On Bisimilarity for Polyhedral Models and SLCS*

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Abstract. The notion of bisimilarity plays an important role in concurrency theory. It provides formal support to the idea of processes having "equivalent behaviour" and is a powerful tool for model reduction. Furthermore, bisimilarity typically coincides with logical equivalence of an appropriate modal logic enabling model checking to be applied on reduced models. Recently, notions of bisimilarity have been proposed also for models of space, including those based on polyhedra. The latter are central in many domains of application that exploit mesh processing and typically consist of millions of cells, the basic components of face-poset models, discrete representations of polyhedral models. This paper builds on the *polyhedral semantics* of the Spatial Logic for Closure Spaces (SLCS) for which the geometric spatial model checker PolyLogicA has been developed, that is based on face-poset models. We propose a novel notion of spatial bisimilarity, called $\pm$-bisimilarity, for face-poset models. We show that it coincides with logical equivalence induced by SLCS on such models. The latter corresponds to logical equivalence (based on SLCS) on polyhedra which, in turn, coincides with simplicial bisimilarity, a notion of bisimilarity for continuous spaces.

Keywords: Bisimulation relations · Spatial bisimilarity · Spatial logics · Logical equivalence · Spatial model checking · Polyhedral models

1 Introduction

The notion of bisimilarity plays an important role in concurrency theory. It provides formal support to the idea of processes having "equivalent behaviour" and is a powerful tool for model reduction. Furthermore, bisimilarity often coincides

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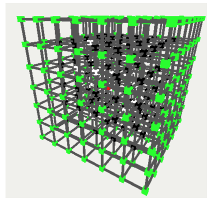
with logical equivalence of appropriate modal logics enabling powerful techniques for enhancing model checking [40, 29, 30]. Recently, notions of bisimilarity have been proposed also for models of space, including those based on polyhedra.

In this work we are following a *topological* approach to spatial logic and spatial model checking. This approach has its origin in the early ideas by McKinsey and Tarski [39], who gave a topological interpretation of the "necessarily" operator of the **S4** modal logic. The approach was extended to consider *Closure Spaces* (CS) [46], a generalisation of topological spaces, covering also discrete spaces such as general graphs, following work by Galton [26, 27] and Smyth and Webster [43], among others. Recent work by Ciancia et al. (see [21, 22]) builds on these theoretical developments using CSs, or better, *Closure Models* (CMs), as the underlying framework for the *Spatial Logic for Closure Spaces* (`SLCS`). A closure model is composed of a CS together with a valuation function mapping every atomic proposition letter p of a given set into the set of points in the space satisfying p. Based on the finite (quasi-discrete) variant of this framework `topochecker`, a spatio-temporal model checker, and `VoxLogicA`[4] a global spatial model checker, have been developed. Spatial logic and spatial model checking have been applied in several application domains such as collective and distributed systems [23, 38, 19, 44, 5]. `VoxLogicA` has been specifically optimised for the analysis of regular point-spaces, such as pixel/voxel-based images, and has been applied successfully in the area of medical imaging [10, 9, 7, 8].

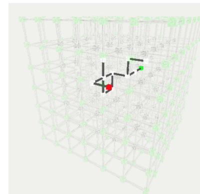
However, for the 2D and 3D visualisation of continuous spatial objects, both in medical imaging and virtual reality, models of *continuous* space are often used. Such spatial models divide the object into suitable areas of different size. These forms of division are known as *mesh techniques* and include triangular surface meshes or tetrahedral volume meshes (see for example [34]). In [11], the theoretical foundations have been developed for polyhedral model checking, including an interpretation of `SLCS` on polyhedral models, a global model checking algorithm for `SLCS` and its implementation in the `PolyLogicA`[4] tool. A visualiser for models and model checking results has been developed as well. Figure 1 provides an example of the use of polyhedral model checking to visualise some part of interest in a 3D tetrahedral volume mesh of a maze composed of 147,245 cells. A cell (see Figure 2), is the basic element of the face-poset model, a discrete representation of a polyhedral model. However, often images consist of a much larger number of cells, typically several millions or more. Figure 1b highlights the polyhedral `SLCS` model checking result of a set of spatial reachability properties characterising the white rooms and their connecting grey corridors from which both a red and a green room can be reached, without passing by black rooms. For details on the property specification and model checking experiments see [11].

Contribution The focus of the current paper is on the development of a suitable notion of spatial bisimulation that can be used to reduce the size of face-poset models, still preserving the `SLCS` properties of polyhedral models they represent.

[4] Available from the `VoxLogicA` repository at https://github.com/vincenzoml/VoxLogicA.



(a)



(b)

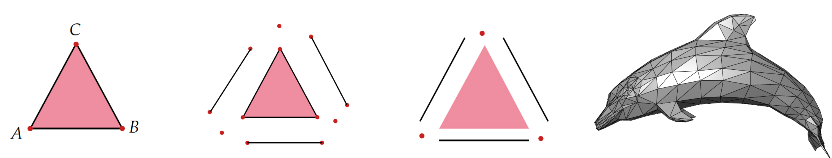
Fig. 1: (1a) 3D maze with green, white and black rooms, and one red room somewhere in the middle. (1b) Polyhedral model checking result highlighting white rooms and their connecting grey corridors from which both a red and a green room can be reached without passing by black rooms. Source [11].

To that aim we introduce a novel notion of bisimilarity on face-poset models, namely $\pm$-bisimilarity, that is defined in terms of "compatible" $\pm$-paths. We show that two cells are logically equivalent according to the relational interpretation of SLCS if and only if they are $\pm$-bisimilar.

Further related work In the domain of geographic information systems (GIS) simplicial complexes are used as an efficient data structure to store large geospatial data sets [13] in 2D or 3D. They also form the core of several important tools in this domain such as the GeoToolKit [6]. Polyhedral model checking techniques could potentially enrich the spatial query languages that are currently used in this database-oriented domain. Polyhedra are also used in the theoretical foundations of real-time and hybrid model checking (see for example [33, 3, 12, 32, 4] and references therein). In that context polyhedra, and their related notions such as template polyhedra [42, 12] and zonotopes [28], are obtained from sets of linear inequalities involving real-time constraints on system behaviour and are a natural representation of sets of states of such systems. However, in the present paper we focus on *spatial* properties of continuous space rather than on behavioural properties of systems. In [31], coalgebraic bisimilarity has been developed for a general kind of models, generalising the topological ones, known as Neighbourhood Frames. To the best of our knowledge, the notions of path and reachability are not part of that framework (that is, bisimilarity in neighbourhood semantics is based on a one-step relation rather than on paths), thus the results therein, although more general than the theory of CSs, cannot be directly reused in the setting of our current work. In [35, 36] the spatial logic SLCS is studied from a model-theoretic perspective. In particular, in [35] the authors focus on issues of expressivity of SLCS in relation to topological connectedness and separation. In [36] it is shown that the logic admits finite models for quasi-discrete neighbourhood models, but it does not do it for general neighbourhood models. The work in [37] introduces bisimulation relations that characterise spatial logics with reachability in simplicial complexes. It uses SLCS, but with a different semantics based on (sets of) simplexes. In the Computer Science literature, spatial logics have been proposed that typically describe situations in which modal operators are interpreted *syntactically* against the *structure of agents* in a process calculus. Some classical examples can be found in [16, 15]. A recent example following such an approach is given in [45]. It concerns model checking of security aspects in cyber-physical systems, in a spatial context based on the idea of bigraphical reactive systems introduced by Milner [41]. The work on *spatial model checking* for logics with reachability originated in [21] and was further developed in [22], which includes also a comparison to the work of Aiello on spatial *until* operators (see e.g. [1]). In [2], Aiello envisaged practical applications of topological logics with an *until* operator to minimisation of images. Recent work in [18, 24] builds on — and extends — that vision, taking CoPa-bisimilarity as a suitable equivalence for spatial minimisation.

2 Background and Notation

We first introduce some background concepts and related notation. For a function $f : X \to Y$, and subsets $A \subseteq X$ and $B \subseteq Y$, we define $f(A)$ and $f^{-1}(B)$ as $\{f(a) \,|\, a \in A\}$ and $\{a \,|\, f(a) \in B\}$, respectively. The *restriction* of f on A is denoted by $f|A$. The set of natural numbers and that of real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. We use the standard interval notation: for $x, y \in \mathbb{R}$ we let $[x, y]$ be the set $\{r \in \mathbb{R} \,|\, x \le r \le y\}$, $[x, y) = \{r \in \mathbb{R} \,|\, x \le r < y\}$ and so on, where $[x, y]$ is equipped with the Euclidean topology inherited from $\mathbb{R}$. We use a similar notation for intervals over $\mathbb{N}$: for $n, m \in \mathbb{N}$ $[m; n]$ denotes the set $\{i \in \mathbb{N} \,|\, m \le i \le n\}$, $[m; n)$ denotes the set $\{i \in \mathbb{N} \,|\, m \le i < n\}$, and similarly for $(m; n]$ and $(m; n)$.



