Real-Time in Stochastic Process Algebra: Keeping Track of Winners and Losers

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Abstract. A stochastic time process algebra that deals with generally distributed delays in the style of real-time process theories is presented. Two types of race condition are distinguished to enable a compositional modeling as well as a non-trivial expansion law. The interplay of real-time and stochastic time is analyzed for the standard bisimulation definitions and for the race condition. Finally, a new notion of context-sensitive interpolation is proposed that captures time-additivity as induced by the race condition.

1 Introduction

Stochastic process algebras support the combined modeling of the functionality and performance of a system in a compositional manner. Markovian process algebras, like EMPA, PEPA, IMC, exploit the memoryless property of the exponential distribution and typically produce Markov chains. However, more general distributions are required, for example, to model contemporary Internet protocols: the transfer control protocol requires real-time time-outs; media streaming and web services are governed by heavy-tail distributions with high variance [1]; etc. Therefore, stochastic process algebras with general distributions have emerged, like TIPP, GSMPA, SPADES, IGSMP and NMSPA [2–6] yielding generalized semi-Markov processes [7] as underlying performance models.

Compositional modeling with general distributions, however, proves to be a non-trivial task. For general distributions one cannot rely on the memoryless property that enables efficient and elegant expansion laws for the parallel composition in the exponential setting. In generalized semi-Markov processes, one exploits clocks to retain the Markov property of history independence. In the same vein, stochastic process algebras typically have a layered semantics: The topmost, symbolic layer uses clocks to manipulate the stochastic delays that guard the actions, whereas the second concrete layer deals with probabilistic timed transitions. Note that uncountably infinite state spaces can occur in case the clocks have continuous probability distributions.

The symbolic representation with clocks is appealing as it enables manipulation of finite structures (e.g., stochastic automata [8] or extensions of generalized semi-Markov processes [5]). In order to obtain a concrete model, the clocks are

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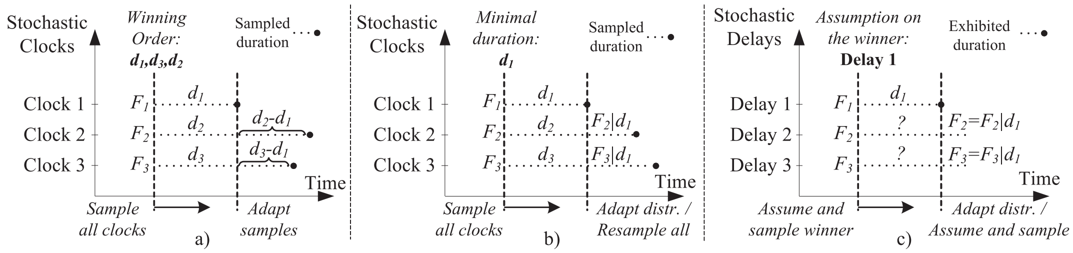

Fig. 1. Race condition: a) residual lifetime semantics with clocks, b) spent lifetime semantics with clocks and c) spent lifetime semantics with stochastic delays

sampled to obtain the probabilistic timed transitions. In general, two execution policies can be adopted [1, 9]: (1) *race condition* [2, 4, 6], which enables the action transitions guarded by the clocks that expire first (standard for the Markovian models) and (2) *pre-selection policy* [3, 5], which preselects the clocks by a probabilistic choice (the execution policy of the generalized semi-Markov processes). In absence of the memoryless property, the samples of the clocks must be updated after each timed transition. The literature provides two techniques for doing this: (1) keeping track of the *residual lifetime* of clocks, i.e., the time that is left before the clock expires or (2) keeping track of the *spent lifetime* of clocks, i.e., the time that the clock has been active.

The residual lifetime semantics [4], depicted in Fig. 1a, supports performance analysis via discrete event simulation, that is extensively exploited when analytical methods cannot be applied. However, it has been criticized for its being unfair as the outcome of the race condition is known upfront due to early sampling of clocks. The spent lifetime semantics [2, 3, 5, 6], depicted in Fig. 1b, has been advocated for its correspondence to standard real-time, as the clocks increase as time passes. Additionally, the approach is considered fair with respect to the race condition as (1) the clocks are pre-sampled to statistically determine the minimal sample and, afterwards, (2) the original samples are discarded, and (3) the probability distributions of the remaining clocks are 'aged' with the minimal sample. However, the fairness comes at a price: re-sampling of the clocks is required after each resolution of the race condition.

An alternative, but equivalent approach to the race condition, see Fig. 1c, is (1) to make a *probabilistic* assumption on the outcome of the race condition by conditioning on the clocks that win the race, and afterward (2) to sample from the (joint) probability distribution of the winning clocks [10]. This alternative approach samples each clock only once. So, we no longer speak of clocks as there is no need to keep track of their lifetimes, but the '*age*' of the distributions is still preserved. We refer to the samples as *stochastic delays*, resembling the notion of timed delays. Note that multiple clocks can simultaneously exhibit the minimal duration only if they sample from discrete probability distributions.

The main goal of our paper is to analyze the similarities of real-time and stochastic time and, subsequently, to exploit the common features to embed real-time in the style of [11, 12] into stochastic time. In [13] a structural translation from stochastic automata to timed automata with deadlines that preserves timed traces and enables embedding of real-time in SPADES is given. A translation from IGSMP into pure real-time models called interactive timed automata is reported in [5]. Thus, stochastic clocks have close ties with timed automata. However, we consider timed delays as a more natural choice for the algebraic approach. In that direction, we look more closely at the race condition. We note that real-time actually induces a *trivial* race condition in which the shortest 'sample' is always exhibited by the same set of delays and always has the same (deterministic) duration. Therefore, we propose to represent timed delays as Dirac stochastic delays.

In the approach presented here, we model stochastic delays as separate constructs guided by discrete (finite or countably infinite) random variables. We cater for weak choice between passage of time and immediate actions and termination [11]. We provide a layered semantics in terms of stochastic transition schemes on the symbolic level (in essence, stochastic automata with spent lifetime semantics) and stochastic transition systems (representing the concrete model based on the probabilistic timed transition systems). Furthermore, we differentiate between two types of stochastic delays: (1) independent, and (2) dependent. In the former the random variables involved are unrelated and is meant for modeling purposes. In the latter, the same samples are produced when guided by the same random variable. Using dependence, the outcomes of the race condition can be made explicit which is helpful when unfolding a parallel composition. After setting up the theory, we introduce timed delays as degenerated stochastic delays and we analyze to what extent the standard properties of real-time are preserved. Finally, we revisit the notion of time additivity and introduce the finer notion of context-sensitive interpolation.

The rest of this paper is organized as follows: Section 2 provides the background and discusses the design choices. Section 3 introduces the basic sequential processes and their semantics. Section 4 deals with the parallel composition and its expansion. Section 5 relates real-time and stochastic time. Section 6 wraps up with concluding remarks.

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2 The Race Condition

We use discrete random variables to represent durations of stochastic delays. The set of distribution functions F such that $F(t) = 0$ for $t \leq 0$ is denoted by $\mathcal{F}$; the set of the corresponding random variables by $\mathcal{V}$. We use X, Y and Z to range over $\mathcal{V}$ and F_X, F_Y and F_Z for their respective distribution functions. If $P(X = t) = 1$, for some $t > 0$, the random variable X is said to be degenerated or Dirac and we write X_t. The set of such random variables is denoted by $\mathcal{V}_{\mathrm{deg}}$. By assumption, the support set $\mathrm{supp}(X) = \{\, t > 0 \mid P(X = t) > 0 \,\}$ of a random variable X is finite or countably infinite. By $\bar{F}_X(t)$ we denote the residual probability distribution $1 - F_X(t)$. For $S \subseteq \mathcal{V}$, $y \in \mathbb{R}$ and $\diamond$ either $<$, $>$ or $=$, we write $S \diamond y$ if $X \diamond y$, for all $X \in S$. Conditional random variables are denoted by $\langle X \,|\, \mathit{Event} \rangle$ with $X \in \mathcal{V}$ and Event such that $P(\mathit{Event}) > 0$. By $\mathrm{PS}(S)$ we denote a standard probability space over the set S.

