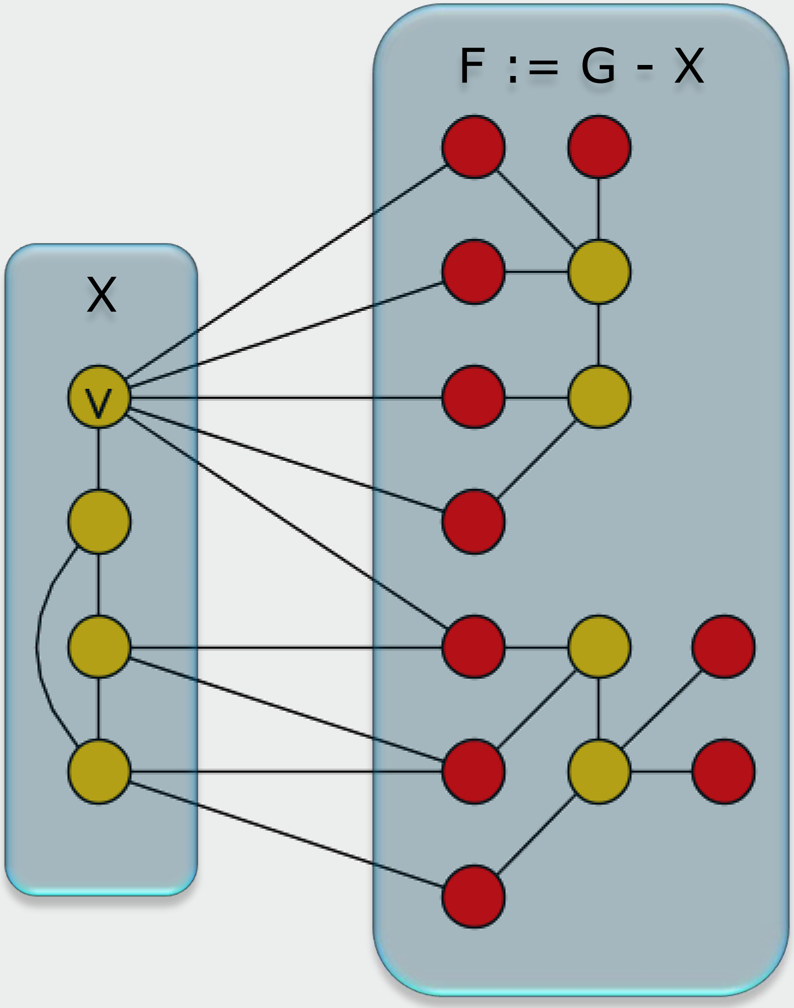

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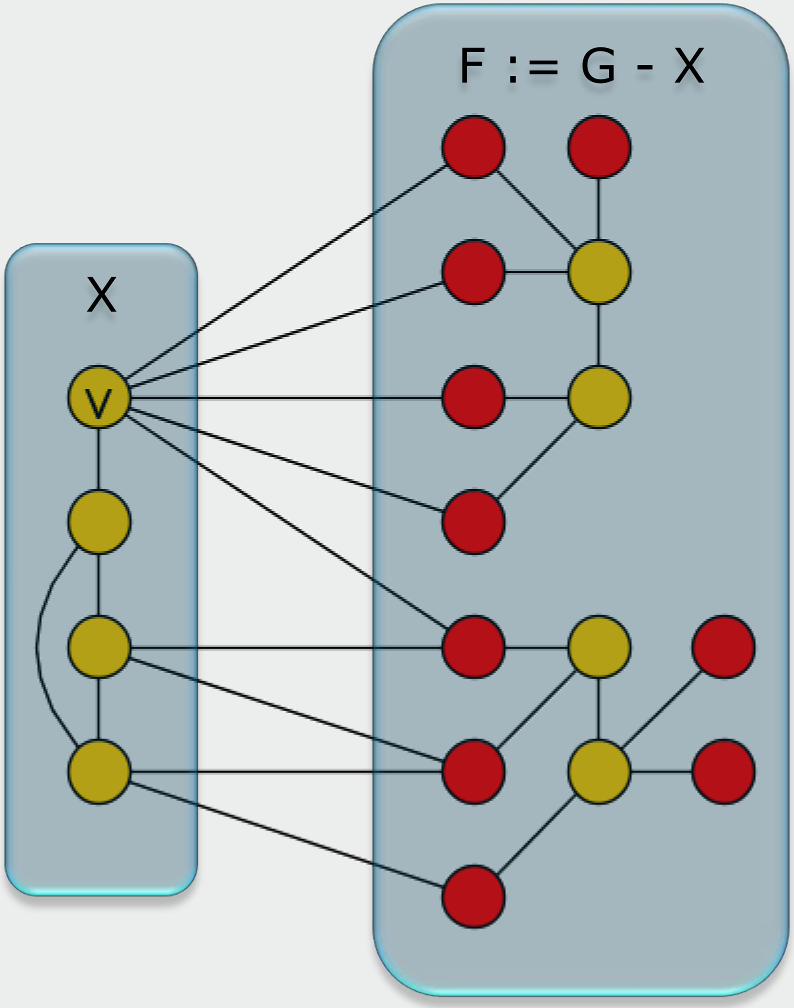

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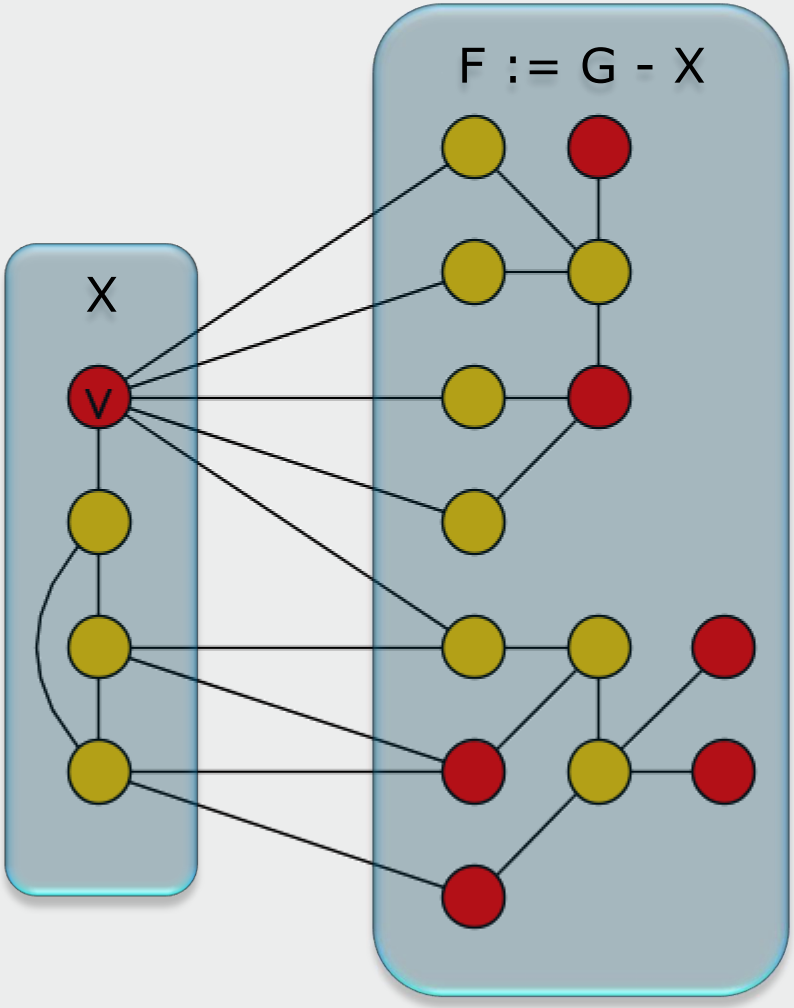

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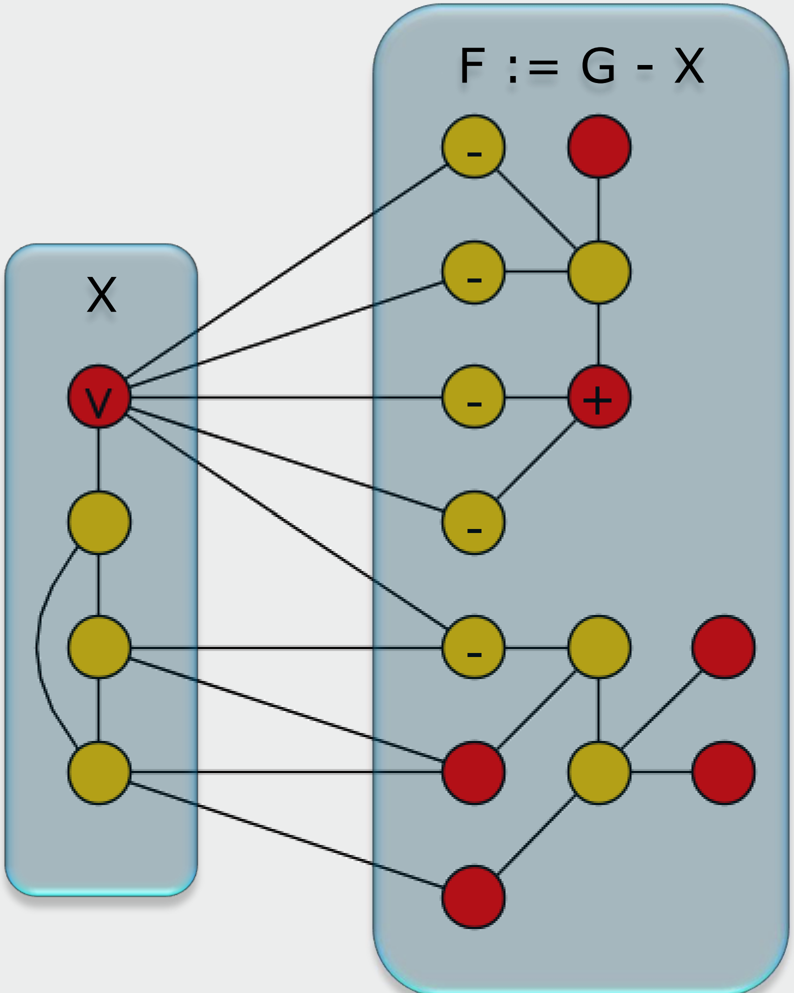