(a) (b) (c) (d)

Fig. 2: (2a) A simplicial complex (actually a simplex itself). (2b) Decomposed into its simplexes as faces. (2c) Partitioned into its cells. (2d) A triangular surface mesh of a dolphin [17].

In the sequel we introduce the notions of simplex, simplicial complex and polyhedron. Intuitively, a polyhedron is composed by the set of points of its simplicial complex, that, in turn, is a finite set of simplexes. Each simplex is the convex hull of a set of affinely independent points, namely the vertices of the simplex. A *cell* of a simplex is the set of points of the (relative) interior of the simplex. For example, a triangle can be partitioned into 7 cells: its interior (an open triangle), three open segments (sides without endpoints) and the three vertices (see Figure 2c). Note that the cells of a simplex can be arranged in a partial order on the basis of the "being a face of" relation on its associated simplex. For instance, in a triangle, each vertex is a face of two open segments (and of the open triangle itself), and each open segment is a face of the open triangle. The notion of cell and face-poset extends to simplicial complexes in a natural way. A polyhedron can then be imagined as the union of the sets of points of the elements of the simplicial complex forming the polyhedron. Figure 2 shows an example of a simplicial complex and its simplexes in the face relation together with a small example of a triangular surface mesh of a dolphin.

Definition 1 (Simplex). *A* simplex σ *of* dimension d *is the convex hull of a finite set* $\{\mathbf{v_0}, \ldots, \mathbf{v_d}\} \subseteq \mathbb{R}^m$ *of* $d+1$ *affinely independent points*[5]*, i.e.* $\sigma = \{\lambda_0 \mathbf{v_0} + \ldots + \lambda_d \mathbf{v_d} \,|\, \lambda_0, \ldots, \lambda_d \in [0,1]$ *and* $\sum_{i=0}^{d} \lambda_i = 1\}$. •

Note that a simplex is a subset of the ambient space $\mathbb{R}^m$ and so it inherits its topological structure. Given a simplex σ with vertices $\mathbf{v_0}, \ldots, \mathbf{v_d}$, any subset of $\{\mathbf{v_0}, \ldots, \mathbf{v_d}\}$ spans a simplex σ' in turn: we say that σ' is a *face* of σ, written $\sigma' \sqsubseteq \sigma$. Clearly, $\sqsubseteq$ is a partial order relation.

Definition 2 (Relative Interior of a Simplex). *Given a simplex* σ *with vertices* $\{\mathbf{v_0}, \ldots, \mathbf{v_d}\}$ *the* relative interior $\widetilde{\sigma}$ *of* σ *is the following set:*
$\{\lambda_0 \mathbf{v_0} + \ldots + \lambda_d \mathbf{v_d} \,|\, \lambda_0, \ldots, \lambda_d \in (0,1]$ *and* $\sum_{i=0}^{d} \lambda_i = 1\}$. •

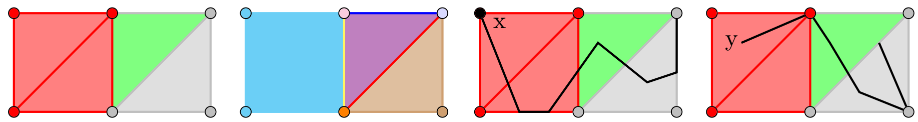
We write $\widetilde{\sigma}' \preceq \widetilde{\sigma}$ whenever $\sigma' \sqsubseteq \sigma$, noting that $\preceq$ is a partial order as well and that $\widetilde{\sigma}' \preceq \widetilde{\sigma}$ if and only if $\widetilde{\sigma}'$ is included in the topological closure of $\widetilde{\sigma}$.

Definition 3 (Simplicial Complex and Polyhedron). *A* simplicial complex K *is a finite collection of simplexes of* $\mathbb{R}^m$ *such that: (i) if* $\sigma \in K$ *and* $\sigma' \sqsubseteq \sigma$ *then also* $\sigma' \in K$*; (ii) if* $\sigma, \sigma' \in K$ *then* $\sigma \cap \sigma' \sqsubseteq \sigma$ *and* $\sigma \cap \sigma' \sqsubseteq \sigma'$.
The polyhedron $|K|$ *of* K *is the set-theoretic union of the simplexes in* K. •

Relations $\sqsubseteq$ and $\preceq$ on simplexes are inherited by simplicial complexes: relation $\sqsubseteq$ on simplicial complex K is the union of the face relations on the simplexes composing K, and similarly for $\preceq$. Note that different simplicial complexes can give rise to the same polyhedron and that the set $\widetilde{K} = \{\widetilde{\sigma} \,|\, \sigma \in K \setminus \{\emptyset\}\}$ of non-empty relative interiors of the simplexes of a simplicial complex K forms a partition of polyhedron $|K|$. The elements of $\widetilde{K}$ are called *cells* and $(\widetilde{K}, \preceq)$ is the face-poset of $|K|$. Note that, by definition of partition, each $x \in |K|$ belongs to a unique cell in the face-poset. Finally, we recall that the polyhedron $|K|$ is a subset of the ambient space $\mathbb{R}^m$ and so inherits its topological structure.

Definition 4 (Topological and Simplicial Path). *A topological* path *in a topological space* P *is a total, continuous function* $\pi : [0,1] \to P$*. Given a polyhedron* $|K|$*, a topological path* $\pi : [0,1] \to |K|$ *is* simplicial *if and only if there is a finite sequence* $r_0 = 0 < \ldots < r_n = 1$ *of values in* $[0,1]$ *and cells* $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n \in \widetilde{K}$ *such that, for all* $i = 1, \ldots, n$*, we have* $\pi((r_{i-1}, r_i)) \subseteq \widetilde{\sigma}_i$. •

[5] $\mathbf{v_0}, \ldots, \mathbf{v_d}$ are affinely independent if $\mathbf{v_1} - \mathbf{v_0}, \ldots, \mathbf{v_d} - \mathbf{v_0}$ are linearly independent. In particular, this condition implies that $d \leq m$.



(a) (b) (c) (d)

Fig. 3: An example of simplicial bisimilarity. Adapted from [11].

In the polyhedral semantics of SLCS proposed in [11], all the points of a polyhedral model that belong to the same cell are required to satisfy the same set of atomic proposition letters. This is reflected in the definition below.

Definition 5 (Polyhedral Model). *For simplicial complex K and set of proposition letters* AP*, a* polyhedral model *is a pair $(|K|, V)$ where $V : \mathtt{AP} \to \mathcal{P}(|K|)$ is a valuation function such that, for all $p \in \mathtt{AP}$, $V(p)$ is a union of cells in $\widetilde{K}$.* •

The notion of simplicial bisimilarity for polyhedra is central in the theory of the polyhedral interpretation of SLCS, together with Theorem 1 below [11]. Simplicial bisimilarity is based on the notion of topological paths and is recalled below as well. The use of paths is reminiscent to the definition of stuttering equivalence for Kripke structures or branching bisimilarity for process calculi [14, 25, 30]. However, here, the notion is cast in the setting of continuous space.

Definition 6 (Simplicial Bisimulation). *Given a Polyhedral Model $\mathcal{X} = (|K|, V)$, a symmetric binary relation $B \subseteq |K| \times |K|$ is a* simplicial bisimulation *if, for all $x_1, x_2 \in |K|$, $B(x_1, x_2)$ implies the following:*

1. *$V^{-1}(\{x_1\}) = V^{-1}(\{x_2\})$;*
2. *for each simplicial path π_1 with $\pi_1(0) = x_1$ there is a simplicial path π_2 with $\pi_2(0) = x_2$ such that $B(\pi_1(t), \pi_2(t))$ for all $t \in [0, 1]$;* •

In [11] it has been shown that, for any given polyhedral model the largest simplicial bisimulation exists. We call it *Simplicial Bisimilarity* and we write $x_1 \sim x_2$ whenever x_1 and x_2 are simplicial bisimilar.