A stochastic delay is a timed delay of a duration guided by a random variable. We observe simultaneous passage of time for a number of stochastic delays until one or some of them expire. This phenomenon is referred to as the *race condition* and the process as the *race*. For multiple *racing* stochastic delays, different stochastic delays can be observed simultaneously as being the shortest. The ones that have the shortest duration are called *winners* and the others are referred to as *losers*. We consider two types of races: (1) *dependent*, in which the stochastic delays simultaneously guided by the same random variable always exhibit the same duration and (2) *independent*, in which the stochastic delays guided by the same random variable are equally distributed, but not necessarily equally sampled. The probability that $W \subseteq V$ are the winners out of the racing delays V with a duration d is denoted by $\mathrm{RC}_d(W, V)$. It is defined as

$$\mathrm{RC}_d(W, V) = \prod_{X \in W} P(X = d) \cdot \prod_{X \in V \setminus W} \bar{F}_X(d).$$

The probability for $W \subseteq V$ being the winners is denoted by $\mathrm{RC}(W, V)$. and can be calculated from $\mathrm{RC}(W, V) = \sum_{d \in \mathrm{supp}(W)} \mathrm{RC}_d(W, V)$.

Design choices Next, we motivate our design choices. We informally write $X.p \parallel Y.q$ for the parallel composition of two processes that are prefixed by stochastic delays guided by the random variables X and Y in Fig. 2 and, similarly, we write $X.p + Y.q + X.r$ for the alternative composition in Fig. 3.

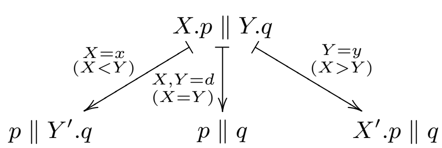

Fig. 2. Parallel composition

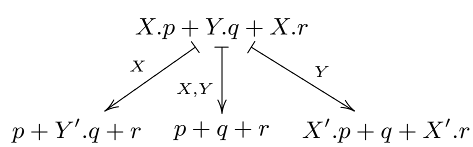

Fig. 3. Alternative composition

As depicted in Fig. 2, the loser becomes dependent on the winner in case of interleaving, whereas the delays are synchronized if they exhibit the same duration. For example, in the leftmost transition in Fig. 2, X is the winner and Y is the loser. By Y' we denote that the distribution of Y has changed. The dependence on the winning duration is expressed using an *aging* function $\_\,|\,\_\colon [\mathcal{F} \times \mathbb{R} \longrightarrow \mathcal{F}]$ given by $(F|d)(t) = (F(t+d) - F(d))/(1 - F(d))$, provided that $F(d) < 1$. We note that iterative aging of a stochastic delay is the same as aging it simultaneously by the sum of the durations, i.e., $(\ldots(F|d_1)\ldots)|d_n = F|\,(\sum_{i=1}^{n} d_i)$ [14].

Expansion Another important issue we consider is the expansion law. Standardly, $X.p \parallel Y.q$ is decomposed as an alternative composition of summands containing $X.p$ and $Y.q$. Clock-based approaches [4, 5, 9] split the stochastic delay X on a starting X^+ and a termination X^- activity and (intuitively) put

$$X.p \parallel Y.q = X^+.X^-.p \parallel Y^+.Y^-.q = X^+.Y^+.(X^-.p \parallel Y^-.q) + Y^+.X^+.(X^-.p \parallel Y^-.q).$$

However, this splitting of a delay into separate activities of start and termination does not completely match the expansion in real-time of the form

$$t.p \parallel s.q = \min(t,s).\big((t-\min(t,s)).p \parallel (s-\min(t,s)).q\big),$$

obtained by prefixing with the minimal delay, where $t, s > 0$.

An initial account of stochastic delays in real-time is given in [15], where name dependence is treated to obtain a suitable expansion. Dependent stochastic delays in an alternative composition are depicted in Fig. 3, where delays guided by the same random variable X observe the same duration. Unfortunately, dependent stochastic delays are not congruent with respect to any composition because of name dependencies [15]. Therefore, we provide two types of delays here: *independent stochastic delays*, notation $\zeta_X.\_$, are in place for modeling purposes; *dependent stochastic delays*, notation $\sigma_X.\_$, support the expansion of the parallel composition. In order to deal with name dependencies in a compositional manner, we introduce the *name resolution operator* $|\_|$. Dependent delays within the scope of this operator are treated as independent with respect to the ones outside the scope. Furthermore, we need to have means to distinguish the winners of a race and to make the losers dependent on them. We introduce a *race condition guard* $\left[\begin{smallmatrix}W\\L\end{smallmatrix}\right]\_$, where $W \subseteq \mathcal{V}$ is the non-empty set of winners and $L \subseteq \mathcal{V}$ is a (disjoint) set of losers. This way we can decompose, e.g.,

$$\sigma_X.p + \sigma_Y.q = \left[\begin{smallmatrix}X\\Y\end{smallmatrix}\right]\sigma_X.(p + \sigma_Y.q) + \left[\begin{smallmatrix}Y\\X\end{smallmatrix}\right]\sigma_Y.(p + \sigma_X.q) + \left[\begin{smallmatrix}X,Y\\\emptyset\end{smallmatrix}\right]\sigma_X.(p+q)$$

where X is the winner in the first summand, Y is the winner in the second and X and Y win the race together in the third. Note the analogy with the trivial race in real-time process theories where $\sigma^t.p + \sigma^{t+s}.q = \sigma^t.(p + \sigma^s.q)$, where $\sigma^t.p$ denotes a timed delay of duration t.

3 Basic Sequential Processes

Let $\mathcal{A}$ be a set of actions and $\mathcal{V}$ a set of random variables that guide the stochastic delays. The collection $\mathrm{BSP}^{\mathrm{dst}}$ of basic sequential processes with discrete stochastic time distinguishes between dependent processes $\mathcal{D}$ with possible unresolved name dependencies and independent processes $\mathcal{I}$. Characteristic for $\mathrm{BSP}^{\mathrm{dst}}$ are the immediate actions and termination, the dependent and independent delays, the race condition guard and a name resolution operator.

Definition 1. *Processes in* $\mathrm{BSP}^{\mathrm{dst}}$ *(the precedence is implied by the ordering) consists of two constants* δ *and* ϵ*, four unary operator schemes (1)* $a.\_$ *for* $a \in \mathcal{A}$*,* *(2) $\sigma_X.\_$ for $X \in \mathcal{V}$, (3) $\zeta_X.\_$ for $X \in \mathcal{V}$ and (4) $\left[{W \atop L}\right]\_$ for $W \in (2^{\mathcal{V}} \setminus \{\emptyset\})$ and $L \subseteq \mathcal{V}$ such that $W \cap L = \emptyset$, one binary operator (5) $\_+\_$ and one unary operator (6) $|\_|$. The syntax is given by P, where*

$$D ::= P \mid a.D \mid \sigma_X.D \mid \left[{W \atop L}\right] D \mid D + D$$
$$P ::= \delta \mid \epsilon \mid |D| \mid a.P \mid \zeta_X.P \mid P + P$$

The sets $\mathcal{I}$ and $\mathcal{D}$ of independent and dependent processes are generated by P and D, respectively, and they are ranged over by p, q, r, etc. We omit $\{$ and $\}$ when clear from the context and write, for example, $\left[{X,Y \atop U,V}\right]p$ instead of $\left[{\{X,Y\} \atop \{U,V\}}\right]p$.