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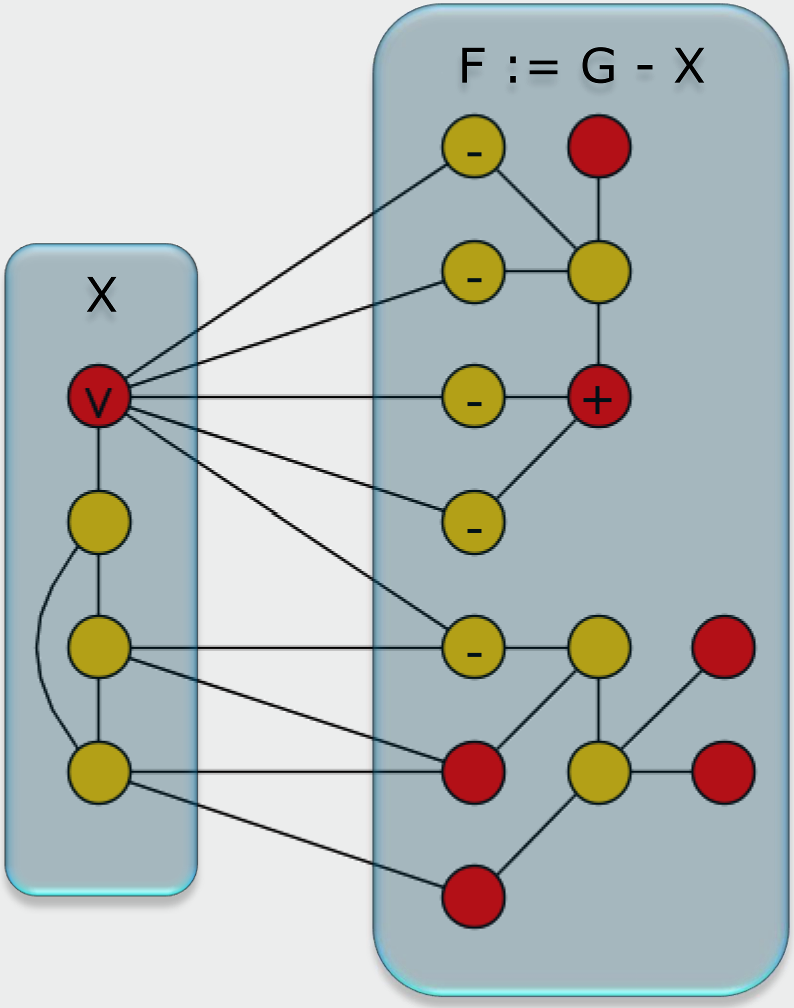

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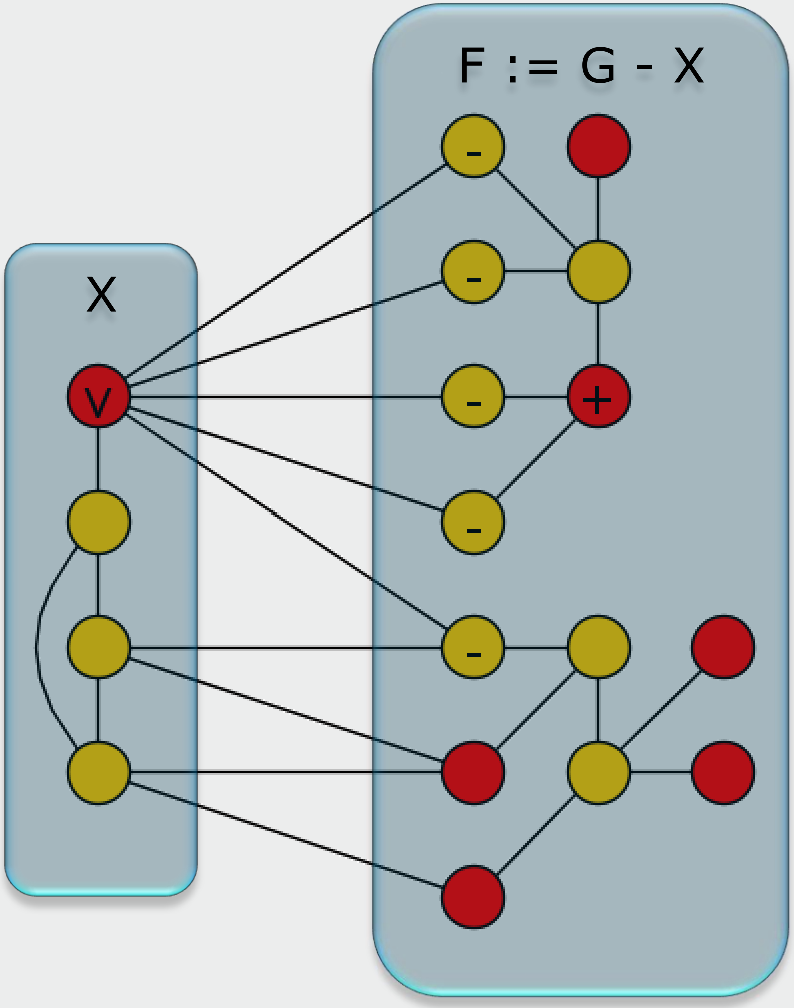

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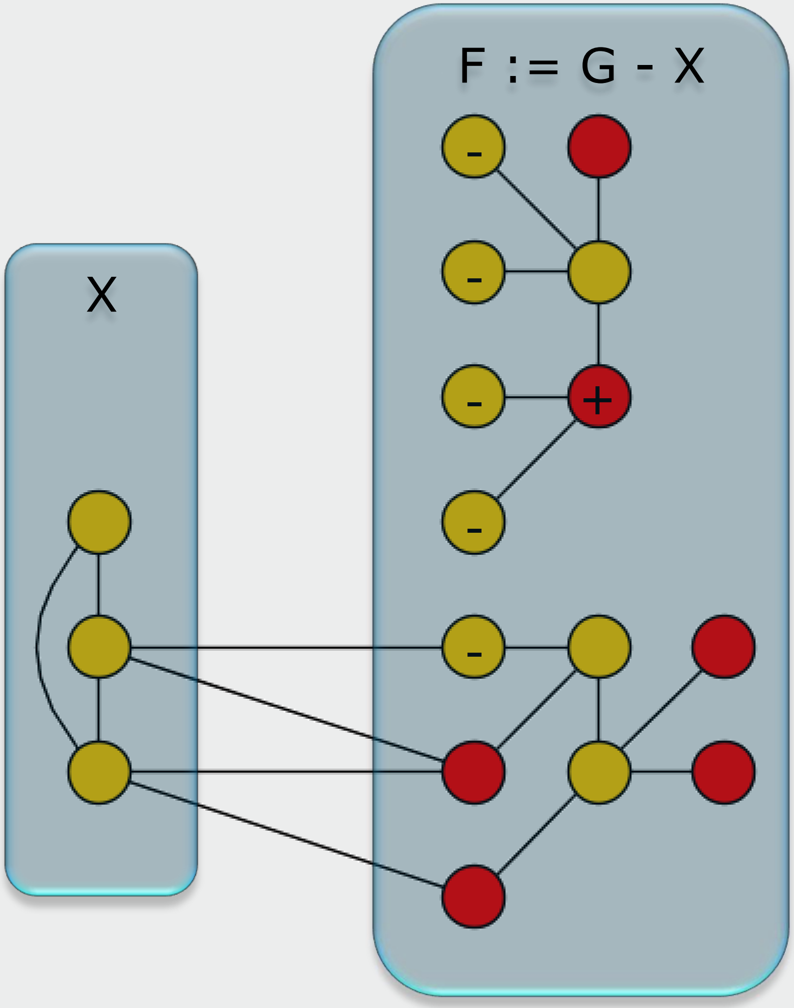

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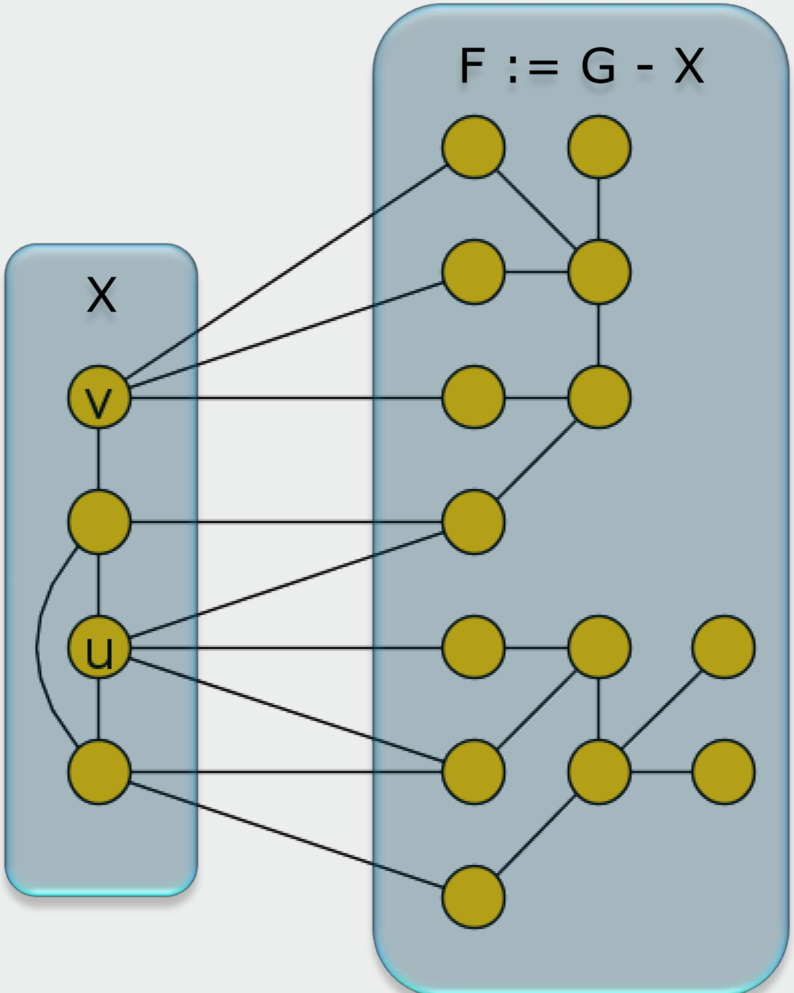