Example Figure 3 illustrates simplicial bisimilarity. Figure 3a shows a polyhedral model composed of four triangles forming two adjacent squares. Atomic proposition letters are represented by colours (e.g. red points satisfy `red`, green points satisfy `green` etc.). Figure 3b shows the nine equivalence classes induced by simplicial bisimilarity in the polyhedral model of Figure 3a. Different classes are shown using different colours.[6] From the figure, it is clear that, for instance, no point x_1 in the yellow class is bisimilar to any point x_2 in the cyan class. This is

[6] Note that the colours of the classes have only an illustrative purpose; in particular they have nothing to do with the colours expressing the evaluation function of atomic proposition letters.

because there are simplicial paths π_1 starting from x_1 that *immediately* enter the green area of Figure 3a (i.e. $V^{-1}(\pi_1(\varepsilon)) = \texttt{green}$ for any small $\varepsilon > 0$) whereas this is impossible for any simplicial path π_2 starting from x_2 ($V^{-1}(\pi_2(\varepsilon)) = \texttt{red}$ for any small $\varepsilon > 0$ and every such path π_2). This implies that $B(x_1, x_2)$ for no simplicial bisimulation B. In fact, the second condition of Definition 6 would be violated since $B(\pi_1(\varepsilon), \pi_2(\varepsilon))$ cannot hold for ε as above. Similarly, the only point x_3 in the orange class can *immediately* enter the red area of Figure 3a via a simplicial path π_3 whereas no other point satisfying `gray` can do that. Note in particular that any point in the top-right segment of the polyhedron can reach the red area via a simplicial path, but any such path must first go through part of the top-right segment of the polyhedron and/or the green area. So, also in this case, the second condition of Definition 6 would be violated. Figures 3c and 3d show an example of pairs of simplicial paths that witness $x \sim y$.

The following definition introduces the variant of SLCS for polyhedral models proposed in [11]. In the present paper, we denote it by $\texttt{SLCS}_\gamma$.

Definition 7 (SLCS on polyhedral models - $\texttt{SLCS}_\gamma$). *The abstract language of* $\texttt{SLCS}_\gamma$ *is the following:* $\Phi ::= p \mid \neg\Phi \mid \Phi_1 \vee \Phi_2 \mid \gamma(\Phi_1, \Phi_2)$.
The satisfaction relation of $\texttt{SLCS}_\gamma$ *with respect to a given polyhedral model* $\mathcal{X} = (|K|, V)$*,* $\texttt{SLCS}_\gamma$ *formula* Φ*, and* $x \in |K|$ *is defined recursively on the structure of* Φ *as follows:*

$$\begin{array}{ll}
\mathcal{X}, x \models p & \Leftrightarrow x \in V(p);\\
\mathcal{X}, x \models \neg\Phi & \Leftrightarrow \mathcal{X}, x \models \Phi \textit{ does not hold};\\
\mathcal{X}, x \models \Phi_1 \vee \Phi_2 & \Leftrightarrow \mathcal{X}, x \models \Phi_1 \textit{ or } \mathcal{X}, x \models \Phi_2;\\
\mathcal{X}, x \models \gamma(\Phi_1, \Phi_2) & \Leftrightarrow \textit{a topological path } \pi : [0,1] \to |K| \textit{ exists such that}\\
& \quad \pi(0) = x, \mathcal{X}, \pi(1) \models \Phi_2, \textit{and } \mathcal{X}, \pi(r) \models \Phi_1 \textit{ for all } r \in (0,1).
\end{array}$$

•

Note that the above definition generalises the classical topological interpretation of the $\Box$ modality as interior. In fact, $\Box\Phi$ is equivalent to $\neg\gamma(\neg\Phi, \texttt{true})$ (see [11]).

Example Again with reference to model $\mathcal{X}$ of Figure 3a, it is easy to see that any point in the yellow class satisfies, for instance, $\gamma(\texttt{green}, \texttt{true})$, and also $\gamma(\texttt{green}, \texttt{red})$ and $\texttt{red} \wedge \gamma(\texttt{green}, \texttt{red})$.

Definition 8 ($\texttt{SLCS}_\gamma$ Logical Equivalence). *Given Polyhedral Model* $\mathcal{X} = (|K|, V)$ *and* $x_1, x_2 \in |K|$ *we say that* x_1 *and* x_2 *are* logically equivalent *with respect to* $\texttt{SLCS}_\gamma$*, written* $x_1 \simeq_{\texttt{SLCS}_\gamma} x_2$*, if and only if, for all* $\texttt{SLCS}_\gamma$ *formulas* Φ *the following holds:* $\mathcal{M}(\mathcal{X}), x_1 \models \Phi$ *if and only if* $\mathcal{M}(\mathcal{X}), x_2 \models \Phi$. •

Logical equivalence coincides with simplicial bisimilarity [11]:

Theorem 1 (Corollary 6.5 of [11]). *Given Polyhedral Model* $\mathcal{X} = (|K|, V)$*,* $x_1, x_2 \in |K|$ *the following holds:* $x_1 \simeq_{\texttt{SLCS}_\gamma} x_2$ *if and only if* $x_1 \sim x_2$. $\Box$

The following definition characterises the discrete representation of polyhedral models we will use in the rest of the paper.

Definition 9 (face-poset model). *Given Polyhedral Model $\mathcal{X} = (|K|, V)$ we define the* face-poset model $\mathcal{M}(\mathcal{X})$ *as the the Kripke model* $(W, \preceq, \mathcal{V})$ *such that: (i)* $W = \widetilde{K}$*; (ii)* $\preceq \subseteq W \times W$ *such that* $\widetilde{\sigma} \preceq \widetilde{\sigma}'$ *if and only if* $\sigma \sqsubseteq \sigma'$*; (iii)* $\widetilde{\sigma} \in \mathcal{V}(p)$ *if and only if* $\widetilde{\sigma} \subseteq V(p)$*.* •

Below, we recall the definition of $\pm$-paths introduced in [11]. They faithfully represent, in the face-poset model, topological paths in the polyhedral one. Consider, for instance, the polyhedron consisting of a segment from point A to point B and its related face-poset. A path starting from, say, point A can "immediately enter" the open segment AB whereas, a path starting from a point within the open segment cannot "immediately proceed" to A (neither to B); it *has* to first traverse a fraction of the open segment AB, then ending in A (or B). This is reflected in the face-poset by requiring that a path therein, i.e. a $\pm$-path, cannot perform a *first* step going against the partial order (going "down"), whereas in its *last* step it cannot follow strictly the partial order (going "up").

Definition 10 ($\pm$-path). *Let* $\mathcal{M}(\mathcal{X}) = (W, \preceq, \mathcal{V})$ *be a finite face-poset model and let* $\preceq^{\pm}$ *be the relation* $\preceq \cup \succeq$*. We say that, for* $\ell \in \mathbb{N}$*, sequence* $\pi : [0; \ell] \to W$ *is a* $\pm$-path *(and we indicate it by* $\pi : [0; \ell] \overset{\pm}{\to} W$*) if* $\ell \geq 2$ *and the following holds:* $\pi(0) \preceq \pi(1) \preceq^{\pm} \pi(2) \preceq^{\pm} \ldots \preceq^{\pm} \pi(\ell - 1) \succeq \pi(\ell)$*.* •

The following definition re-interprets `SLCS` on finite face-posets and is based on $\pm$-paths [11]. In order to avoid confusion, in the sequel, we will call the resulting logic $\texttt{SLCS}_{\pm}$.

Definition 11 (`SLCS` on finite face-posets - $\texttt{SLCS}_{\pm}$). *The satisfaction relation of* $\texttt{SLCS}_{\pm}$ *with respect to a given finite face-poset model* $\mathcal{M}(\mathcal{X}) = (W, \preceq, \mathcal{V})$*,* $\texttt{SLCS}_{\pm}$ *formula* Φ*, and* $w \in W$ *is defined recursively on the structure of* Φ*:*

$$
\begin{array}{lll}
\mathcal{M}(\mathcal{X}), w \models p & \Leftrightarrow & w \in \mathcal{V}(p); \\
\mathcal{M}(\mathcal{X}), w \models \neg \Phi & \Leftrightarrow & \mathcal{M}(\mathcal{X}), w \models \Phi \text{ does not hold}; \\
\mathcal{M}(\mathcal{X}), w \models \Phi_1 \vee \Phi_2 & \Leftrightarrow & \mathcal{M}(\mathcal{X}), w \models \Phi_1 \text{ or } \mathcal{M}(\mathcal{X}), w \models \Phi_2; \\
\mathcal{M}(\mathcal{X}), w \models \gamma(\Phi_1, \Phi_2) & \Leftrightarrow & \text{a } \pm\text{-path } \pi : [0; \ell] \overset{\pm}{\to} W \text{ exists such that } \pi(0) = w, \\
 & & \mathcal{M}(\mathcal{X}), \pi(\ell) \models \Phi_2, \text{ and} \\
 & & \mathcal{M}(\mathcal{X}), \pi(i) \models \Phi_1 \text{ for all } i \in (0; \ell).
\end{array}
$$