In the setting above, the interpretation of the alternative composition relies on the context: (1) a non-deterministic choice is made between actions; (2) a weak choice is allowed between actions, successful termination and stochastic delays; (3) a dependent race is imposed on dependent stochastic delays; (4) an independent race is enabled between independent stochastic delays. By construction, all dependent delays are captured in scope of the name resolution operator.

Stochastic transition schemes Semantics is given in terms of *stochastic transition schemes* based on the operational rules given in Table 1 below. In essence, they represent stochastic automata with spent lifetime semantics reflecting the race condition. The explicit is given in terms of stochastic transition systems introduced later, which handle stochastic delays as probabilistic timed delays. They are induced from the schemes by solving the races and sampling, cf. Fig. 1c.

As we implement spent lifetime semantics, we have to keep track of the dependencies of the losers on the winners. This is done in an *environment* that for each variable holds a set of stochastic delays on which it depends. We put $\alpha \colon \mathcal{V} \to 2^{\mathcal{V}}$ and write $\mathcal{E}_{\mathrm{d}}$ for the set of all such environments. W.l.o.g., the initial state has no dependencies between the delays, so the initial environment $\alpha_\emptyset$ with $\alpha_\emptyset(X) = \emptyset$, for $X \in \mathcal{V}$, applies.

Definition 2. *A stochastic transition system scheme is a tuple $(\mathcal{S} \times \mathcal{E}_{\mathrm{d}}, \langle s, \alpha_\emptyset \rangle, \mathcal{A}, \mathcal{V}, \to, \mapsto, \downarrow, \mathrm{R})$, where*

- *$\mathcal{S} \times \mathcal{E}_{\mathrm{d}}$ is a set of states in environments;*
- *$\langle s, \alpha_\emptyset \rangle \in \mathcal{S} \times \mathcal{E}_{\mathrm{d}}$ is an initial state;*
- *$\to \subseteq \mathcal{S} \times \mathcal{A} \times \mathcal{S}$ is the labeled transition relation;*
- *$\mapsto \subseteq (\mathcal{S} \times \mathcal{E}_{\mathrm{d}}) \times (2^{\mathcal{V}} \setminus \{\emptyset\}) \times (\mathcal{S} \times \mathcal{E}_{\mathrm{d}})$ is the stochastic delay transition relation.*
- *$\downarrow \subseteq \mathcal{S}$ is the immediate termination predicate;*
- *$\mathrm{R} \colon \mathcal{S} \to 2^{\mathcal{V}}$ is the racing delays function, where $S_{\langle u,\alpha \rangle} \subseteq R(u)$, for $S_{\langle u,\alpha \rangle} = \bigcup_{S : \langle u,\alpha \rangle \xmapsto{S} \langle u',\alpha' \rangle} S$.*

We note that the labeled transitions do not alter the environment as they depend only on the structure of the process term. The stochastic delay transitions define the winners of a race. The potential transitions are only performed internally by the operational rules, as they are used to resolve the name dependencies in the dependent races. They are not visible in the graph that represents the stochastic transition scheme as all dependent delays are in the scope of the independent race operator. However, we consider them in the stochastic bisimulation relation below as separate constructs in order to allow manipulation of dependent terms in the equational theory. Finally, whether or not a state has the termination option is given by the termination predicate $\downarrow$. For the stochastic transition scheme of $p \in \mathcal{I}$, defined by the operational rules in Table 1 below, we write $\mathrm{STS}(p)$.

Structural operational semantics We next discuss the operational rules presented in Table 1. For a process $p \in \mathcal{I}$, the racing delays $\mathrm{R}(p)$ are defined as the dependent and independent stochastic delays that are directly connected by the topmost alternative composition, i.e., $\mathrm{R}(p) = \mathrm{I}(p) \cup \mathrm{D}(p)$, where

$$\mathrm{I}(\zeta_X.p) = \{X\}, \quad \mathrm{I}(|p|) = \mathrm{D}(d), \quad \mathrm{I}(p+q) = \mathrm{I}(p) \cup \mathrm{I}(q)$$
$$\mathrm{D}(\sigma_X.p) = \{X\}, \quad \mathrm{D}(\left[\begin{smallmatrix} W \\ L \end{smallmatrix}\right] p) = W \cup L, \quad \mathrm{D}(p+q) = \mathrm{D}(p) \cup \mathrm{D}(q)$$

and $\mathrm{I}(p), \mathrm{D}(p) = \emptyset$ for the other cases. It is assumed that all stochastic delays have unique names, so there are no clashes in the environment [15]. For conciseness, we put $f\{f_1/D_1 \dots f_n/D_n\}(x) = f_i(x)$ if $x \in D_i$, and $f(x)$ otherwise, for functions $f, f_1, \dots, f_n \colon C \to D$ and disjoint subsets $D_1, \dots, D_n \subseteq D$ for $n \in \mathbb{N}$. Also, we put $\alpha_X(Y) = \alpha(Y) \cup \{X\}$ for $Y \in \mathcal{V}$.

We have the standard rules for termination options and action prefix. Rule 7 states that the independent stochastic delay transition $\zeta_X.p$ allows passage of time as guided by X. Rule 13 shows that the dependent stochastic delay prefix $\sigma_X.p$ enables a potential delay guided by X. Non-determinism can be resolved by an action transition, as captured by rule 5. The racing delays of the losing summand are made dependent on the winners by adding a winning random variable to their dependence set as given by rule 10. Rule 8 represents weak choice between immediate labeled transitions and passage of time. Rule 12 states that a joint race can be won by the union of the winners of the both summands. The weak choice in the dependent case is given by rule 14. The importance of random variable names is readily observed in the rules 16 and 18: The additional condition $S \cap \mathrm{R}(q) = \emptyset$ in rule 16 guarantees that no winner and loser are guided by the same random variable. In rule 18 the losers and winners of both summands are compared. Rules 19 to 21 describe mixed races: If the race is won by both summands, the joint delay is dependent because of the name dependence of one of the summands. Rules 25, 26 and 28 show that the scope operator does not affect the termination options, the action prefix or independent delays, but it turns dependent stochastic delays into (actual) independent ones as given by rule 27. Rules 29 and 30 are straightforward. Finally, rule 31 provides means to explicitly specify and make the losers dependent on the winners. The conditions assure well-formedness, whereas the environment is updated only for the outdated losers that exist in the dependent racing delays of the resulting term p'.

Stochastic transition schemes defined by the structural operational semantics in Table 1 induce complete probability measures using the race condition because

$$\textbf{1}\ \langle \epsilon,\alpha\rangle\!\downarrow \qquad \textbf{2}\ \frac{\langle p,\alpha\rangle\!\downarrow}{\langle p+q,\alpha\rangle\!\downarrow} \qquad \textbf{3}\ \frac{\langle q,\alpha\rangle\!\downarrow}{\langle p+q,\alpha\rangle\!\downarrow} \qquad \textbf{4}\ \langle a.p,\alpha\rangle \xrightarrow{a} \langle p,\alpha\rangle$$

$$\textbf{5}\ \frac{\langle p,\alpha\rangle \xrightarrow{a} \langle p',\alpha\rangle}{\langle p+q,\alpha\rangle \xrightarrow{a} \langle p',\alpha\rangle} \qquad \textbf{6}\ \frac{\langle q,\alpha\rangle \xrightarrow{b} \langle q',\alpha\rangle}{\langle p+q,\alpha\rangle \xrightarrow{b} \langle p',\alpha\rangle} \qquad \textbf{7}\ \langle \zeta_X.p,\alpha\rangle \overset{X}{\longmapsto} \langle p,\alpha\rangle$$

$$\textbf{8}\ \frac{\langle p,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \not\longmapsto}{\langle p+q,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle} \qquad \textbf{9}\ \frac{\langle p,\alpha\rangle \not\longmapsto,\ \langle q,\alpha\rangle \overset{T}{\longmapsto} \langle q',\alpha'\rangle}{\langle p+q,\alpha\rangle \overset{T}{\longmapsto} \langle q',\alpha'\rangle}$$