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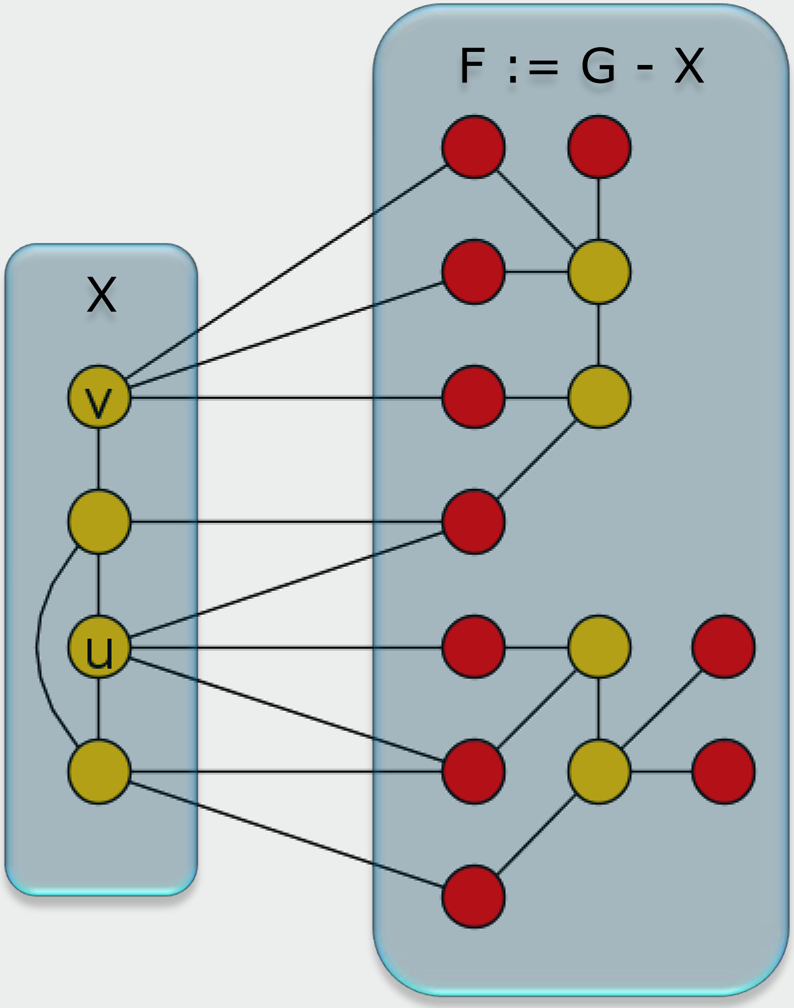

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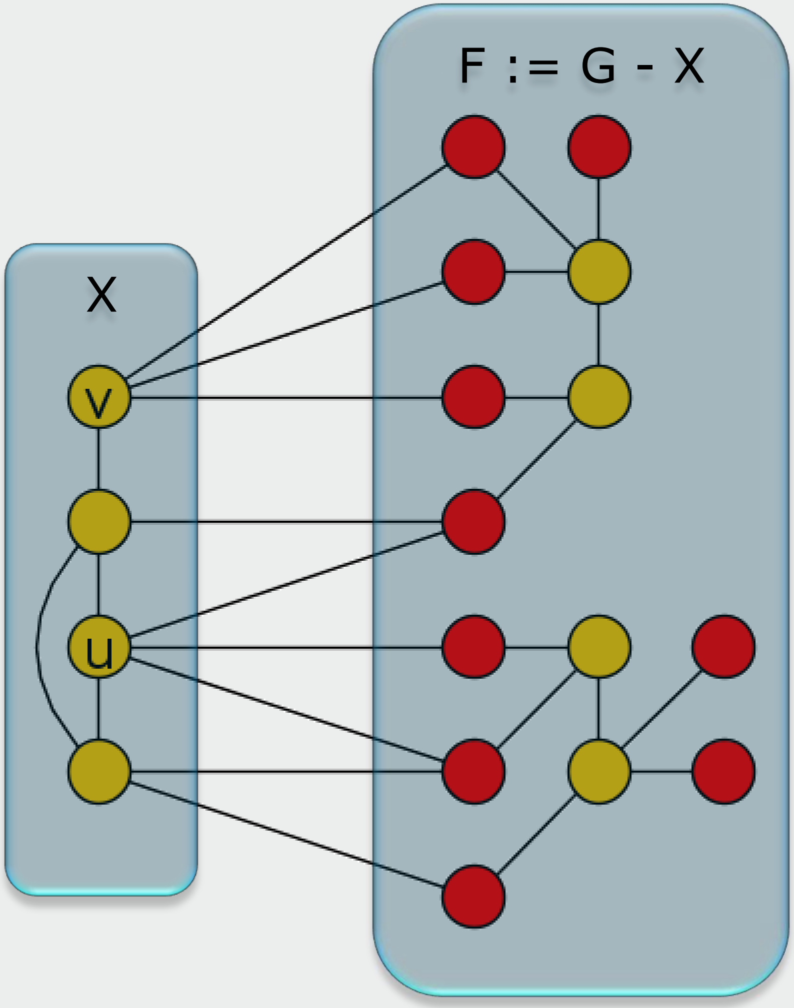

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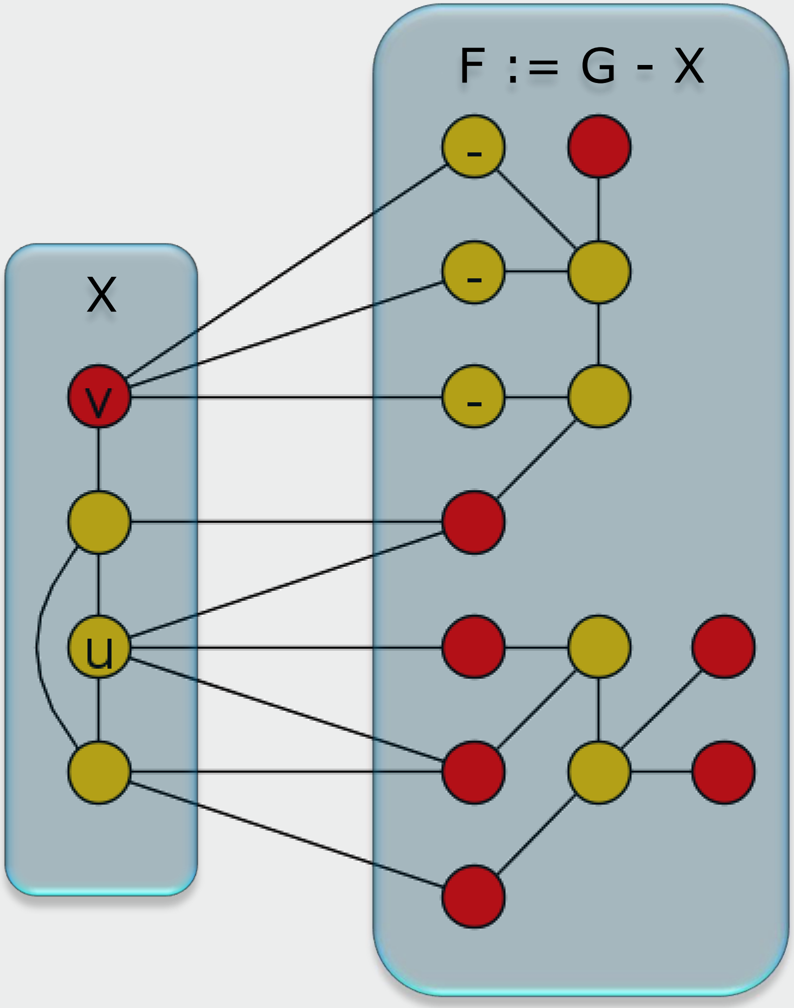

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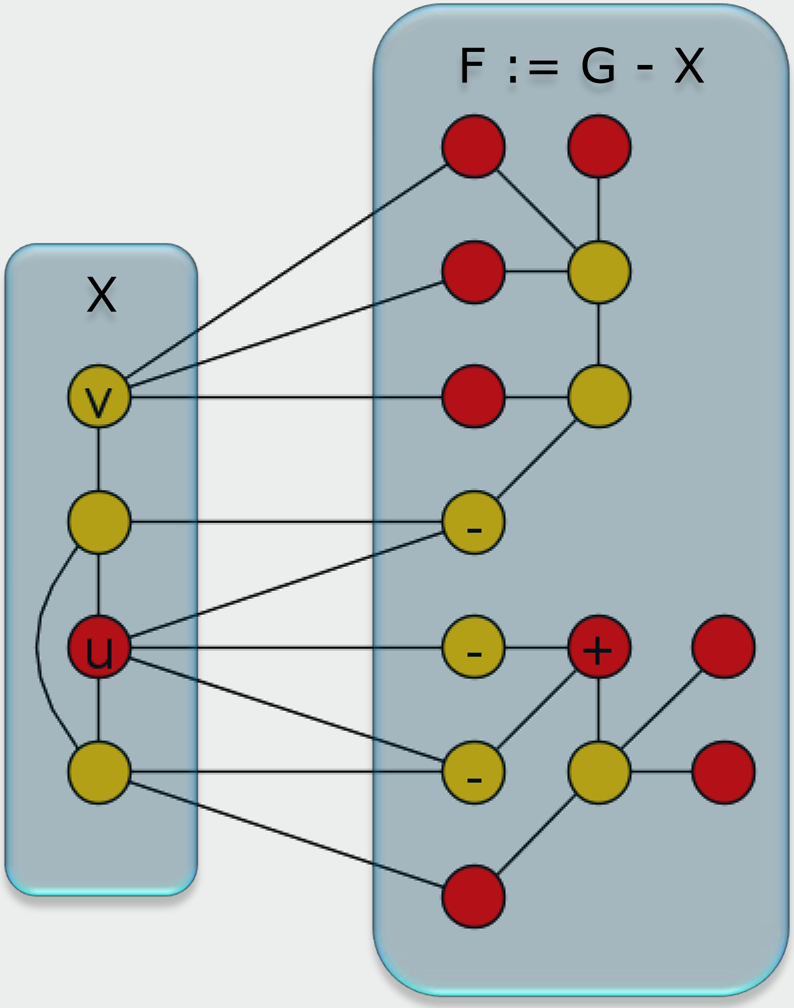