Definition 12 (Logical Equivalence). *Given finite face-poset model* $\mathcal{M}(\mathcal{X}) = (W, \preceq, \mathcal{V})$ *and* $w_1, w_2 \in W$ *we say that* w_1 *and* w_2 *are* logically equivalent *with respect to* $\texttt{SLCS}_{\pm}$*, written* $w_1 \simeq_{\texttt{SLCS}_{\pm}} w_2$ *if and only if, for all* $\texttt{SLCS}_{\pm}$ *formulas* Φ *the following holds:* $\mathcal{M}(\mathcal{X}), w_1 \models \Phi$ *if and only if* $\mathcal{M}(\mathcal{X}), w_2 \models \Phi$*.* •

A fundamental result, see [11], follows, where with slight overloading, for $x \in |K|$, we let $\mathcal{M}(x)$ denote the unique cell $\widetilde{\sigma} \in \widetilde{K}$ such that $x \in \widetilde{\sigma}$ (see Figure 4 for an illustration).

Theorem 2 (Theorem 4.4 of [11]). *Let* $\mathcal{X} = (|K|, V)$ *a polyhedral model and* $\mathcal{M}(\mathcal{X})$ *the associated face-poset model as by Definition 9. For all* $x \in |K|$ *and formula* Φ *the following holds:* $\mathcal{X}, x \models \Phi$ *if and only if* $\mathcal{M}(\mathcal{X}), \mathcal{M}(x) \models \Phi$*.* □

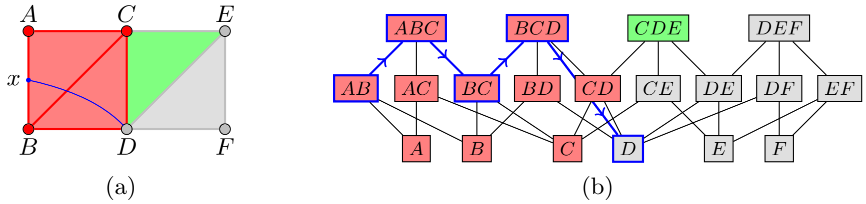


Fig. 4: (4a) A polyhedral model $\mathcal{X}$ with atomic propositions **red**, **green** and **gray**, and a path from a point x to vertex D. (4b) Hasse diagram of face-poset model $\mathcal{M}(\mathcal{X})$ and a path (in blue) corresponding to the path in $\mathcal{X}$.

The following definition introduces some notation for *sequences*, which $\pm$-paths are a particular case of, and that will be useful in the rest of the paper.

Definition 13 (Sequences). *Given a set X, a* sequence *over X from x, of* length *$\ell \in \mathbb{N}$, is a total function $s : [0;\ell] \to X$ such that $s(0) = x$. For sequence s of length ℓ, we often use the notation $(x_i)_{i=0}^{\ell}$ where $x_i = s(i)$ for $i \in [0;\ell]$. Given sequences $s' = (x'_i)_{i=0}^{\ell'}$ and $s'' = (x''_i)_{i=0}^{\ell''}$, with $x'_{\ell'} = x''_0$, the* sequentialisation *$s' \cdot s'' : [0;\ell' + \ell''] \to X$ of s' with s'' is the sequence from x'_0 defined as follows:*

$$(s' \cdot s'')(i) = \begin{cases} s'(i), & \text{if } i \in [0;\ell'],\\ s''(i-\ell'), & \text{if } i \in [\ell';\ell'+\ell'']. \end{cases}$$

For sequence $s = (x_i)_{i=0}^{n}$ and $k \in [0;n]$ we define the k-shift operator $\_\uparrow k$ as follows: $s\uparrow k = (x_{j+k})_{j=0}^{n-k}$ and, for $0 < m \leq n$, we let $s\leftarrow m$ denote the sequence obtained from s by inserting a copy of $s(m)$ immediately before $s(m)$ itself, i.e. $s\leftarrow m = (s[0;m]) \cdot ((s(m), s(m)), (s\uparrow m))$. Finally, a (non-empty) prefix *of s is a sequence $s|[0;k]$, for some $k \in [0;n]$.* •

For example, for sequence (a,b,c) of length 2 and sequence (c,d) of length 1, we have $(a,b,c)\cdot(c,d) = (a,b,c,d)$, of length 3, $(a)\cdot(a,b) = (a,b)$, $(a)\cdot(a) = (a)$. Note the difference between sequentialisation and concatenation '++': for instance, $(a,b){+\!+}(c) = (a,b,c)$ whereas $(a,b)\cdot(c)$ is undefined since $b \neq c$, $(a){+\!+}(a)$ is (a,a) whereas $(a)\cdot(a) = (a)$. We have $(a,b,c)\uparrow 1 = (b,c)$ and $(a,b,c)\uparrow 2 = (c)$ while $(a,b,c)\leftarrow 1 = (a,b,b,c)$. Sequences $(a), (a,b), (a,b,c)$ are all the (non-empty) prefixes of (a,b,c).

3 ±-bisimilarity and the Coincidence Result

In this section, we present the novel notion of $\pm$-bisimulation, that is based on the notion of $\pm$-path *compatibility*, inspired by compatibility of paths in quasi-discrete closure models introduced in [24]. We additionally show that $\pm$-bisimularity coincides with logical equivalence for $\mathtt{SLCS}_{\pm}$.

Definition 14 (±-path compatibility). *Given face-poset model $\mathcal{M}(\mathcal{X}) = (W, \preceq, \mathcal{V})$ and binary relation $B \subseteq W \times W$, two $\pm$-paths $\pi_1 = (w'_i)_{i=0}^{k_1}$, $\pi_2 =$*

$(w''_j)_{j=0}^{k_2}$ are called compatible *with respect to B in $\mathcal{M}(\mathcal{X})$ if, for some $N > 0$, two total monotone non-decreasing surjections $z_1 : [0; k_1] \to [1; N]$ and $z_2 : [0; k_2] \to [1; N]$ exist such that $z_1(1) = z_2(1), z_1(k_1 - 1) = z_2(k_2 - 1)$ and $B(w'_i, w''_j)$ for all indices $i \in [0; k_1]$ and $j \in [0; k_2]$ satisfying $z_1(i) = z_2(j)$.* •

The functions z_1 and z_2 are referred to as *matching functions*. Note that both the number N and functions z_1 and z_2 need not be unique. The minimal number $N > 0$ for which matching functions exist is defined to be the *number of zones* of the two $\pm$-paths π_1 and π_2. It is easy to see that, whenever two $\pm$-paths are compatible, for any pair of matching function z_1 and z_2 the following holds, by virtue of monotonicity and surjectivity: $z_1(0) = z_2(0) = 1$ and $z_1(k_1) = z_2(k_2) = N$. Hence $B(w'_0, w''_0)$ and $B(w'_{k_1}, w''_{k_2})$, and of course $B(w'_1, w''_1)$ and $B(w'_{k_1-1}, w''_{k_2-1})$.

Given binary relation $B \subseteq W \times W$, compatibility of $\pm$-paths with respect to B is a binary relation over $\pm$-paths. We write $\pi_1 \, \texttt{comp}^B \pi_2$ whenever $\pm$-paths π_1 and π_2 are compatible with respect to B. Lemma 1 below, proved in Appendix B, states some properties of $\pm$-paths compatibility that turn useful in the sequel.