$$\textbf{10}\ \frac{\langle p,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\longmapsto} \langle q',\alpha''\rangle,\ X\in S}{\langle p+q,\alpha\rangle \overset{S}{\longmapsto} \langle p'+q,\alpha'\{\alpha'_X/\mathrm{R}(q)\}\rangle} \qquad \textbf{11}\ \frac{\langle p,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\longmapsto} \langle q',\alpha''\rangle,\ Y\in T}{\langle p+q,\alpha\rangle \overset{T}{\longmapsto} \langle p+q',\alpha''\{\alpha''_Y/\mathrm{R}(p)\}\rangle}$$

$$\textbf{12}\ \frac{\langle p,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\longmapsto} \langle q',\alpha''\rangle}{\langle p+q,\alpha\rangle \overset{S\cup T}{\longmapsto} \langle p'+q',\alpha\{\alpha'/\mathrm{R}(p),\alpha''/\mathrm{R}(q)\}\rangle}$$

$$\textbf{13}\ \langle \sigma_X.p,\alpha\rangle \overset{X}{\mapstochar\dashrightarrow} \langle p,\alpha\rangle \qquad \textbf{14}\ \frac{\langle p,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \not\mapstochar\dashrightarrow}{\langle p+q,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle p',\alpha'\rangle} \qquad \textbf{15}\ \frac{\langle p,\alpha\rangle \not\mapstochar\dashrightarrow,\ \langle q,\alpha\rangle \overset{T}{\mapstochar\dashrightarrow} \langle q',\alpha'\rangle}{\langle p+q,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle q',\alpha'\rangle}$$

$$\textbf{16}\ \frac{\langle p,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\mapstochar\dashrightarrow} \langle q',\alpha''\rangle,\ S\cap\mathrm{R}(q)=\emptyset,\ X\in S}{\langle p+q,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle p'+q,\alpha'\{\alpha'_X/\mathrm{R}(q)\}\rangle}$$

$$\textbf{17}\ \frac{\langle p,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\mapstochar\dashrightarrow} \langle q',\alpha''\rangle,\ S\cap\mathrm{R}(q)=\emptyset,\ X\in S}{\langle p+q,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle p'+q,\alpha'\{\alpha'_X/\mathrm{R}(q)\}\rangle}$$

$$\textbf{18}\ \frac{\langle p,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\mapstochar\dashrightarrow} \langle q',\alpha''\rangle,\ S\cap(\mathrm{R}(q)\setminus T)=\emptyset,\ (\mathrm{R}(p)\setminus S)\cap T=\emptyset}{\langle p+q,\alpha\rangle \overset{S\cup T}{\mapstochar\dashrightarrow} \langle p'+q',\alpha\{\alpha'/\mathrm{R}(p)\}\{\alpha''/\mathrm{R}(q)\}\rangle}$$

$$\textbf{19}\ \frac{\langle p,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\mapstochar\dashrightarrow} \langle q',\alpha''\rangle,\ X\in S}{\langle p+q,\alpha\rangle \overset{S}{\longmapsto} \langle p'+q,\alpha'\{\alpha'_X/\mathrm{R}(q)\}\rangle} \qquad \textbf{20}\ \frac{\langle p,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\mapstochar\dashrightarrow} \langle q',\alpha''\rangle,\ Y\in T}{\langle p+q,\alpha\rangle \overset{T}{\mapstochar\dashrightarrow} \langle p'+q,\alpha''\{\alpha''_Y/\mathrm{R}(p)\}\rangle}$$

$$\textbf{22}\ \frac{\langle p,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\longmapsto} \langle q',\alpha''\rangle,\ X\in S}{\langle p+q,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle p'+q,\alpha'\{\alpha'_X/\mathrm{R}(q)\}\rangle} \qquad \textbf{23}\ \frac{\langle p,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\longmapsto} \langle q',\alpha''\rangle,\ Y\in T}{\langle p+q,\alpha\rangle \overset{T}{\longmapsto} \langle p'+q,\alpha''\{\alpha''_Y/\mathrm{R}(p)\}\rangle}$$

$$\textbf{21}\ \frac{\langle p,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\mapstochar\dashrightarrow} \langle q',\alpha''\rangle}{\langle p+q,\alpha\rangle \overset{S\cup T}{\mapstochar\dashrightarrow} \langle p'+q',\alpha\{\alpha'/\mathrm{R}(p)\}\{\alpha''/\mathrm{R}(q)\}\rangle} \qquad \textbf{24}\ \frac{\langle p,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\longmapsto} \langle q',\alpha''\rangle}{\langle p+q,\alpha\rangle \overset{S\cup T}{\mapstochar\dashrightarrow} \langle p'+q',\alpha\{\alpha'/\mathrm{R}(p)\}\{\alpha''/\mathrm{R}(q)\}\rangle}$$

$$\textbf{25}\ \frac{\langle p,\alpha\rangle\!\downarrow}{\langle |p|,\alpha\rangle\!\downarrow} \qquad \textbf{26}\ \frac{\langle p,\alpha\rangle \xrightarrow{a} \langle p',\alpha'\rangle}{\langle |p|,\alpha\rangle \xrightarrow{a} \langle |p'|,\alpha'\rangle} \qquad \textbf{27}\ \frac{\langle p,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle p',\alpha'\rangle}{\langle |p|,\alpha\rangle \overset{S}{\longmapsto} \langle |p'|,\alpha'\rangle} \qquad \textbf{28}\ \frac{\langle p,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle}{\langle |p|,\alpha\rangle \overset{S}{\longmapsto} \langle |p'|,\alpha'\rangle}$$

$$\textbf{29}\ \frac{\langle p,\alpha\rangle\!\downarrow}{\langle \left[\begin{smallmatrix}W\\L\end{smallmatrix}\right] p,\alpha\rangle\!\downarrow} \qquad \textbf{30}\ \frac{\langle p,\alpha\rangle \xrightarrow{a} \langle p',\alpha'\rangle}{\langle \left[\begin{smallmatrix}W\\L\end{smallmatrix}\right] p,\alpha\rangle \xrightarrow{a} \langle p',\alpha'\rangle}$$

$$\textbf{31}\ \frac{\langle p,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle p',\alpha'\rangle,\ \mathrm{D}(p)\subseteq S\subseteq W,\ \mathrm{I}(P)=\emptyset,\ X\in W}{\langle \left[\begin{smallmatrix}W\\L\end{smallmatrix}\right] p,\alpha\rangle \overset{W}{\mapstochar\dashrightarrow} \langle p',\alpha'\{\alpha'_X/\big(\mathrm{R}(p')\cap\{Y\in L \mid \alpha(Y)\cap W=\emptyset\}\big)\}\rangle}$$

Table 1. Structural operational semantics for $\mathrm{BSP}^{\mathrm{dst}}$

of Rules 10-12, 16-18 and 19-24, which enable all possible outcomes of a race.

Moreover, there are no multiple equal transitions from the same state as each stochastic delay transition is uniquely labeled by the winning set.