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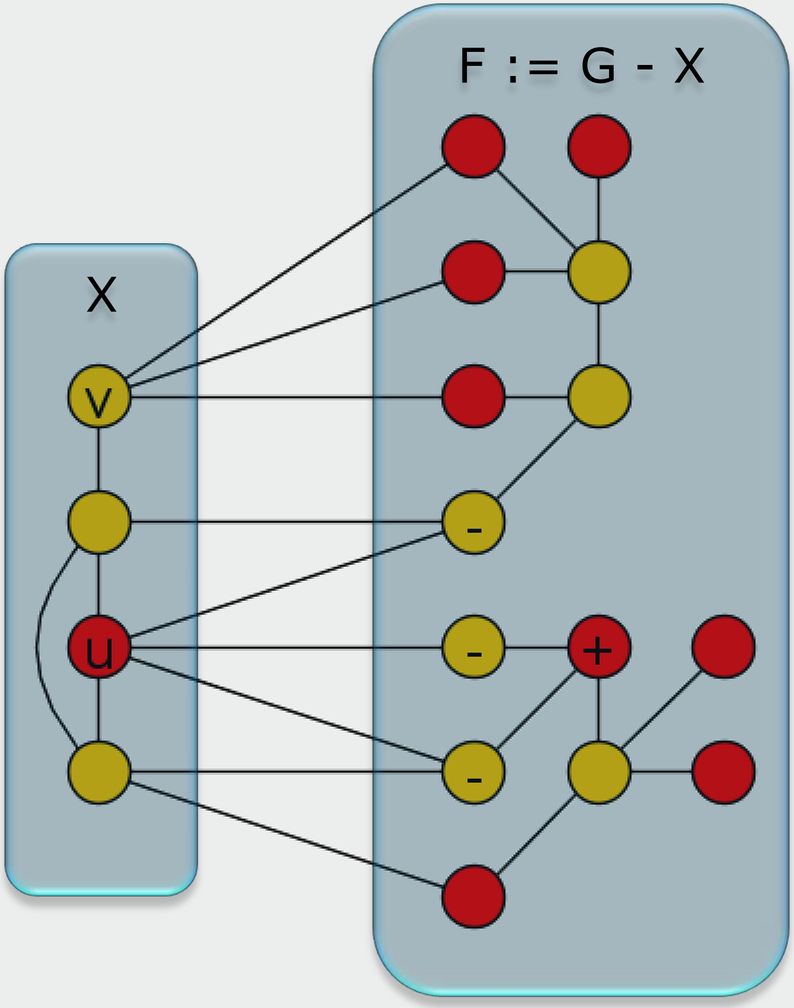

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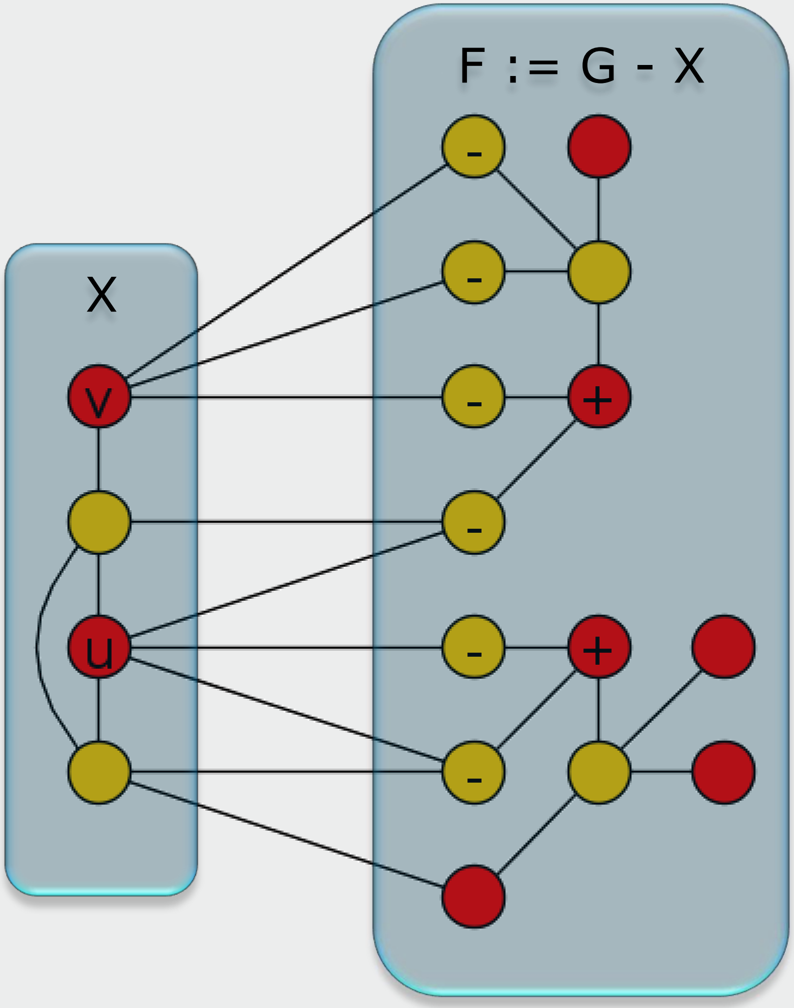

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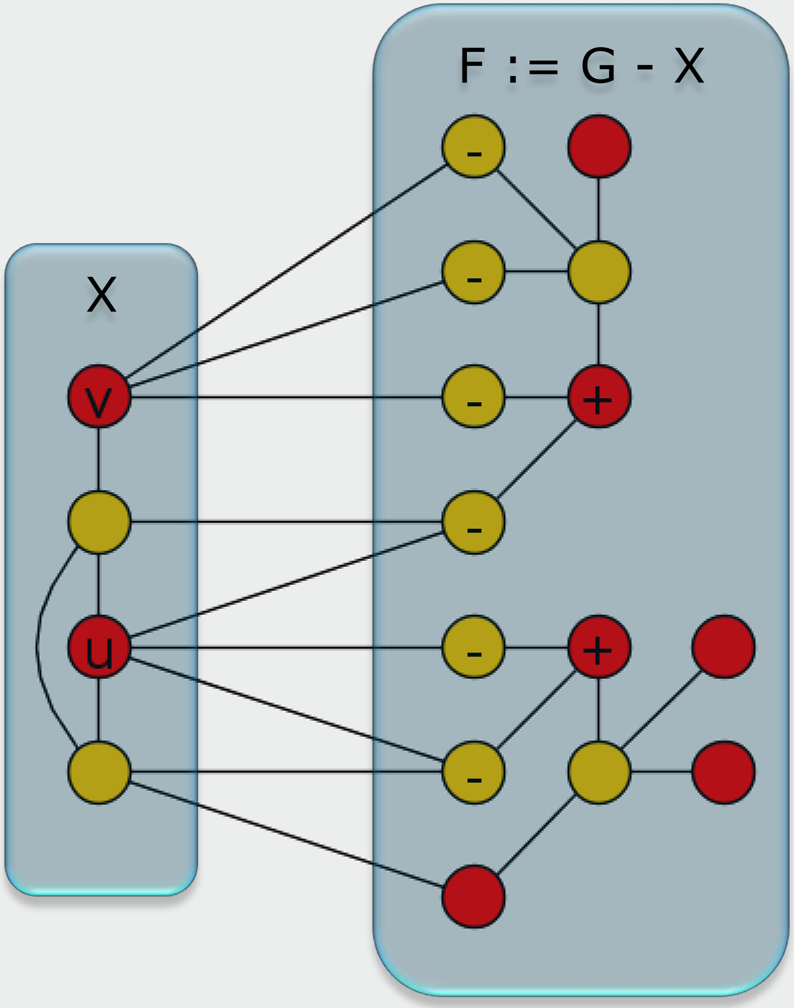

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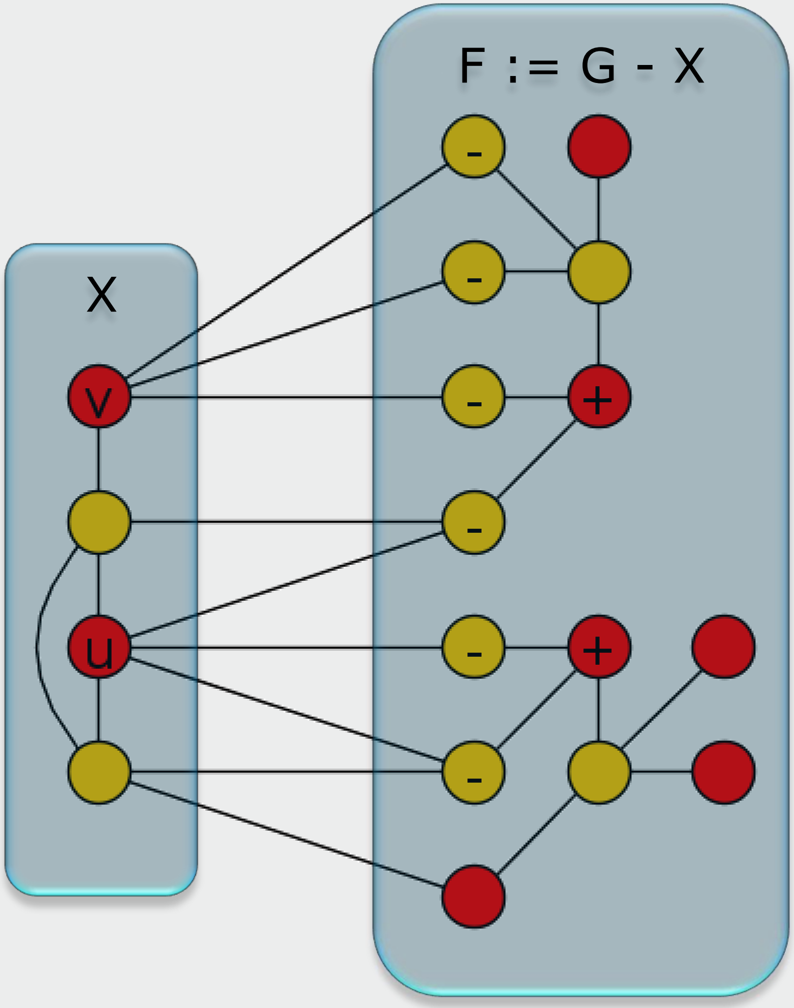