Lemma 1. *Let $\mathcal{M}(\mathcal{X}) = (W, \preceq, \mathcal{V})$ be a face-poset, $B \subseteq W \times W$ a relation, π, π_1, π_2 $\pm$-paths with π of length $\ell > 0$, $s_1 : [0; \ell_1] \to W, s_2 : [0; \ell_2] \to W$ sequences of length $\ell_1, \ell_2 \in \mathbb{N}$ respectively, $m \in (0; \ell)$. The following holds:*

1. *$\pi \, \texttt{comp}^B (\pi {\leftarrow} m)$.*
2. *If B is an equivalence relation, then:*
   (a) *so is $\texttt{comp}^B$, and*
   (b) *the sequentialisation of two sequences of equivalent elements, and non-decreasing first step, with two compatible $\pm$-paths results in compatible $\pm$-paths. Formally: if $\pi_1 \, \texttt{comp}^B \pi_2$, $s_h(0) \preceq s_h(1)$ and $s_h(\ell_h) = \pi_h(0)$ for $h \in [1; 2]$, with $B(s_1(i), s_2(j))$ for all $i \in [0; \ell_1)$ and $j \in [0; \ell_2)$, then $s_1 \cdot \pi_1$ and $s_2 \cdot \pi_2$ are $\pm$-paths that are compatible with respect to B.*

Definition 15 ($\pm$-bisimulation). *Let $\mathcal{M}(\mathcal{X}) = (W, R, \mathcal{V})$ be a finite face-poset model. A symmetric binary relation $B \subseteq W \times W$ is a poset $\pm$-bisimulation if, for all $w_1, w_2 \in W$, if $B(w_1, w_2)$ then the following holds:*

1. *$\mathcal{V}^{-1}(\{w_1\}) = \mathcal{V}^{-1}(\{w_2\})$;*
2. *for each $\pm$-path π_1 from w_1 there is a $\pm$-path π_2 from w_2 such that $\pi_1 \, \texttt{comp}^B \pi_2$.*

We say that w_1 and w_2 are $\pm$-bisimilar, written $w_1 \rightleftharpoons_{\pm} w_2$, if there is a $\pm$-bisimulation B such that $B(w_1, w_2)$. •

Example With reference to the polyhedral model $\mathcal{X}$ of Figure 3a, in Figure 5b the $\pm$-bisimilarity equivalence classes are shown in different colours for $\mathcal{M}(\mathcal{X})$. In Figure 5a we recall the simplicial bisimilarity quotient of model $\mathcal{X}$, adding some names for reference in the sequel. There is no $\pm$-path starting from any of the cells in the cyan class that is compatible with $\pm$-path $\pi_{CD} = (CD, CDE, CDE)$ from cell CD in the yellow class as it is easy to see in Figure 4b. The same applies for $\pm$-path $\pi_C = (C, CDE, CDE)$ from cell C.[7] Similarly, let us consider cell D. We have already seen that there is no other point in the polyhedral model that is simplicial bisimilar to point D. Let us consider $\pm$-path $\pi_D = (D, CD, CD)$. In the sequel we show there cannot be any $\pm$-path from any other cell satisfying `gray` that is compatible with π_D. In fact, any other such a $\pm$-path π should be such that $\pi(1)$ satisfies `red` (this is required by the fact that $z_D(1) = z(1)$ for any pair of matching functions for π_D and π) and $\pi(j)$ should not satisfy `green` for any j (since no element of π_D satisfies `green`). On the other hand, any $\pm$-path π' starting from any other cell satisfying `gray` and reaching a cell satisfying `red` is such that $\pi'(1)$ does *not* satisfy `red`. Furthermore, many such $\pm$-paths have an element that satisfies `green`. Thus, there is no $\pm$-path starting from any other `gray` cell that is compatible with (D, CD, CD) and D is in fact in a different class than any other `gray` cell.

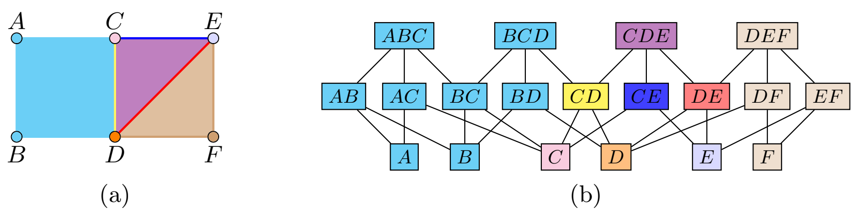


Fig. 5: Equivalence classes of the polyhedral model of Figure 3a w.r.t. simplicial bisimilarity (5a) and those of its face-poset model w.r.t. $\pm$-bisimilarity (5b).

We are now in a position to state and prove the two main technical results of this paper, viz. soundness of $\pm$-bisimilarity and the fact that logical equivalence is a $\pm$-bisimulation.

Theorem 3. *For w_1, w_2 in finite face-poset model $\mathcal{M}(\mathcal{X})$, the following holds: if $w_1 \rightleftharpoons_\pm w_2$ then $w_1 \simeq_{\mathtt{SLCS}_\pm} w_2$.*

Proof. Let $\mathcal{M}(\mathcal{X}) = (W, \preceq, \mathcal{V})$ be a face-poset model. We proceed by induction on the structure of Φ in $\mathtt{SLCS}_\pm$. We only cover the case $\gamma(\Phi_1, \Phi_2)$ since the others are straightforward. Let w_1 and w_2 be two points of $\mathcal{M}(\mathcal{X})$ such that $w_1 \rightleftharpoons_\pm w_2$. Suppose $w_1 \models \gamma(\Phi_1, \Phi_2)$. Let $\pi_1 = (w'_i)_{i=0}^{k_1}$ be a $\pm$-path from w_1 satisfying $\pi_1(k_1) \models \Phi_2$ and $\pi_1(i) \models \Phi_1$ for all $i \in (0; k_1)$. Since $w_1 \rightleftharpoons_\pm w_2$, a $\pm$-path $\pi_2 = (w''_i)_{i=0}^{k_2}$ from w_2 exists that is compatible with π_1 with respect to $\rightleftharpoons_\pm$. Let, for appropriate $N > 0$, $z_1 : [0; k_1] \to [1; N]$ and $z_2 : [0; k_2] \to [1; N]$ be matching functions for π_1 and π_2. Without loss of generality, $z_2^{-1}(\{N\}) = \{k_2\}$.

Since $z_1(k_1) = z_2(k_2) = N$, we have $\pi_1(k_1) \rightleftharpoons_\pm \pi_2(k_2)$. Thus $\pi_2(k_2) \models \Phi_2$ by Induction Hypothesis. Moreover, if $j \in (0; k_2)$, then $z_2(j) < N$ by assumption and there is $i \in (0; k_1)$ such that $z_1(i) = z_2(j)$, that is $\pi_1(i) \rightleftharpoons_\pm \pi_2(j)$.

[7] Recall that partial orders are transitive and reflexive.

Since $\pi_1(i) \models \Phi_1$, it follows that $\pi_2(j) \models \Phi_1$ by Induction Hypothesis. Therefore $\pm$-path π_2 witnesses $w_2 \models \gamma(\Phi_1, \Phi_2)$. $\square$

Theorem 4. *For finite face-poset model $\mathcal{M}(\mathcal{X})$, $\simeq_{\mathtt{SLCS}_\pm}$ is a $\pm$-bisimulation.*

Proof. Let $\mathcal{M}(\mathcal{X}) = (W, \preceq, \mathcal{V})$ be a finite face-poset model. We check that $\simeq_{\mathtt{SLCS}_\pm}$ satisfies requirement (2) of Definition 15. Requirement (1) is immediate. Let, for points $x, y \in W$, the $\mathtt{SLCS}_\pm$-formula $\delta_{x,y}$ be such that $\delta_{x,y}$ is $\mathtt{true}$ if $x \simeq_{\mathtt{SLCS}_\pm} y$, and $x \models \delta_{x,y}$ and $y \models \neg\delta_{x,y}$ if $x \not\simeq_{\mathtt{SLCS}_\pm} y$. Put $\chi(x) = \bigwedge_{y \in W} \delta_{x,y}$. It is easy to see that, for $x, y \in W$, it holds that

$$y \models \chi(x) \text{ if and only if } x \simeq_{\mathtt{SLCS}_\pm} y. \tag{1}$$

Let Π be the set of all finite sequences $(x_i)_{i=0}^n$ over $\mathcal{M}(\mathcal{X})$. Note that such sequences might not be $\pm$-paths. Furthermore, let function $\mathtt{zones} : \Pi \to \mathbb{N}$ be such that, for sequence $s = (x_i)_{i=0}^n$,

$$\begin{array}{ll} \mathtt{zones}(s) = 1 & \text{if } n = 0 \\ \mathtt{zones}(s) = \mathtt{zones}(s{\uparrow}1) & \text{if } n > 0 \text{ and } x_0 \simeq_{\mathtt{SLCS}_\pm} x_1 \\ \mathtt{zones}(s) = \mathtt{zones}(s{\uparrow}1) + 1 & \text{if } n > 0 \text{ and } x_0 \not\simeq_{\mathtt{SLCS}_\pm} x_1 \end{array}$$

A sequence s is said to have k zones, if $\mathtt{zones}(s) = k$.

Claim For all $k \geqslant 1$, for all $x_1, x_2 \in W$, if $x_1 \simeq_{\mathtt{SLCS}_\pm} x_2$ and π_1 is a $\pm$-path from x_1 and π_1 has k zones, then a $\pm$-path π_2 from x_2 exists such that π_2 is compatible with π_1 with respect to $\simeq_{\mathtt{SLCS}_\pm}$. The claim is proven by induction on k.