Stochastic transition systems A *stochastic transition system* represents an instantiation of a stochastic transition scheme with respect to a given assignment $\varphi\colon \mathcal{V} \to \mathcal{F}$ of probability distributions. The race condition is used to derive the underlying probability spaces that define the probabilistic behavior of each stochastic delay transition. In order to compute the probability distributions of the stochastic delays we have to keep track of the exhibited durations. More precisely, we need both the original distribution function and its *age*, i.e., the amount of time that the stochastic delay participated in races that it lost [14]. The age of a stochastic delay is calculated from the durations of the stochastic delays on which it depends. Therefore, we extend the environments with a function $\beta\colon \mathcal{V} \to \mathrm{I\!R}^+ \cup \{\bot\}$. Also, we extend the set of possible durations to $\mathrm{I\!R}_\bot = \mathrm{I\!R}^+ \cup \{\bot\}$ with the special symbol $\bot$ to denote that a stochastic delay has not yet expired. By convention $F|\bot = F$ and $x+\bot = x$, for $x \in \mathrm{I\!R}_\bot$. Thus, an environment is a pair of two functions (α, β). Moreover, we assume, for each $X \in \mathcal{V}$ and $t > 0$, that the probability distribution function $F_X(t) = (\varphi(X)|\mathrm{t}(X))(t)$ is defined. Here, the function $\mathrm{t}\colon \mathcal{V} \to \mathrm{I\!R}_\bot$ provides the total age of X, i.e., $\mathrm{t}(X) = \sum_{Y \in \alpha(X)} \beta(Y) + \mathrm{t}(Y)$. If $\alpha(X) = \emptyset$ then $\mathrm{t}(X) = \bot$. We denote the set of all well-defined environments by $\mathcal{E}_\mathrm{t}$. Again, w.l.o.g., we assume that in the initial state there are no expired stochastic delays, so the initial environment is $(\alpha_\emptyset, \beta_\bot)$, where $\beta_\bot(X) = \bot$ for all $X \in \mathcal{V}$.

Definition 3. *A stochastic transition system is a tuple* $(\mathcal{S} \times \mathcal{E}_\mathrm{t}, \langle s, (\alpha_\emptyset, \beta_\bot)\rangle, \mathcal{A}, \mathcal{V}, \varphi, \to, \mapsto, \downarrow)$*, where*

- $\mathcal{S} \times \mathcal{E}_\mathrm{t}$ *is a set of states in well-defined environments;*
- $\varphi\colon \mathcal{V} \to \mathcal{F}$ *assigns the distributions to the random variables;*
- $\langle s, (\alpha_\emptyset, \beta_\bot)\rangle \in \mathcal{S} \times \mathcal{E}_\mathrm{t}$ *is the initial state;*
- $\to \subseteq (\mathcal{S} \times \mathcal{E}_\mathrm{t}) \times \mathcal{A} \times (\mathcal{S} \times \mathcal{E}_\mathrm{t})$ *is the labeled transition relation;*
- $\mapsto\colon \mathcal{S} \times \mathcal{E}_\mathrm{t} \to \mathrm{PS}(\mathrm{I\!R}^+ \times (\mathcal{S} \times \mathcal{E}_\mathrm{t}))$ *is the stochastic delay transition function;*
- $\downarrow \subseteq \mathcal{S} \times \mathcal{E}_\mathrm{t}$ *is the immediate termination predicate.*

Each stochastic transition scheme coupled with an assignment of probability distributions to the stochastic delays induces a stochastic transition system. The labeled transitions and the termination predicate are defined by the structural operational semantics in Table 1 as they do not depend on any environment. The probability measure of the stochastic delays is induced by the race condition. The formal definition is as follows:

Definition 4. *The stochastic transition scheme* $S = (\mathcal{S} \times \mathcal{E}_\mathrm{d}, \langle s, \alpha_\emptyset\rangle, \mathcal{A}, \mathcal{V}, \to, \mapsto, \downarrow, \mathrm{R})$ *coupled with an assignment function* $\varphi\colon \mathcal{V} \to \mathcal{F}$*, induces a stochastic transition system* $(\mathcal{S} \times \mathcal{E}_\mathrm{t}, \langle s, (\alpha_\emptyset, \beta_\bot)\rangle, \mathcal{A}, \to, \mapsto, \downarrow)$*, where* $\mapsto (\langle u, (\alpha, \beta)\rangle) = (\mathrm{I\!R}^+ \times (\mathcal{S} \times \mathcal{E}_\mathrm{t}), P)$ *is the probability space induced by the race condition. The probability measure* P *is given by* $P((t, \langle u', (\alpha', \beta')\rangle)) = \mathrm{RC}_t(S, \mathrm{R}(u))$*, for*

$t \in \mathrm{supp}(S)$, *where* $\langle u, \alpha \rangle \overset{S}{\longmapsto} \langle u', \alpha' \rangle$ *and* $\beta(X) \in \mathrm{supp}(\varphi(X)|\mathrm{t}(X))$ *is well-defined for* $X \in \mathcal{V}$. *The stochastic transition system induced by* S *and* φ *is denoted by* (S, φ).

Stochastic bisimulation In defining a suitable process equivalence for stochastic transition systems, we follow the standard approach. See, e.g., [16, 8, 5]. We require the bisimulation to be an equivalence, such that every two states from the same class (1) perform the same labeled transitions, (2) perform stochastic delay transitions to every other class with the same duration and the same accumulative probability, and (3) have the same termination options.

Definition 5. *Let* R *be an equivalence relation on* $\mathcal{S} \times \mathcal{E}_{\mathrm{t}}$. *The accumulative transition probability for stochastic delay from a state* $u \in \mathcal{S}$ *to an equivalence class* C *of* R *with duration* d *is given by* $\mathrm{P}_{\mathrm{acc}}(u, C, d) = \sum_{u' \in C} P(d, u')$, *where* $\mapsto (u) = (\mathbb{R}^{+} \times (\mathcal{S} \times \mathcal{E}_{\mathrm{t}}), P)$. *The relation* R *is a stochastic bisimulation if, for all* uRv, $a \in \mathcal{A}$, $d > 0$ *and* $C \in (\mathcal{S} \times \mathcal{E}_{\mathrm{t}})/R$:

1. *if* $u \xrightarrow{\ell} u'$ *then* $v' \in \mathcal{S}$ *for some* v' *such that* $v \xrightarrow{\ell} v'$ *and* $u'Rv'$;
2. $\mathrm{P}_{\mathrm{acc}}(u, d, C) = \mathrm{P}_{\mathrm{acc}}(v, d, C)$;
3. *if* $u\downarrow$ *then* $v\downarrow$.

If u, v *are bisimilar we write* $u \leftrightarrow v$. *Two transition systems* T *and* T' *are bisimilar if their initial states are, notation* $T \leftrightarrow T'$. *Two independent terms* $p, q \in \mathcal{I}$ *are bisimilar, notation* $p \leftrightarrow q$, *if for every* $\varphi \colon \mathcal{V} \to \mathcal{F}$, $(\mathrm{STS}(p), \varphi) \leftrightarrow (\mathrm{STS}(q), \varphi)$.

We note that the dependent stochastic delays play no role in the definition of the bisimulation relation, as the language prohibits their existence outside the scope of the name resolution operator. However, the equational theory given below, deals with constituent dependent terms. To extend the stochastic bisimulation to dependent processes, two additional transfer conditions must hold for $\alpha \in \mathcal{E}_{\mathrm{d}}$:

- if $\langle p, \alpha \rangle \overset{S}{\dashrightarrow} \langle p', \alpha' \rangle$ then $\langle q, \alpha \rangle \overset{S}{\dashrightarrow} \langle q', \alpha' \rangle$ and $\langle p', \alpha' \rangle R \langle q', \alpha' \rangle$ for $p', q' \in \mathcal{D}$;
- if $\langle p, \alpha \rangle \overset{S}{\dashrightarrow} \langle p', \alpha' \rangle$ then $\langle q, \alpha \rangle \overset{S}{\dashrightarrow} \langle q', \alpha' \rangle$ and $\langle p', \alpha' \rangle \leftrightarrow \langle q', \alpha' \rangle$ for $p', q' \in \mathcal{I}$.

For a congruence proof for $\leftrightarrow$, we refer to our previous work [14] for the independent stochastic delays (in a slightly different setting).

Theorem 6. *The bisimulation relation* $\leftrightarrow$ *is a congruence for* $\mathrm{BSP}^{\mathrm{dst}}$. □

For the dependent delays the congruence property should be clear as bisimilar dependent processes must perform exactly the same potential transitions as in the case of standard strong bisimulation.