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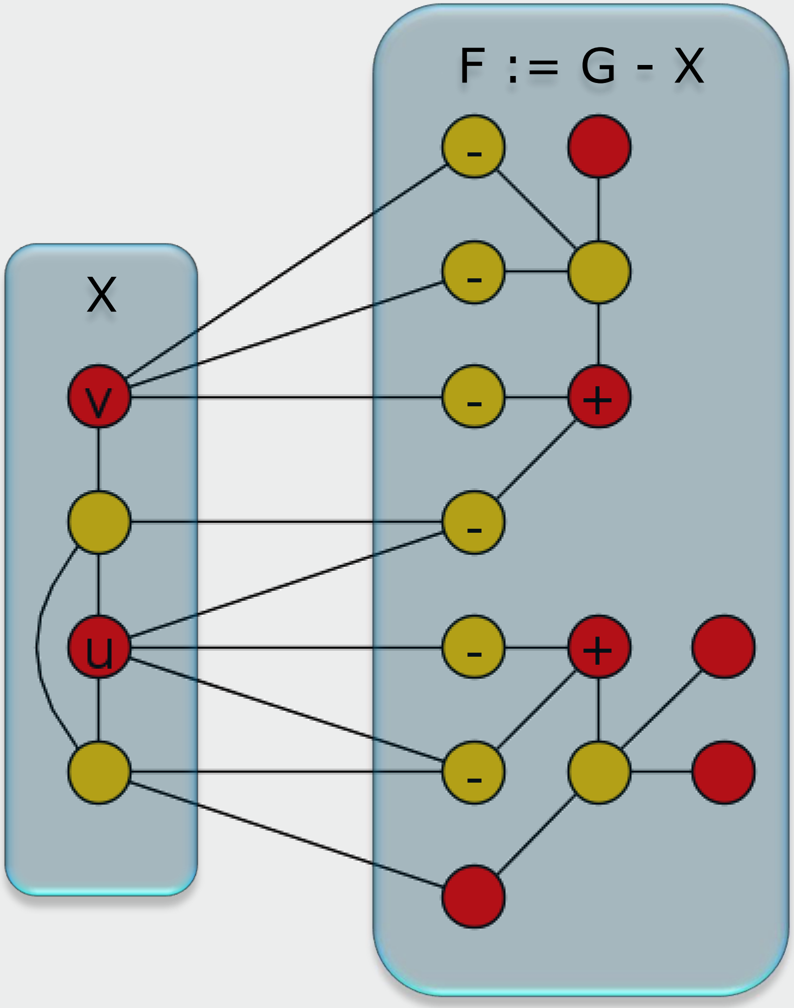

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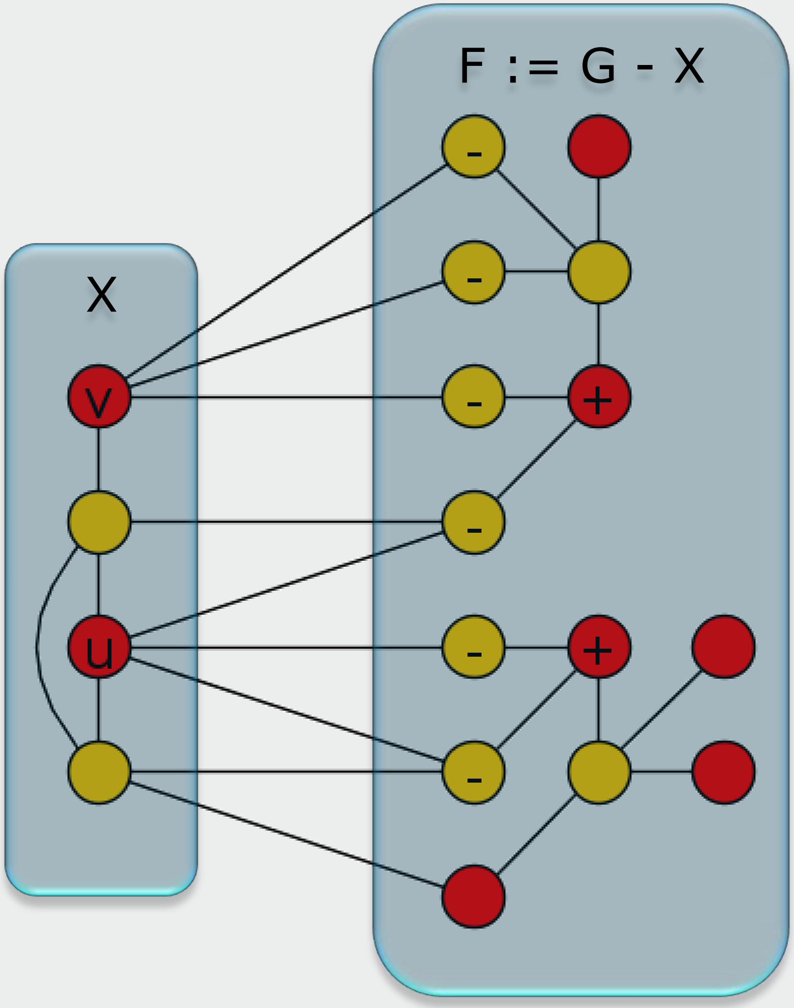

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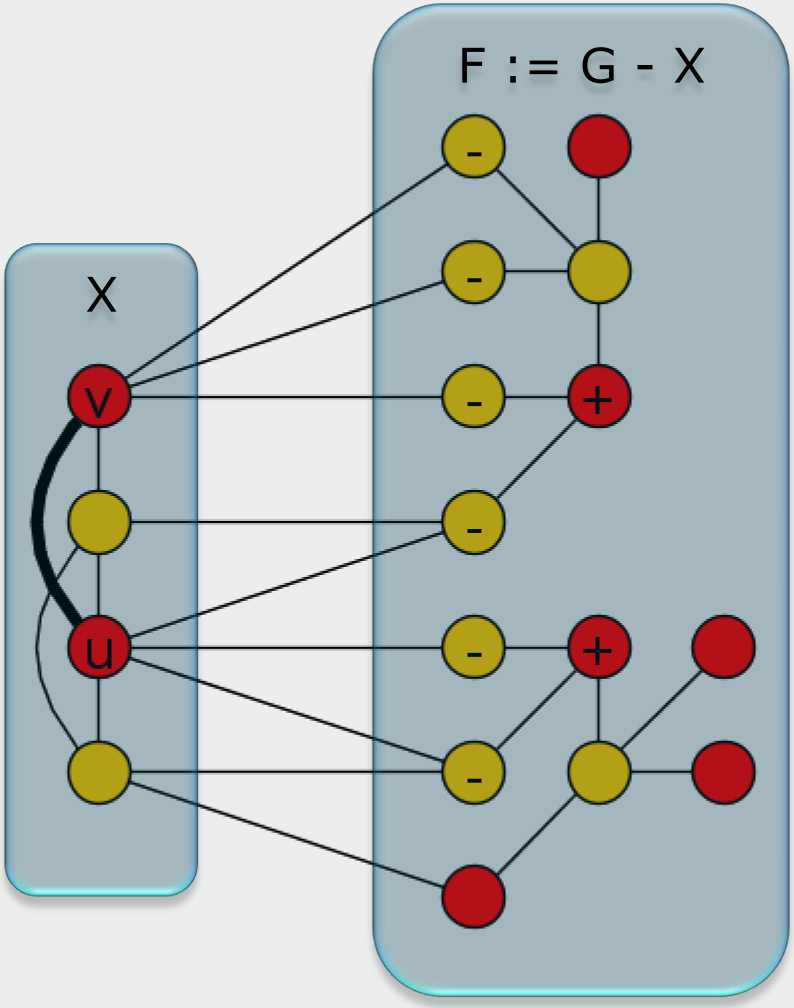
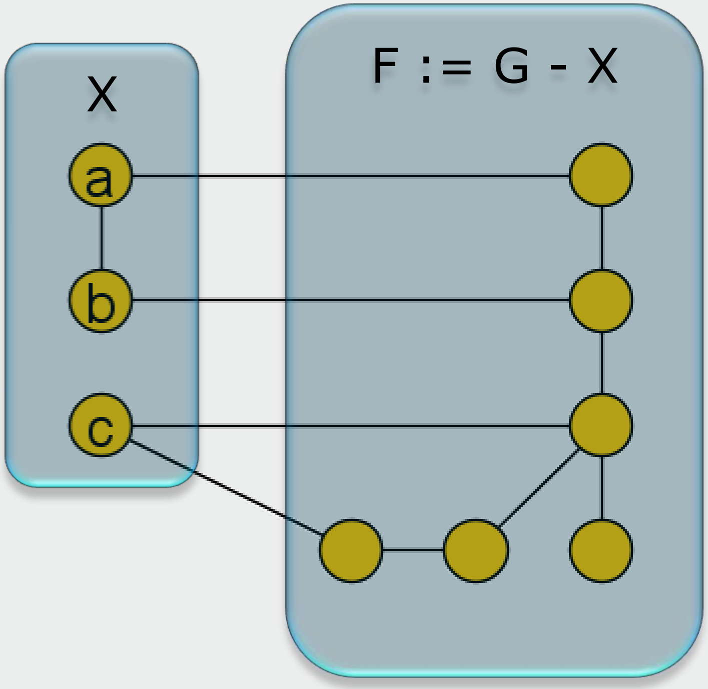