Base case, $k = 1$: If $x_1 \simeq_{\mathtt{SLCS}_\pm} x_2$ and $\pi_1 = (x'_i)_{i=0}^n$ is a $\pm$-path from x_1 that has 1 zone only, then $x_1 \simeq_{\mathtt{SLCS}_\pm} x'_i$ for all $i \in [0; n]$. Let π_2 be the $\pm$-path (x_2, x_2, x_2). Since $x_1 \simeq_{\mathtt{SLCS}_\pm} x_2$, also $x_2 \simeq_{\mathtt{SLCS}_\pm} x'_i$ for all $i \in [0; n]$. Hence, π_2 is compatible with π_1 with respect to $\simeq_{\mathtt{SLCS}_\pm}$ with matching functions $z_1(i) = 1$ for all $i \in [0; n]$ and $z_2(j) = 1$ for all $j \in [0; 2]$.

Induction step, $k{+}1$: Suppose $x_1 \simeq_{\mathtt{SLCS}_\pm} x_2$ and $\pi_1 = (x'_i)_{i=0}^n$ is a $\pm$-path from x_1 of $k{+}1$ zones. Let $m > 0$ be such that $x_1 \simeq_{\mathtt{SLCS}_\pm} x'_i$ for all $i \in [0; m)$ and $x_1 \not\simeq_{\mathtt{SLCS}_\pm} x'_m$. We distinguish two cases:

Case A: $m = 1$ (Figure 6 shows an example for $m = 1$ and length $n = 3$). In this case, it holds that $x_1 \models \gamma(\chi(x'_1), \mathtt{true})$. Since $x_2 \simeq_{\mathtt{SLCS}_\pm} x_1$, we also have $x_2 \models \gamma(\chi(x'_1), \mathtt{true})$. Therefore, a $\pm$-path π' exists from x_2 such that $\pi'(1) \models \chi(x'_1)$, i.e. $\pi'(1) \simeq_{\mathtt{SLCS}_\pm} x'_1$ by Equation 1 (Figure 6a). Let us, first of all, consider the sequence $\pi'_1 = (x'_1, x'_1) \cdot (\pi_1{\uparrow}1)$, obtained by inserting a copy of x'_1 before $(\pi_1{\uparrow}1)$ (Figure 6b and Figure 6c). Note that π'_1 is a $\pm$-path of length n. In fact, $\pi'_1(0) {\preceq} \pi'_1(1)$, since $\pi'_1(0) = \pi'_1(1)$ by construction. Furthermore, $\pi'_1(n-1) = \pi_1(n-1) {\succeq} \pi_1(n) = \pi'_1(n)$, where $\pi_1(n-1) {\succeq} \pi_1(n)$ because π_1 is a $\pm$-path. Finally, all the subsequent intermediate elements of π'_1 are in the $\preceq^\pm$ relation by construction. Moreover, note that π'_1 has the same number of zones as $\pi_1{\uparrow}1$, that is k. So, by the Induction Hypothesis, since $\pi'(1) \simeq_{\mathtt{SLCS}_\pm} x'_1$, there is a $\pm$-path π'' from $\pi'(1)$ such that $\pi'' \, \mathtt{comp}^{\simeq_{\mathtt{SLCS}_\pm}} \pi'_1$

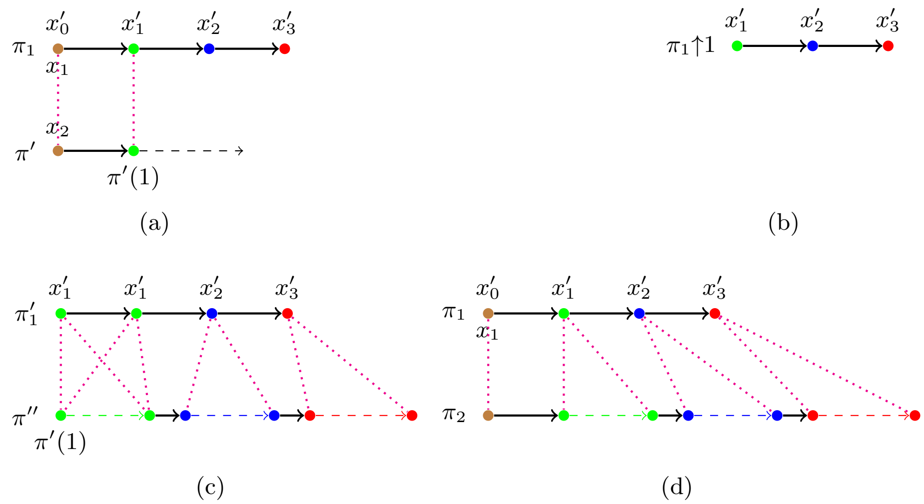


Fig. 6: Example illustrating the proof of Theorem 4, for $n = 3$, **Case A:** $m = 1$. In the figure, different zones are shown by using different colours, and we assume $\texttt{zones}(\pi_1) = 4$. Dotted lines in magenta indicate pairs belonging to $\simeq_{\texttt{SLCS}_\pm}$.

(see Figure 6c). Now, using Lemma 1.2b, for sequences $\pi'|[0;1]$ and $\pi_1|[0;1]$ and $\pm$-paths π'' and π'_1 respectively, we get $(\pi'|[0;1] \cdot \pi'')\, \texttt{comp}^{\simeq_{\texttt{SLCS}_\pm}} (\pi_1|[0;1]) \cdot \pi'_1$. Finally, noting that $(\pi_1|[0;1]) \cdot \pi'_1$ is exactly $\pi_1{\leftarrow}1$ and using Lemma 1.1, we get $(\pi_1|[0;1]) \cdot \pi'_1\, \texttt{comp}^{\simeq_{\texttt{SLCS}_\pm}} \pi_1$. Since $\simeq_{\texttt{SLCS}_\pm}$ is an equivalence relation, we finally get, using Lemma 1.2a, $(\pi'|[0;1] \cdot \pi'')\, \texttt{comp}^{\simeq_{\texttt{SLCS}_\pm}} \pi_1$ and we choose $\pi_2 = \pi'|[0;1] \cdot \pi''$ (see Figure 6d)).

Case B: $m > 1$. If $m > 1$ then it holds that $x_1 \models \gamma(\chi(x_1), \chi(x'_m))$. Since, by hypothesis, $x_2 \simeq_{\texttt{SLCS}_\pm} x_1$ also $x_2 \models \gamma(\chi(x_1), \chi(x'_m))$. Thus, a $\pm$-path π', of some length $\ell \geq 2$, from x_2 exists, such that $\pi'(\ell) \models \chi(x'_m)$ and $\pi'(j) \models \chi(x_1)$ for all $j \in (0;\ell)$. We have that $x'_m \simeq_{\texttt{SLCS}_\pm} \pi'(\ell)$ and $x_1 \simeq_{\texttt{SLCS}_\pm} \pi'(j)$ for all $j \in (0;\ell)$, by Equation 1. In the sequel, we focus on the case $1 < m < n$. The proof for the case $1 < m = n$ is straightforward and is shown in Appendix A.

Suppose $m > 1$ and $m < n$ (Figure 7 shows an example for $m = 2$ and $n = 3$). In a similar way as before, we first consider the sequence $\pi'_1 = (x'_m, x'_m) \cdot (\pi_1{\uparrow}m)$ and let h be the length of π'_1. Note that π'_1 is a $\pm$-path. In fact $(\pi_1{\uparrow}m) = (\ldots x'_{n-1}, x'_n)$ has length at least 1—it has at least two elements, because $m < n$ and the length of (x'_m, x'_m) is 1. So, by definition of sequentialisation π'_1 has length at least 2—it has at least three elements. Moreover $\pi'_1(0) = \pi'_1(1)$ by construction, so $\pi'_1(0) \preceq \pi'_1(1)$ and $\pi'_1(h-1) = \pi_1(n-1) \succeq \pi_1(n) = \pi'_1(h)$, since π_1 is a $\pm$-path. Finally, all the subsequent intermediate elements of π'_1 are in the $\preceq^\pm$ relation by construction. Note, furthermore, that π'_1 has the same number of zones as $(\pi_1{\uparrow}m)$, namely k. So, by the Induction Hypothesis, since $\pi'(\ell) \simeq_{\texttt{SLCS}_\pm}$