α-conversion A major requirement for the operational rules in Table 1 to be well-defined is uniqueness of random variable names. However, the uniqueness of variable names is not a prerequisite in the syntax, so we have to tackle this problem in another way. We allow renaming of random variables with fresh random variables and we ensure that the original and the replacement have the same probability distribution to preserve equivalent stochastic behaviour. In the process of renaming, all dependent stochastic delays in the same race are renamed together because of name dependencies.

For a technical underpinning of the renaming of the variables, we define a relation $\simeq_\alpha \subseteq \mathcal{I} \times \mathcal{I}$. As an example, we want $p = \zeta_X.\epsilon + \zeta_X.\epsilon$ to be congruent to $p' = \zeta_X.\epsilon + \zeta_Y.\epsilon$, $p'' = \zeta_Y.\epsilon + \zeta_X.\epsilon$ and $p'''\zeta_Y.\epsilon + \zeta_Y.\epsilon$ provided that $F_X = F_Y$ because variable names play no role in the stochastic bisimulation, but they do introduce conflicts in the environments.

As a technical aid we need an auxiliary function $\mathrm{A}(p)\colon \mathcal{I} \to 2^{\mathcal{V}}$ that extracts all stochastic delays of a process term p:

$$\mathrm{A}(\delta) = \mathrm{A}(\epsilon) = \emptyset,\ \mathrm{A}(a.p) = \mathrm{A}(|p|) = \mathrm{A}(p),\ \mathrm{A}(\sigma_X.p) = \mathrm{A}(\zeta_X.p) = \{X\} \cup \mathrm{A}(p),$$
$$\mathrm{A}(\left[\begin{smallmatrix} W \\ L \end{smallmatrix}\right] p) = W \cup L \cup \mathrm{A}(p),\ \mathrm{A}(p+q) = \mathrm{A}(p) \cup \mathrm{A}(q)$$

Furthermore, by $[C \to D]$ we denote the set of all bijections from C to D. We define an auxiliary predicate $\mathrm{cf}_r(p, p')$, where $r \in [\mathrm{D}(p) \to \mathrm{D}(p')]$ is well-defined. The predicate cf_r ensures that dependent stochastic delays are correctly renamed and it is given by

$$\mathrm{cf}_r(\delta,\delta) = \mathrm{cf}_r(\epsilon,\epsilon) = \mathrm{cf}_r(a.p, a.p') = \mathrm{cf}_r(\zeta_X.p, \zeta_Y.p') = \mathrm{cf}_r(|p|, |p'|) = \text{true}$$
$$\mathrm{cf}_r(\sigma_X.p, \sigma_Y.p') \text{ if } r(X) = Y$$
$$\mathrm{cf}_r(\left[\begin{smallmatrix} W \\ L \end{smallmatrix}\right] p, \left[\begin{smallmatrix} W' \\ L' \end{smallmatrix}\right] p') \text{ if } r(W) = W',\ r(L) = L'$$
$$\mathrm{cf}_r(p+q, p'+q') \text{ if } \mathrm{cf}_r(p,p'),\ \mathrm{cf}_r(q,q')$$

For technical convenience, we use the notion of a *maximal distinct representation* [15] in which all stochastic delays have unique names (modulo permutations of $\mathcal{V}$). For example, $\zeta_X.(\zeta_X.\zeta_X.\epsilon + \zeta_X.\delta) + \zeta_X$ has $\zeta_X.(\zeta_Y.\zeta_Z.\epsilon + \zeta_U.\delta) + \zeta_V$ as a maximal distinct representation as long as the distribution function and the dependence sets of the variables are the same. The relation $\mathrm{mdr}_r \subseteq \mathcal{S} \times \mathcal{S}$, for $r\colon \mathcal{V} \to \mathcal{V}$ holds if the first term is a maximal distinct representation of the second. We use r to keep track of the last renaming of variables (required for $\left[\begin{smallmatrix} W \\ L \end{smallmatrix}\right] p$) and initially we put for all $X \in \mathcal{V}$ and some $Y \in \mathcal{V}$, $r(X) = Y$ such that $F_X = F_Y$.

$\mathrm{mdr}_r(\delta,\delta)$ $\quad$ $\mathrm{mdr}_r(\epsilon,\epsilon)$ $\quad$ $\mathrm{mdr}_r(a.p, a.p')$ if $\mathrm{mdr}_r(p', p)$ $\quad$ $\mathrm{mdr}_r(|p'|, |p|)$ if $\mathrm{mdr}_r(p', p)$
$\mathrm{mdr}_{r'}(\zeta_Y.p', \zeta_X.p)$ if $Y \notin \mathrm{A}(p')$, $F_Y = F_X$, $\mathrm{mdr}_r(p', p)$, $r' = r\{Y/X\}$
$\mathrm{mdr}_{r'}(\sigma_Y.p', \sigma_X.p)$ if $Y \notin \mathrm{A}(p')$, $F_X = F_Y$, $\mathrm{mdr}_r(p', p)$, $r' = r\{Y/X\}$
$\mathrm{mdr}_{r''}(p'+q', p+q)$ if $\mathrm{cf}_{r''}(p+q, p'+q')$, $r'' \in [\mathrm{R}(p+q)) \to \mathrm{R}(p'+q')]$,
$\quad (\mathrm{A}(p'+q') \setminus \mathrm{R}(p'+q')) \cap (\mathrm{A}(p+q) \setminus \mathrm{R}(p+q)) = \emptyset$,
$\quad \mathrm{mdr}_r(p', p)$, $\mathrm{mdr}_{r'}(q', q)$, $r'' = r'\{r/\mathrm{A}(p)\}$
$\mathrm{mdr}_r(\left[\begin{smallmatrix} r(W) \\ r(L) \end{smallmatrix}\right] p', \left[\begin{smallmatrix} W \\ L \end{smallmatrix}\right] p)$ if $\mathrm{mdr}_r(p', p)$.

With all the machinery in place, α-conversion becomes easy. Two process terms can be α-converted if they have the same maximal distinct representations.

Definition 7. *Two* $\mathrm{BSP}^{\mathrm{dst}}$*-terms* p *and* q *are* α*-convertible, notation* $p \simeq_\alpha q$*, if* $\{p' \in \mathcal{I} \cup \mathcal{D} \mid \mathrm{mdr}(p', p)\} = \{q' \in \mathcal{I} \cup \mathcal{D} \mid \mathrm{mdr}(q', q)\}$.

Intuitively, the definition states that the renaming of variables is allowed as long as the stochastic delays have the same stochastic behavior. As a consequence, α-conversion does not alter the stochastic behavior of the stochastic transition schemes and $\simeq_\alpha$ is a congruence. This can be proven rigorously by structural induction and case analysis for every rule of the operational semantics and, therefore, the proof is omitted here. Note that the α-conversion implicitly introduces side conditions that restrict the assignment of distributions to the random variables. Here, we get a little bit sloppy and do not rigorously formalize the existence of the side conditions, but, as it is standard, we assume that they are fulfilled when assigning distributions to random variables.

Now, we define a conflict-free stochastic transition scheme.

Definition 8. *The conflict-free stochastic transition scheme of* $p \in \mathcal{I}$ *is given by* $\mathrm{STS}(p')$*, where* $\mathrm{mdr}(p', p)$.

We overload the notation $\mathrm{STS}(p)$ to denote a conflict-free stochastic transition scheme of $p \in \mathcal{I}$.