Deleting trees from F: An example

- Consider this tree T in forest F



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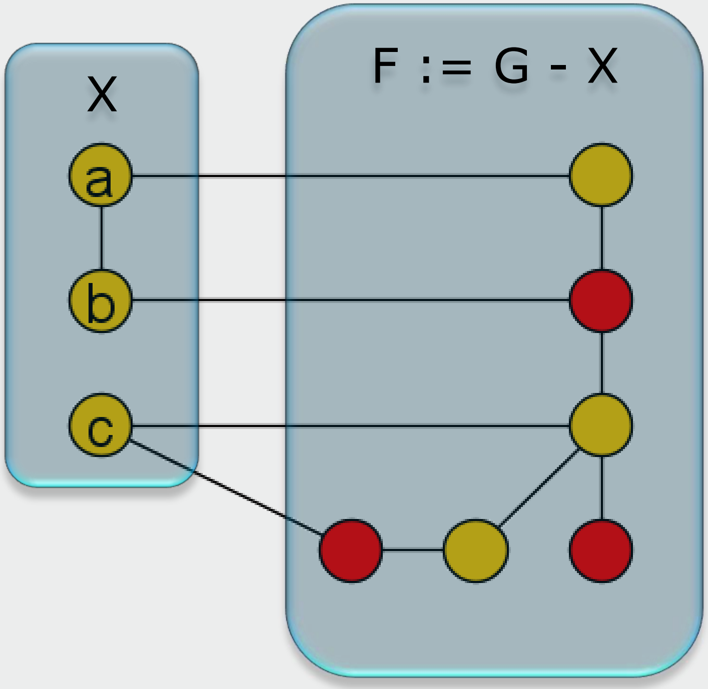

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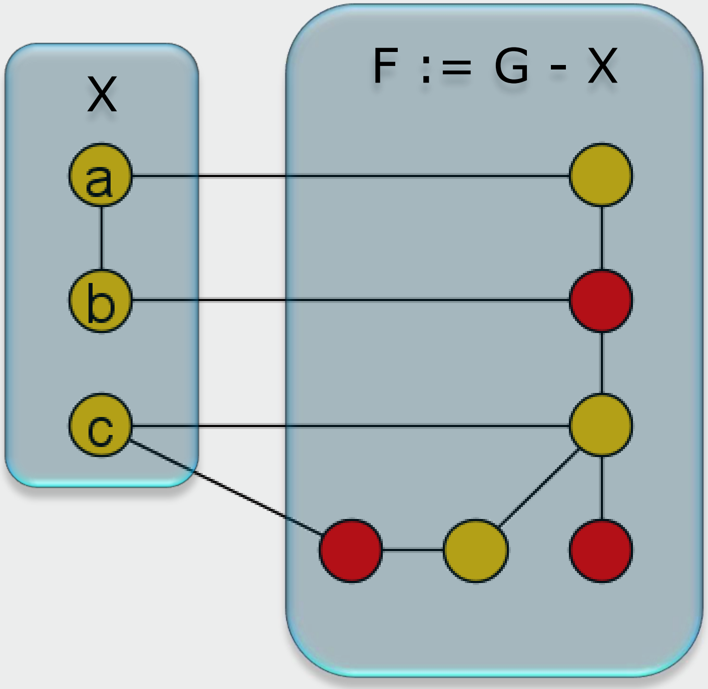

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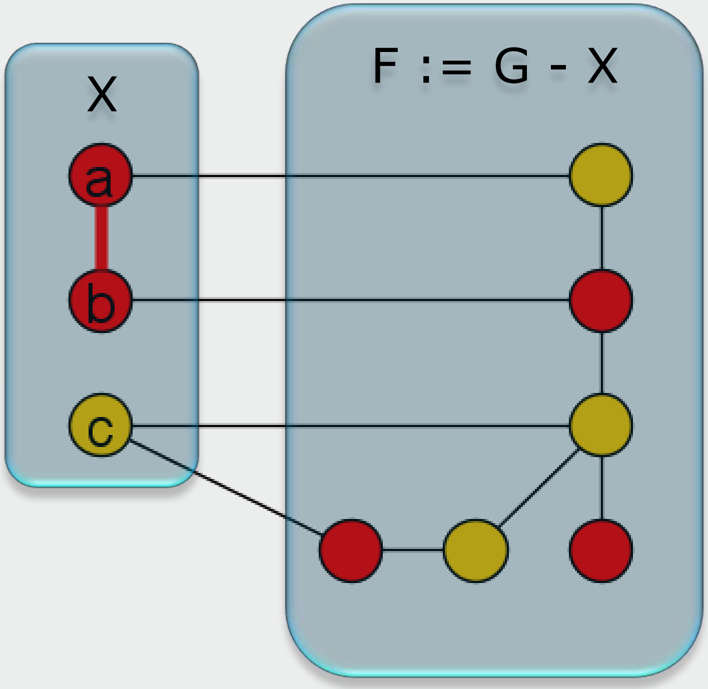

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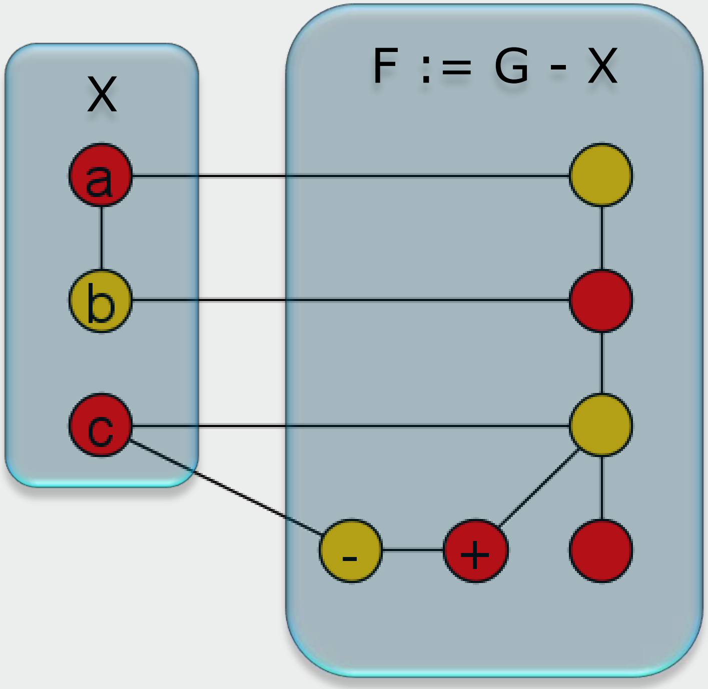

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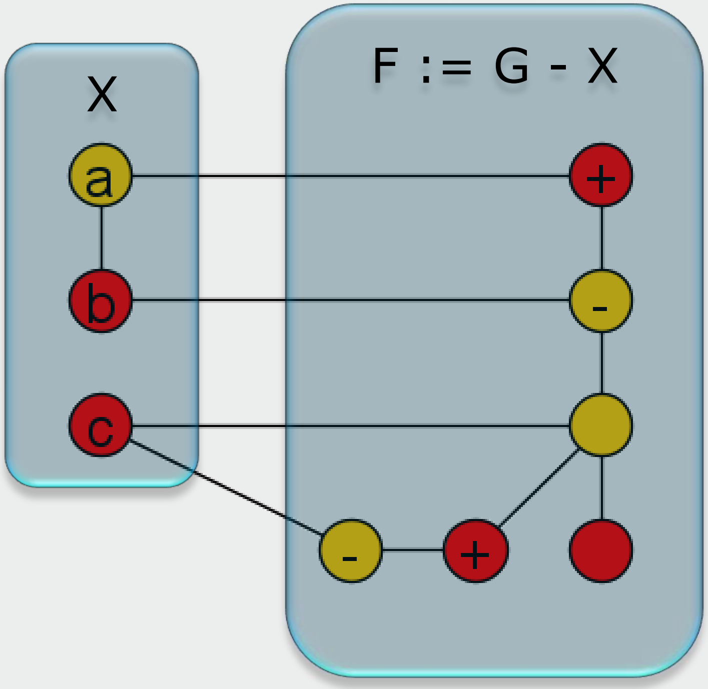

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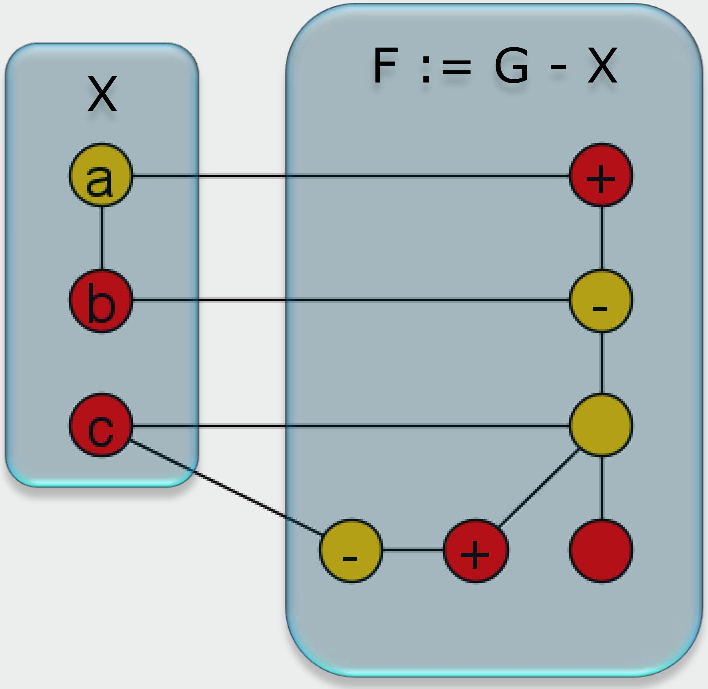