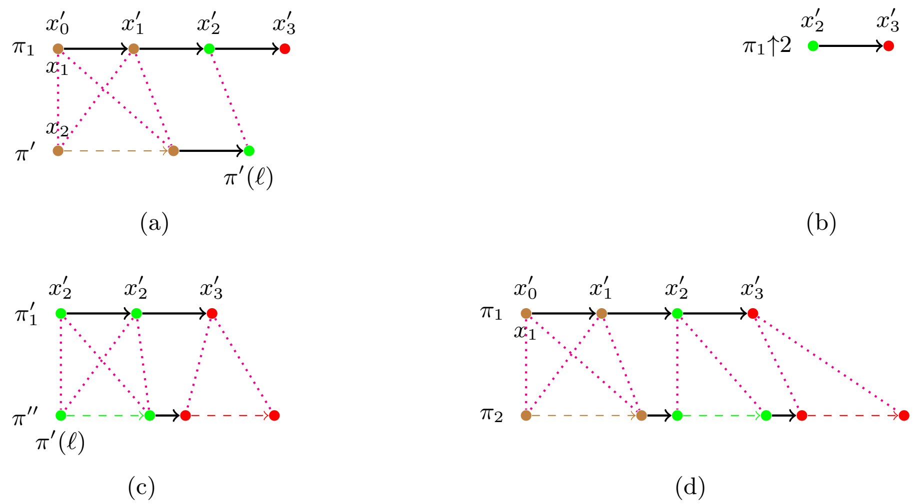


Fig. 7: Example illustrating the proof of Theorem 4, for $n = 3$, **Case B** and $1 < m < n$. In the figure, different zones are shown by using different colours, and we assume $\texttt{zones}(\pi_1) = 3$. Dotted lines in magenta indicate pairs that belong to $\simeq_{\texttt{SLCS}_\pm}$.

x'_m we know that there is a $\pm$-path π'' from $\pi'(\ell)$ such that $\pi'' \; \texttt{comp}^{\simeq_{\texttt{SLCS}_\pm}} \; \pi'_1$ (see Figure 7c). Now, using Lemma 1.2b, for sequences π' and $\pi_1|[0;m]$ and $\pm$-paths π'' and π'_1 respectively, we get $(\pi' \cdot \pi'') \; \texttt{comp}^{\simeq_{\texttt{SLCS}_\pm}} (\pi_1|[0;m]) \cdot \pi'_1$. Finally, noting that $(\pi_1|[0;m]) \cdot \pi'_1$ is exactly $\pi_1{\leftarrow}m$ and using Lemma 1.1, we get $(\pi_1|[0;m]) \cdot \pi'_1 \; \texttt{comp}^{\simeq_{\texttt{SLCS}_\pm}} \pi_1$. Since $\simeq_{\texttt{SLCS}_\pm}$ is an equivalence relation, we finally get, using Lemma 1.2a, $\pi' \cdot \pi'' \; \texttt{comp}^{\simeq_{\texttt{SLCS}_\pm}} \pi_1$ and we choose $\pi_2 = \pi' \cdot \pi''$ (see Figure 7d).

This proves the claim. From the claim it follows immediately that $\simeq_{\texttt{SLCS}_\pm}$ satisfies Definition 15(2). □

On the basis of Theorem 3 and Theorem 4, we have that the largest $\pm$-bisimulation exists, it is a $\pm$-bisimilarity, it is an equivalence relation, and it coincides with logical equivalence in the face-poset induced by $\texttt{SLCS}_\pm$:

Corollary 1. *For every finite face-poset $\mathcal{M}(\mathcal{X}) = (W, \preceq, \mathcal{V}), w_1, w_2 \in W$, the following holds: $w_1 \rightleftharpoons_\pm w_2$ if and only if $w_1 \simeq_{\texttt{SLCS}_\pm} w_2$.* □

Example As expected, with reference to the face-poset model $\mathcal{M}(\mathcal{X})$ of Figure 4b for polyhedral model $\mathcal{X}$ of Figure 3a, it is easy to see that cells C and CD satisfy $\gamma(\texttt{green}, \texttt{true})$, and also $\gamma(\texttt{green}, \texttt{red})$ and $\texttt{red} \land \gamma(\texttt{green}, \texttt{red})$.

In conclusion, recalling that for all $x \in \mathcal{X}$ and $\texttt{SLCS}_\gamma$ formula Φ, we have that $\mathcal{X}, x \models \Phi$ if and only if $\mathcal{M}(\mathcal{X}), \mathcal{M}(x) \models \Phi$, we get the following final result

Corollary 2. *For all polyhedral models $\mathcal{X}$, $x, x_1, x_2 \in \mathcal{X}$: $x_1 \sim x_2$ if and only if $x_1 \simeq_{\mathtt{SLCS}_\gamma} x_2$ if and only if $\mathcal{M}(x_1) \rightleftharpoons_\pm \mathcal{M}(x_2)$ if and only if $\mathcal{M}(x_1) \simeq_{\mathtt{SLCS}_\pm} \mathcal{M}(x_2)$.* □

Example Figure 8 shows the minimal model $\min(\mathcal{M}(\mathcal{X}))$, modulo ±-bisimilarity, of $\mathcal{M}(\mathcal{X})$ (see Figure 4b). Model $\min(\mathcal{M}(\mathcal{X}))$ has been obtained in a similar way as described in Proposition 1 of [20] . Note that the model is transitive and reflexive, because of Corollary 1 above, and the reflexivity and idempotency axioms of topological modal logic. Thus, in Figure 8 the model is represented by its Hasse diagram. Each element of $\min(\mathcal{M}(\mathcal{X}))$ is coloured according to the atomic proposition satisfied by the members of the corresponding ±-bisimilarity class and its border has the colour of the class (see Figure 5b). The ±-path $(1, 1, 1, 1, 3)$ in the minimal model corresponds to (AB, ABC, BC, BCD, D) shown in Figure 4b and $(2, 5, 2)$ witnesses formula $\mathtt{red} \wedge \gamma(\mathtt{green}, \mathtt{red})$ in the minimal model.

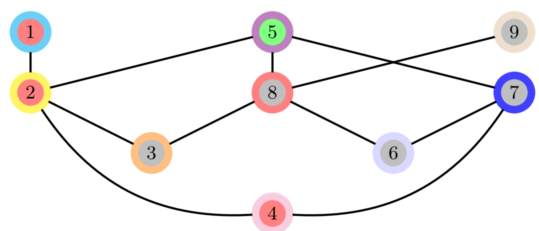


Fig. 8: Hasse diagram of the minimal model, modulo ±-bisimilarity, of the model of Figure 4b.

4 Conclusions and Future Work

We have introduced a novel notion of spatial bisimilarity, namely ±-bisimilarity on face-poset models representing polyhedra models. We have shown that it coincides with logical equivalence based on the variant of SLCS proposed in [11]. Consequently, two points in a polyhedral model are simplicial bisimilar if and only if their corresponding cells in the face-poset are ±-bisimilar.

Part of future work will be to investigate the relationship between bisimilarity notions developed for face-poset models, and those developed in the context of closure models, e.g. those studied in [18, 24]. Furthermore, we plan to develop slightly weaker notions of ±-bisimilarity, together with their associated spatial logics. Such coarser equivalences are of interest for further model reduction. We will follow an approach along the lines of the work in [20] for CMs. Finally, the issue of the impact of adding a "converse" operator for γ to the logic — in a similar vein as for other reachability operators, in e.g. [8, 18, 24] — on the associated bisimilarity and its geometrical interpretation is another subject for future study.

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Appendices A and B, containing some proofs, are included here for convenience of the reviewers. They are not meant to be part of the final version of the paper, if accepted, where a reference to a technical report containing all the proofs will be inserted.

A Proof of Theorem 4 - Case $1 < m = n$

Suppose $m > 1$ and $m = n$ (Figure 9 shows an example for $m = n = 3$).
In this case, $\pi_2 = \pi'$ is a $\pm$-path that is compatible with π_1. Let, in fact, $z_1 : [0; n] \to W$ and $z_2 : [0; \ell] \to W$ be defined as follows:

$$z_1(i) = \begin{cases} 1 \text{ if } i \in [0; n), \\ 2 \text{ if } i = n. \end{cases} \qquad z_2(j) = \begin{cases} 1 \text{ if } j \in [0; \ell), \\ 2 \text{ if } j = \ell. \end{cases}$$

We have that $z_1(1) = z_2(1)$, $z_1(n-1) = z_2(\ell - 1)$, and $\pi_2(j) \simeq_{\mathtt{SLCS}_\pm} \pi_1(i)$ whenever $z_1(i) = z_2(j)$. In fact:

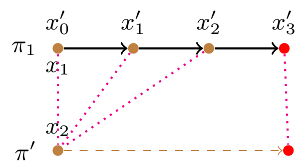

Fig. 9: Example illustrating the proof of Theorem 4, for $n = 3$, **Case B** and $1 < m = n$. In the figure, different zones are shown by using different colours, and we assume $\texttt{zones}(\pi_1) = 2$. Dotted lines in magenta indicate pairs that are elements of $\simeq_{\texttt{SLCS}_\pm}$.