Name resolution Another issue that we consider due to the name dependencies is replacement of independent processes of the form $|p|$ by p. This substitution is important for axiomatization purposes. The replacement is safe if (1) there is a name resolution scope on a higher level, (2) the dependent delays of p do not have common variable names with the dependent delays of the context in the alternative composition, and (3) the dependent delays of p do not occur in the winning or the losing set of the race condition guard. For example, we cannot replace $|\sigma_X.p|$ by $\sigma_X.p$ in the context of (1) $a.\,|\sigma_X.p|$, (2) $||\sigma_X.p| + \sigma_X.q + r|$ or (3) $\left|\left[{}^{Y}_{X,Z}\right]\sigma_Y.(|\sigma_X.p| + \sigma_Z.q)\right|$ because in (1) the dependent delay would exist outside the scope of any name resolution operator, in (2) there is a clash in the random variable names which leads to a different stochastic behavior and in (3) there is a clash with the losing set. In all conflicting cases α-conversion is used to obtain a conflict-free renaming of the term.

For technical underpinning of the name resolution, we require the notion of a context. We say that p is in the context $C(\Box)$ of the term q if $C(p) \equiv q$, where $\equiv$ denotes syntactic equality. The context C is given by the BNF:

$$C ::= \Box \mid \delta \mid \epsilon \mid a.C \mid \sigma_X.C \mid \zeta_X.C \mid \left[{}^{W}_{L}\right] C \mid |C| \mid C + C,$$

where $a \in \mathcal{A}$, $X \in \mathcal{V}$, $W, L \subseteq \mathcal{V}$, such that $W \neq \emptyset$ and $W \cap L = \emptyset$ and C contains only one $\Box$. For example, the context of p in $a.(p + q)$ is $a.(\Box + q)$.

Using the contexts, we can formally express the three conditions for safe replacement from above.

Theorem 9. *The context* $C(\Box)$ *is safe for substitution of* $|p|$ *by* p *if the following conditions are fulfilled:*

1. $C(\square) \equiv C'(|C''(\square)|)$;
2. If $C(\square) \equiv C'(q + \square)$ or $C(\square) \equiv C'(\square + q)$ then $\mathrm{D}(p) \cap \mathrm{D}(q) = \emptyset$;
3. If $C(\square) \equiv C'(\left[{W \atop L}\right](q + \sigma_X.C''(\square) + r))$ and $X \in W$ then $\mathrm{D}(p) \cap L = \emptyset$. $\square$

It is not difficult to perceive that the conflict-free stochastic transition schemes of $C(|p|)$ and $C(p)$ are isomorphic if the conditions for safe replacement are fulfilled. As for α-conversion, the proof requires simple, but meticulous case analysis for every operational rule in Table 1 and, therefore, it is omitted.

Typical specifications Often, system specifications contain only independent stochastic delays, action transitions and termination options as given by T:

$$T ::= \delta \mid \epsilon \mid a.T \mid \zeta_X.T \mid T + T, \qquad \text{for } a \in \mathcal{A} \text{ and } X \in \mathcal{V}.$$

However, in the setting $\mathrm{BSP}^{\mathrm{dst}}$, the language also allows specifications in terms of dependent stochastic delays. It turns out that this 'a-typical' specifications have greater expressiveness than the 'typical' specifications as they have the capability of defining outcomes of incomplete races (which induce incomplete probability spaces), like, for example, $\left|\left[{X \atop Y}\right](\sigma_X.\delta + \sigma_Y.\delta)\right|$. As we focus on the interplay between stochastic and real time, in the current setting we develop an equational theory only for the typical specifications, as is done elsewhere [1, 9].

Equational Theory First, we define the term algebra of $\mathrm{BSP}^{\mathrm{dst}}$.

Definition 10. *The term algebra of* $\mathrm{BSP}^{\mathrm{dst}}$ *is*

$$\mathbb{P}(\mathrm{BSP}^{\mathrm{dst}}) = (\mathcal{I}/\simeq_\alpha, \delta, \epsilon, a.\_ \text{ for } a \in \mathcal{A}, \zeta_X.\_ \text{ for } X \in \mathcal{V}, \_ + \_).$$

Next, we formally define the term model of $\mathrm{BSP}^{\mathrm{dst}}$.

Definition 11 (Term model of $\mathrm{BSP}^{\mathrm{dst}}$). *The term model of* $\mathrm{BSP}^{\mathrm{dst}}$ *is the quotient algebra* $\mathbb{P}(\mathrm{BSP}^{\mathrm{dst}})_{/\leftrightarrows}$.

The equational theory for typical specifications up to α-conversion is given in Table 2. We discuss some of the axioms. The main property of dependent delays is given by $A6$. As usual, $\zeta_X.p + \zeta_X.q \not\leftrightarrows \zeta_X.(p + q)$ for independent stochastic delays, unless F_X is Dirac. Independent delays can be replaced by dependent delays in an immediate scope of the name resolution operator, axiom $I5$. Merger of two name resolution operators is conditioned by disjoined dependent racing delays, axiom $I6$. Axiom $R1$ shows when the race condition guard is well-defined, whereas $R2$ induces an initial race on one stochastic delay. Iterative application of the race condition guard is given by $R5$. Axioms $R6$ allows renaming of winners as they observe the same duration in the race they win together. Axioms $R7$, $R8$ and $R9$ define resolution of three types of races: ($R7$) common winners induce a race in which the winners are joined, provided that there are no clashes between the losers; ($R8$) if the losers of the first race are the winners of the second then the winners of the first race are the overall winners, again provided that there are no clashes; ($R9$) if there are no common winners or losers then the race can have every possible outcome. Next, we give an example of a derivation:

Example 12. We have the following derivation that transforms a typical specification $\zeta_X.\zeta_Y.\epsilon + \zeta_X.\delta + a.\epsilon$ to a term with dependent stochastic delays and completely resolved race conditions.

$$\begin{aligned}
&\zeta_X.\zeta_Y.\epsilon + \zeta_X.\delta + a.\epsilon \overset{I5}{=} |\sigma_X.\,|\sigma_Y.\epsilon|| + |\sigma_X.\delta| + a.\epsilon \overset{NR}{=} |\sigma_X.\sigma_Y.\epsilon| + |\sigma_X.\delta| + a.\epsilon \overset{I6}{=} \\
&|\sigma_X.\sigma_Y.\epsilon + \sigma_Z.\delta + a.\epsilon| \overset{R2}{=} \left|\left[\begin{smallmatrix}X\\\emptyset\end{smallmatrix}\right]\sigma_X.\sigma_Y.\epsilon + \left[\begin{smallmatrix}Z\\\emptyset\end{smallmatrix}\right]\sigma_Z.\delta\right| + a.\epsilon \overset{R9,R2,A5,I3,R6,F_X=F_Z}{=} \\
&\left|\left[\begin{smallmatrix}X\\Z\end{smallmatrix}\right]\sigma_X.\big(\left[\begin{smallmatrix}Y\\\emptyset\end{smallmatrix}\right]\sigma_Y.\epsilon + \left[\begin{smallmatrix}Z\\\emptyset\end{smallmatrix}\right]\sigma_Z.\delta\big) + \left[\begin{smallmatrix}X,Z\\\emptyset\end{smallmatrix}\right]\sigma_X.\sigma_Y.\epsilon + \left[\begin{smallmatrix}Z\\X\end{smallmatrix}\right]\sigma_Z.\sigma_X.\sigma_Y.\epsilon + a.\epsilon\right| \overset{R9,A5,R2}{=} \\
&\Big|\left[\begin{smallmatrix}X\\Z\end{smallmatrix}\right]\sigma_X.\big(\left[\begin{smallmatrix}Y\\Z\end{smallmatrix}\right]\sigma_Y.\big(\epsilon + \left[\begin{smallmatrix}Z\\\emptyset\end{smallmatrix}\right]\sigma_Z.\delta\big) + \left[\begin{smallmatrix}Y,Z\\\emptyset\end{smallmatrix}\right]\sigma_Y.\epsilon + \left[\begin{smallmatrix}Z\\Y\end{smallmatrix}\right]\sigma_Z.\left[\begin{smallmatrix}Y\\\emptyset\end{smallmatrix}\right]\sigma_Y.\epsilon\big) + \\
&\qquad\qquad \left[\begin{smallmatrix}X,Z\\\emptyset\end{smallmatrix}\right]\sigma_X.\left[\begin{smallmatrix}Y\\\emptyset\end{smallmatrix}\right]\sigma_Y.\epsilon + \left[\begin{smallmatrix}Z\\X\end{smallmatrix}\right]\sigma_Z.\left[\begin{smallmatrix}X\\\emptyset\end{smallmatrix}\right]\sigma_X.\left[\begin{smallmatrix}Y\\\emptyset\end{smallmatrix}\right]\sigma_Y.\epsilon + a.\epsilon\Big|,
\end{aligned}$$

where *NR* indicates name resolution.