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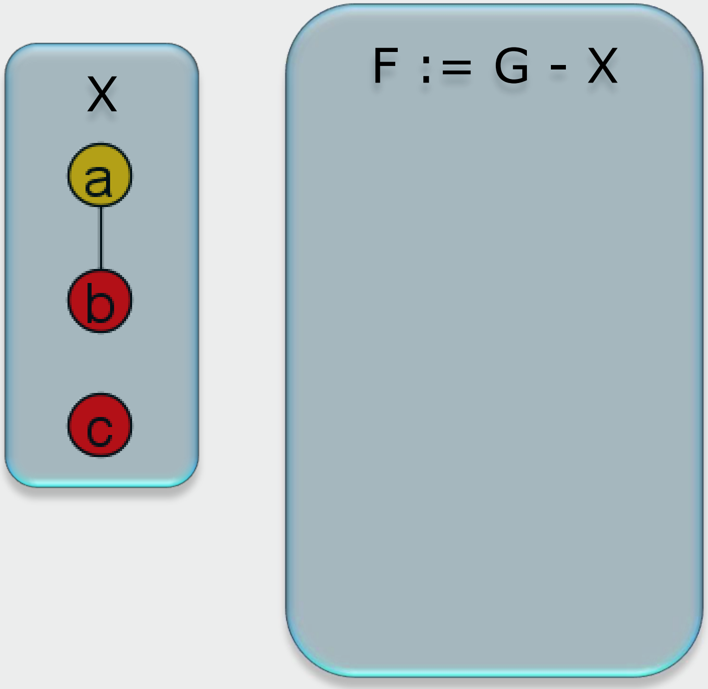

Deleting trees from F: the rule

- If there is a tree T in the forest F, such that:
  - for all non-adjacent pairs {u,v} in X:
    MIS(T) = MIS(T – N(u,v))
- Then delete T from the instance, decrease k by MIS(T)

- Justified by the following lemma:
  - If there is an independent set X' ⊆ X such that
    MIS(T) > MIS(T – N(X'))
  - then there is such a set of size at most 2

Overview of the reduction process

- Two more rules to simplify the trees in F
- Effect of the rules:
  - For each vertex v in X, the amount you have to "pay" in F for using v is at most |X|
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- Long proof shows that |F| is $O(|X|^3)$ after reduction
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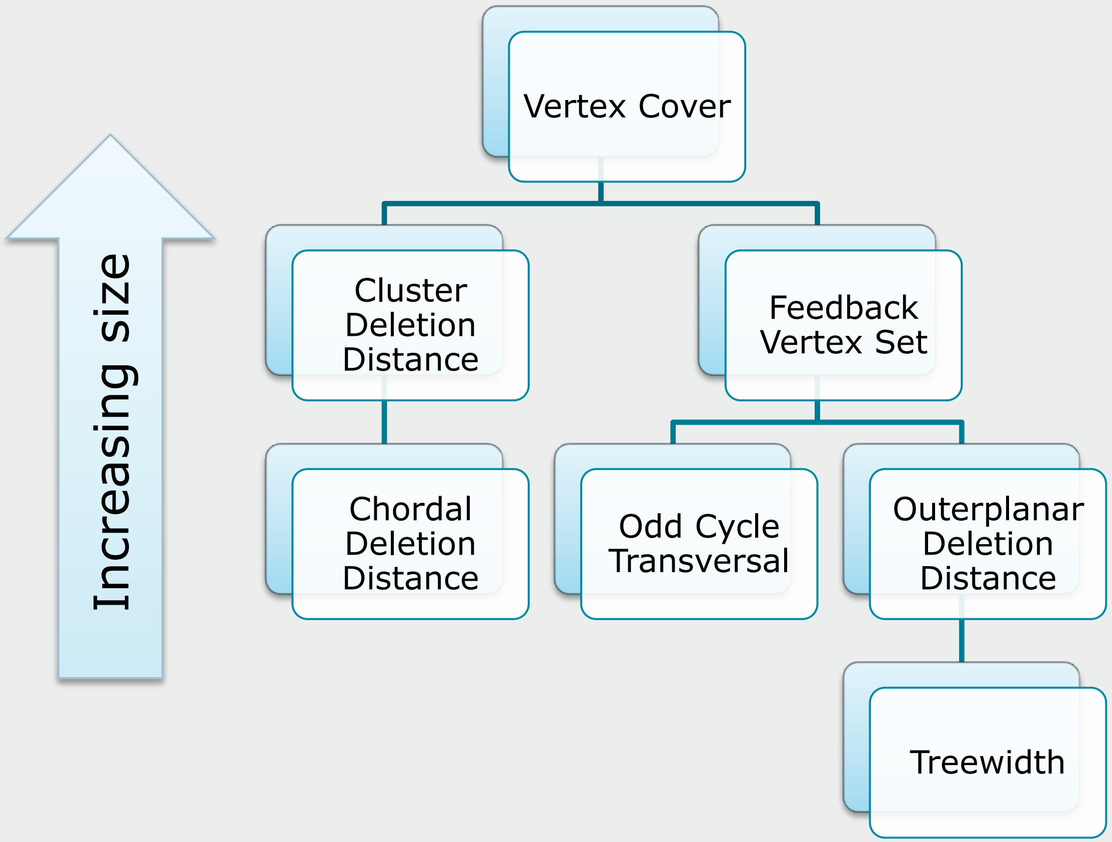
CONCLUSION AND DISCUSSION

Kernelizability of (Unweighted) Vertex Cover



All parameterizations are fixed-parameter tractable

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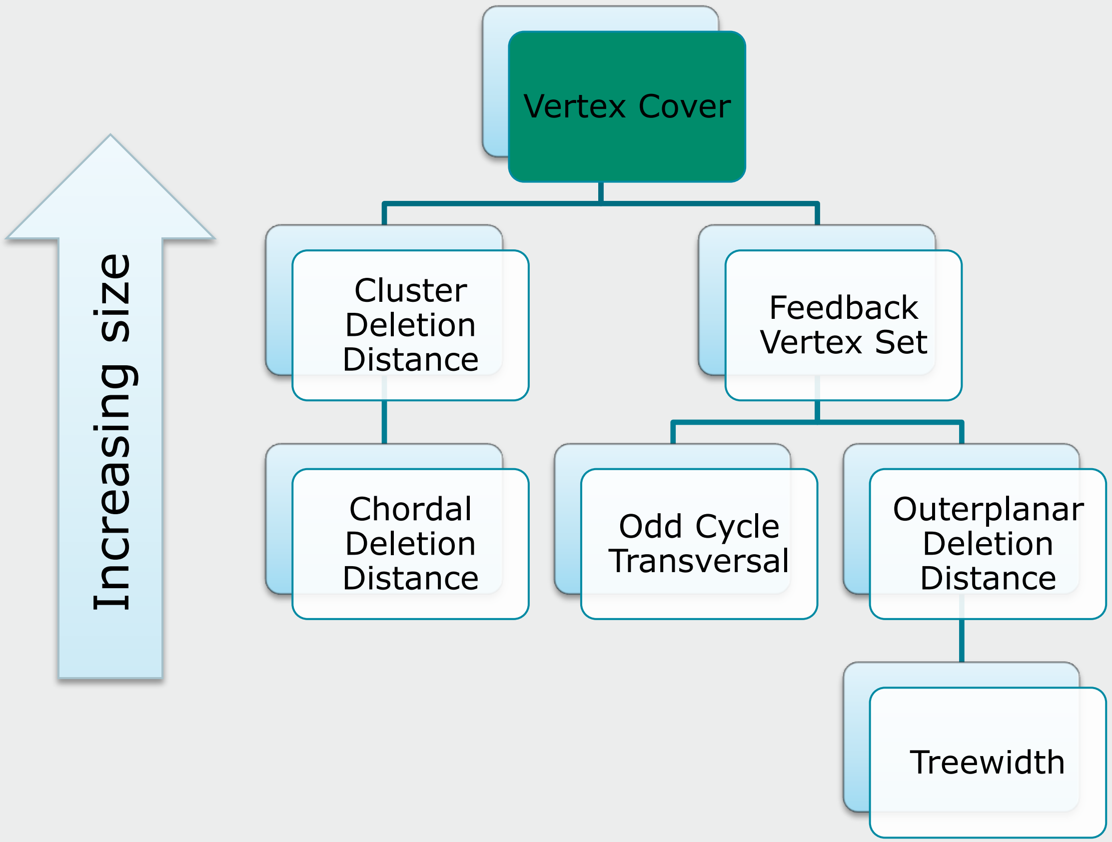

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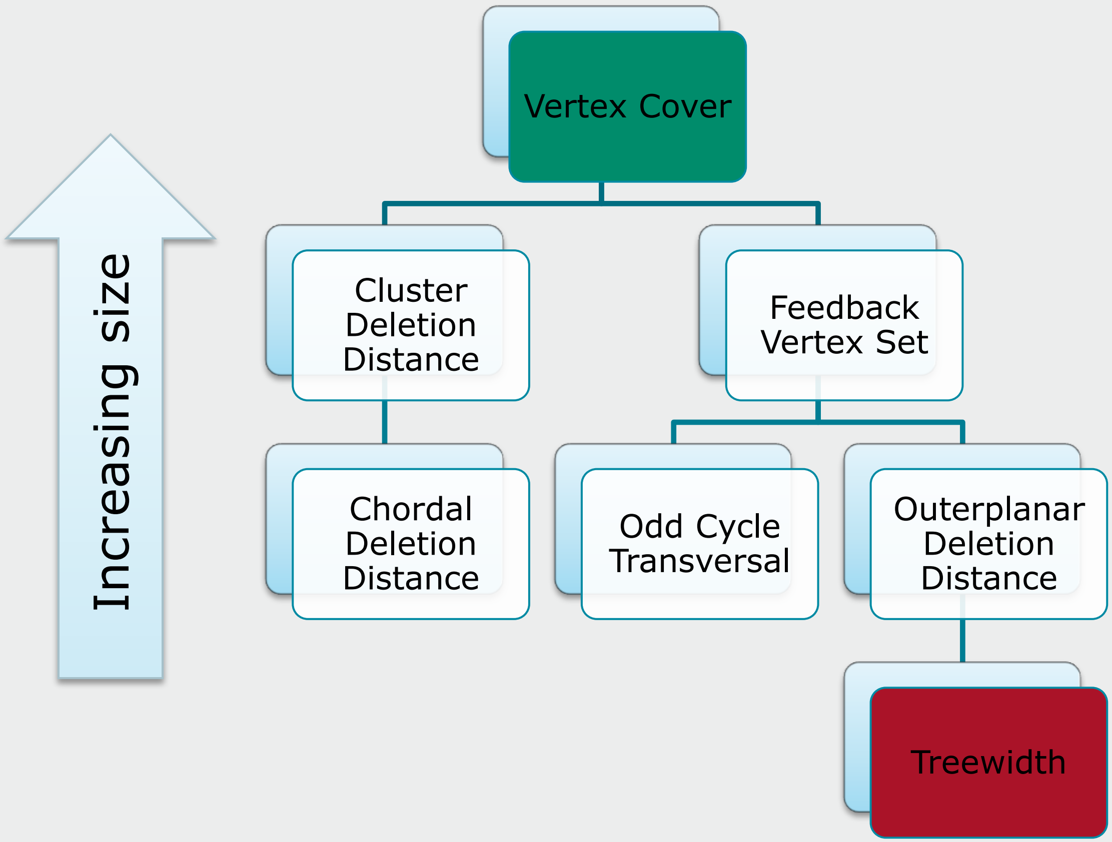