– $\pi_2(\ell) \simeq_{\texttt{SLCS}_\pm} \pi_1(m)$, since $\pi_2(\ell) = \pi'(\ell)$, $\pi_1(m) = x'_m$, and $\pi'(\ell) \models \chi(x'_m)$;
– for $i \in [0;n)$ and $j \in [0;\ell)$ we have $\pi_2(j) \simeq_{\texttt{SLCS}_\pm} \pi_1(i)$ since $\pi_2(j) = \pi'(j) \simeq_{\texttt{SLCS}_\pm} x_1 \simeq_{\texttt{SLCS}_\pm} \pi_1(i)$, since $\pi_2(0) = x_2 \simeq_{\texttt{SLCS}_\pm} x_1 = \pi_1(0)$ by hypothesis and, for $j \in (0;\ell)$ and $i \in [0;n)$, as a consequence of $\pi'(j) \models \chi(x_1)$ $\pi_1(i) \models \chi(x_1)$, as shown above.

B Auxiliary Lemmas

Lemma 1 *Let $\mathcal{M}(\mathcal{X}) = (W, \preceq, \mathcal{V})$ be a face-poset, $B \subseteq W \times W$ a relation, π, π_1, π_2 $\pm$-paths with π of length $\ell > 0$, $s_1 : [0;\ell_1] \to W, s_2 : [0;\ell_2] \to W$ sequences of length $\ell_1, \ell_2 \in \mathbb{N}$ respectively, $m \in (0;\ell)$. The following holds:*

1. *$\pi \; \texttt{comp}^B (\pi{\leftarrow}m)$.*
2. *If B is an equivalence relation, then:*
   (a) *so is $\texttt{comp}^B$, and*
   (b) *the sequentialisation of two sequences of equivalent elements and non-decreasing first step with two compatible $\pm$-paths results in compatible $\pm$-paths. Formally: if $\pi_1 \, \texttt{comp}^B \pi_2$, $s_h(0) \preceq s_h(1)$ and $s_h(\ell_h) = \pi_h(0)$ for $h \in [1;2]$ with $B(s_1(i), s_2(j))$ for all $i \in [0;\ell_1)$ and $j \in [0;\ell_2)$, then $s_1 \cdot \pi_1$ and $s_2 \cdot \pi_2$ are $\pm$-paths that are compatible with respect to B.*

Proof. For what concerns point 1 just consider functions $z : [0;\ell] \to [1;\ell+1]$ and $z' : [0;\ell+1] \to [1;\ell+1]$ defined as follows

$$z(i) = i + 1. \qquad z'(j) = \begin{cases} z(j) \text{ if } i \leq m, \\ z(j-1) \text{ if } i > m. \end{cases}$$

It is easy to check that z and z' are matching functions for π and $\pi{\leftarrow}m$ with respect to B.
As far as point 2 is concerned, we only prove Point (2a), i.e. that if B is an equivalence relation, then so is $\texttt{comp}^B$. The other part of the statement, i.e. Point (2b), follows directly from the conditions on sequences s_1 and s_2 and the

relevant definitions.
The proof for reflexivity and symmetry of $\texttt{comp}^B$ is straightforward. We prove transitivity. Let $\pi_1 : [0;\ell_1] \stackrel{\pm}{\to} W$, $\pi_2 : [0;\ell_2] \stackrel{\pm}{\to} W$, and $\pi_3 : [0;\ell_3] \stackrel{\pm}{\to} W$. Suppose $\pi_1 \, \texttt{comp}^B \pi_2$ and $\pi_2 \, \texttt{comp}^B \pi_3$. Let $f_1 : [0;\ell_1] \to [1,N]$ and $f_2 : [0;\ell_2] \to [1,N]$ be the relevant matching functions for $\pi_1 \, \texttt{comp}^B \pi_2$ and, using Lemma 2 below, let $f_2 : [0;\ell_2] \to [1,N]$ and $f_3 : [0;\ell_3] \to [1,N]$ be the matching functions relevant for $\pi_2 \, \texttt{comp}^B \pi_3$. We show that f_1 and f_3 are matching functions for $\pi_1 \, \texttt{comp}^B \pi_3$. Let $i_1 \in [0;\ell_1]$ and $i_3 \in [0;\ell_3]$ s.t. $f_1(i_1) = f_3(i_3)$. Since $\pi_1 \, \texttt{comp}^B \pi_2$—and f_1 is total and f_2 is surjective—there is i_2 s.t. $f_1(i_1) = f_2(i_2)$, and so $B(\pi_1(i_1), \pi_2(i_2))$. Moreover, since $f_1(i_1) = f_3(i_3)$, we also get $f_2(i_2) = f_3(i_3)$ and, consequently, $B(\pi_2(i_2), \pi_3(i_3))$, given that $\pi_2 \, \texttt{comp}^B \pi_3$. By transitivity of B we have $B(\pi_1(i_1), \pi_3(i_3))$, which brings to the assert. □

Lemma 2. *Let $\mathcal{M}(\mathcal{X}) = (W, \preceq, \mathcal{V})$ a face-poset and $B \subseteq W \times W$ an* equivalence *relation. Let $\pi_1 : [0;\ell_1] \stackrel{\pm}{\to} W$ and $\pi_2 : [0;\ell_2] \stackrel{\pm}{\to} W$ such that $\pi_1 \, \texttt{comp}^B \pi_2$ with N zones and $z_1 : [0;\ell_1] \to [1,N]$ and $z_2 : [0;\ell_2] \to [1,N]$ be the relevant matching functions. Let furthermore $\pi_3 : [0;\ell_3] \stackrel{\pm}{\to} W$ such that $\pi_2 \, \texttt{comp}^B \pi_3$ with N' zones and matching functions $z'_2 : [0;\ell_2] \to [1,N']$ and $z_3 : [0;\ell_3] \to [1,N']$. Then $N' = N$ and $z'_2 = z_2$.* □

Proof. A sketch of the proof follows. Let $\pi_1 \, \texttt{comp}^B \pi_2$ with N and matching functions $z_1 : [0;\ell_1] \to [1;N]$ and $z_2 : [0;\ell_2] \to [1;N]$, and suppose B is an equivalence over W. By definition of matching functions and the fact that B is an equivalence, it follows that any set $S_{h,k} = \{\pi_h(i) \,|\, z_h(i) = k\}$, for $h \in [1;2]$ and $k \in [1;N]$ is a subset of an equivalence class of B. Note that such $S_{h,k}$ is zone k of π_h and that such a zone is the *longest* subsequence of π_h composed of immediately successive (i.e. adjacent in the ±-path) elements of π_h that are equivalent w.r.t B. Suppose now $\pi_2 \, \texttt{comp}^B \pi_3$, with N' zones and $z'_2 : [0;\ell_2] \to [1,N']$ and $z_3 : [0;\ell_3] \to [1,N']$. Define $S'_{h,k}$ as $S_{h,k}$, but w.r.t. π_2 and π_3. If $N' < N$ this would mean that in π_2 there would be adjacent equivalent elements that fall into different zones, which would contradict the fact that the number of zones of π_2 is N. Similarly, if $N' > N$ then there would be a k such that $S_{2,k}$ contains points that are not equivalent w.r.t. B: again a contradiction. Thus $N = N'$ and, consequently, $z'_2 = z_2$. □

Remark 1. Note that the reasoning in the proof of Lemma 2 above is valid only if B is an equivalence relation. Consider for instance the following ±-paths, π_x, π_y, π_z, for appropriate x_i, y_i, z_i, for $i \in [0;2]$: $\pi_x = (x_0, x_1, x_2), \pi_y = (y_0, y_1, y_2), \pi_z = (z_0, z_1, z_2)$, where $B = \{(x_i, y_j) \,|\, i,j \in [0;2]\} \cup \{(y_k, z_k) \,|\, k \in [0;2]\}$.

Clearly B is not an equivalence relation. If we consider π_x and π_y we see they are compatible and there is only one zone. If instead we consider π_y and π_z, we see that also they are compatible, but the number of zones is necessarily three.