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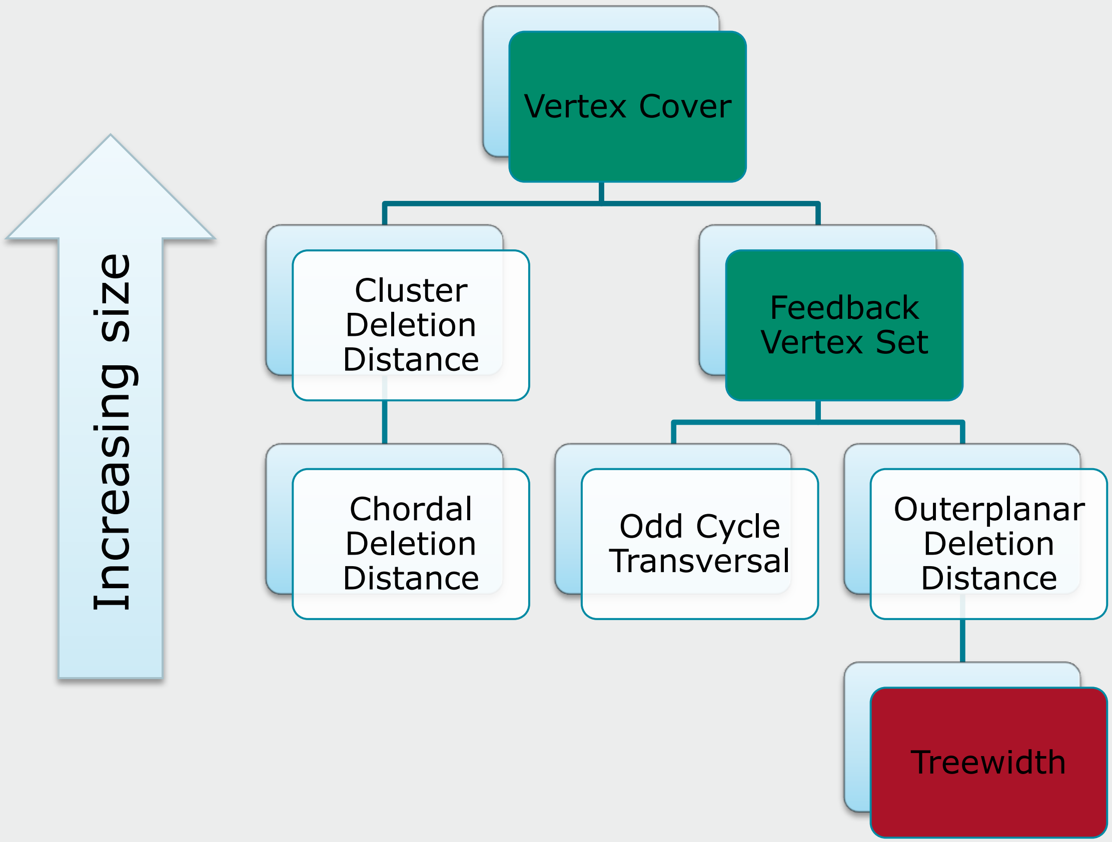

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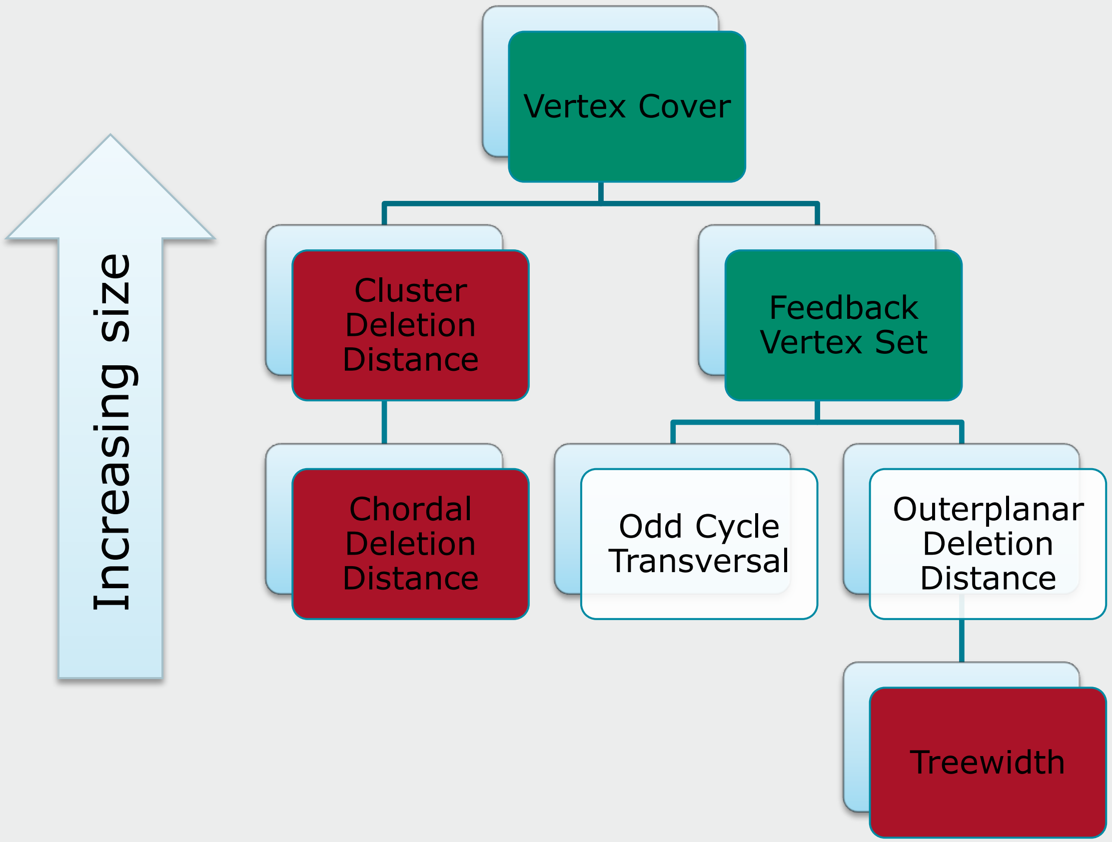

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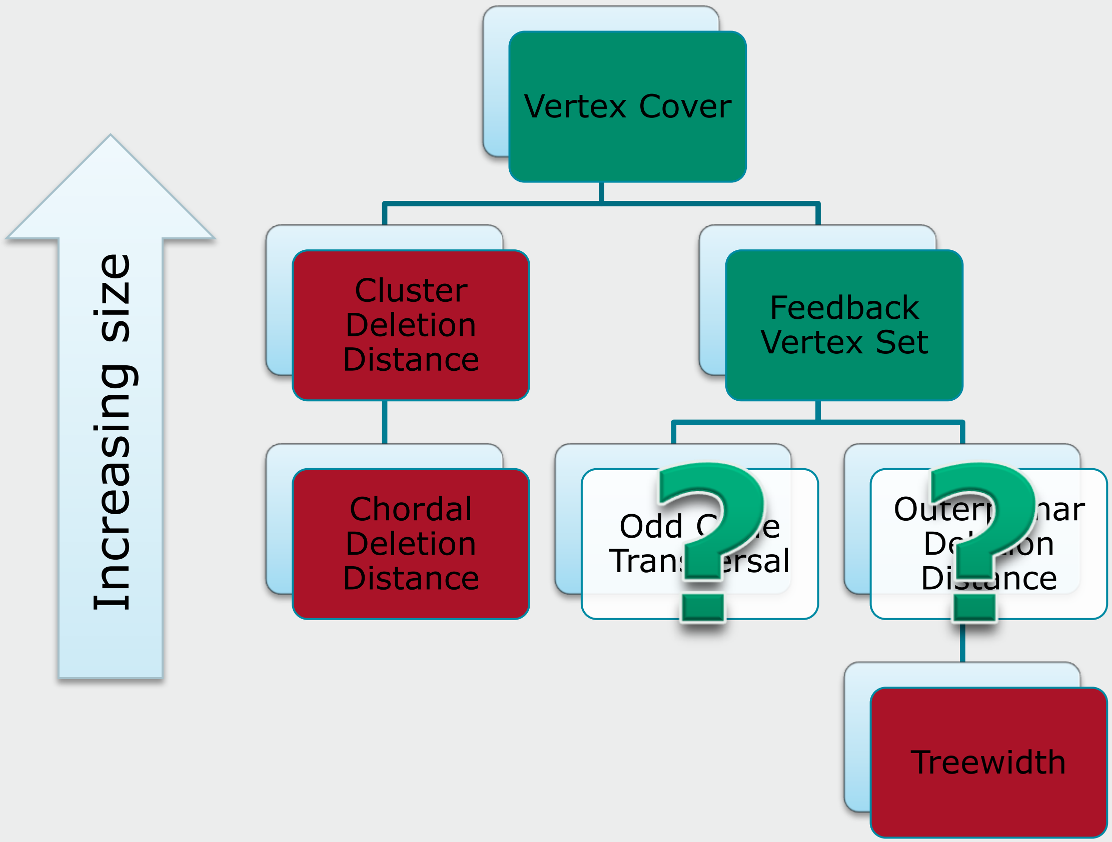

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Thank you!