The soundness and completeness property are stated in the following theorem.

Theorem 13. *The equational theory in Table 2 is sound and ground-complete for the typical specifications of* $\mathrm{BSP}^{\mathrm{dst}}$.

Proof. Soundness We give proof of soundness of some characteristic axioms. The rest of the axioms can be proven in the same style. It suffices to give a bisimulation relation R for every $\ell, r \in \mathcal{I}$, such that ℓ and r present the left and the right side of the axiom. In case the terms cannot perform some type of transition or cannot terminate, we omit those cases in the analysis. As a reminder, dependent stochastic delays have to perform the same stochastic delay transitions.

I5 Let $R = \{(\zeta_X.x, |\sigma_X.x|) \mid x \in \mathcal{D}\} \cup \{(|\sigma_X.x|, \zeta_X.x) \mid x \in \mathcal{D}\} \cup \{(x, x) \mid x \in \mathcal{D}\}$ $\cup\ \{(x, |x|) \mid x \in \mathcal{D},\} \cup \{(|x|, x) \mid x \in \mathcal{D}\}$.

The only possible stochastic delay transition for both terms is guided by X, i.e., $\langle \zeta_X.p, \alpha\rangle \overset{X}{\longmapsto} \langle p, \alpha\rangle$ and $\langle |\sigma_X.\,|p||, \alpha\rangle \overset{X}{\longmapsto} \langle p, \alpha\rangle$. Note that if $\zeta_X.p$ is not in a context in the scope of the name resolution operator then $p \in \mathcal{I}$. Otherwise, Theorem 9 for name resolution applies for the replacement of $|p|$ by p.

I6 Let $R = \{(|x + y|, |x| + |y|) \mid x, y \in \mathcal{D}\} \cup \{(|x| + |y|, |x + y|) \mid x, y \in \mathcal{D}\} \cup \{(x, x) \mid x \in \mathcal{D}\}$.

1. If $|\langle p + q, \alpha\rangle| \overset{a}{\longrightarrow} \langle r, \alpha\rangle$ then either $\langle |p|, \alpha\rangle \overset{a}{\longrightarrow} \langle r, \alpha\rangle$ or $\langle |q|, \alpha\rangle \overset{a}{\longrightarrow} \langle r, \alpha\rangle$ for $p, q, r \in \mathcal{D}$ and $\alpha \in \mathcal{E}_{\mathrm{d}}$. In any case $\langle |p| + |q|, \alpha\rangle \overset{a}{\longrightarrow} r$. Similarly for the other direction.
2. The independent racing delays for both terms are the same, so the induce the same stochastic delay transitions. As $\mathrm{D}(p) \cap \mathrm{D}(q) = \emptyset$, both terms induce the same potential stochastic delay transitions. This amounts to the same races, thus the same stochastic behavior.
3. If $\langle |p + q|, \alpha\rangle\!\downarrow$ then either $\langle p, \alpha\rangle\!\downarrow$ or $\langle q, \alpha\rangle\!\downarrow$. In any case $\langle |p| + |q|, \alpha\rangle\!\downarrow$. Similarly for the other direction.

R8 Let $R = \{(\left[\begin{smallmatrix}W\\L\end{smallmatrix}\right]\sigma_X.x, \left[\begin{smallmatrix}W\\L\end{smallmatrix}\right]\sigma_Y.x) \mid X, Y \in W,\ x \in \mathcal{D}\} \cup \{(\left[\begin{smallmatrix}W\\L\end{smallmatrix}\right]\sigma_Y.x, \left[\begin{smallmatrix}W\\L\end{smallmatrix}\right]\sigma_X.x) \mid X, Y \in W,\ x \in \mathcal{D}\} \cup \{(x, x) \mid x \in \mathcal{D}\}$.

1. The only transition that both terms can perform is the potential stochastic delay transition $\langle \left[{W \atop L}\right] \sigma_X.p, \alpha\rangle \overset{W}{\dashrightarrow} \langle p, \alpha\rangle$ and $\langle \left[{W \atop L}\right] \sigma_Y.p, \alpha\rangle \overset{W}{\dashrightarrow} \langle p, \alpha\rangle$.

R9 Let $R = \{(\left[{W \atop L}\right] \sigma_X.x + \left[{U \atop T}\right] \sigma_Y.y, \left[{W \atop L\cup U\cup T}\right] \sigma_X.(x + \left[{U \atop T}\right] \sigma_Y.y)) \mid L \cap U \neq \emptyset,\ L \cap T = W \cap (U \cup T) = \mathrm{R}(p) \cap U = \emptyset,\ X \in W,\ Y \in U,\ x, y \in \mathcal{D}\} \cup \{(\left[{W \atop L\cup U\cup T}\right] \sigma_X.(x + \left[{U \atop T}\right] \sigma_Y.y), \left[{W \atop L}\right] \sigma_X.x + \left[{U \atop T}\right] \sigma_Y.y) \mid L \cap U \neq \emptyset,\ L \cap T = W \cap (U \cup T) = \mathrm{R}(p) \cap U = \emptyset,\ X \in W,\ Y \in U,\ x, y \in \mathcal{D}\} \cup \{(x, x) \mid x \in \mathcal{D}\}$.

Here we have $\langle \left[{W \atop L}\right] \sigma_X.p, \alpha\rangle \overset{W}{\dashrightarrow} \langle p, \alpha\{\alpha_X/(\mathrm{R}(p) \cap L)\}\rangle$ and $\langle \left[{U \atop T}\right] \sigma_Y.q, \alpha\rangle \overset{U}{\dashrightarrow} \langle q, \alpha\{\alpha_Y/(\mathrm{R}(q) \cap T)\}\rangle$ as possible transitions of the summands. Because $L \cap U \neq \emptyset$, the only possible transition of their sum is: $\langle \left[{W \atop L}\right] \sigma_X.p + \left[{U \atop T}\right] \sigma_Y.q, \alpha\rangle \overset{W}{\dashrightarrow} \langle p + \left[{U \atop T}\right] \sigma_Y.q, \alpha\{\alpha_X/(\mathrm{R}(p) \cap L)\}\{\alpha_X/(U \cup T)\}\rangle$. We note that the rest of the conditions are required for well-formedness of the operational rule. Next, the term on the right side performs the transition $\left[{W \atop L\cup U\cup T}\right] \sigma_X.(x + \left[{U \atop T}\right] \sigma_Y.y) \overset{W}{\dashrightarrow} \langle p + \left[{U \atop T}\right] \sigma_Y.q, \alpha\{\alpha_X/((\mathrm{R}(p) \cup U \cup T) \cap (L \cup U \cup T))\}\rangle$. Note that $\alpha\{\alpha_X/(\mathrm{R}(p) \cap L)\}\{\alpha_X/(U \cup T)\} = \alpha\{\alpha_X/((\mathrm{R}(p) \cup U \cup T) \cap (L \cup U \cup T))\}$.

Ground-completeness We note that we get a little bit sloppy when defining the typical specifications, i.e., we write $\mathbb{P}(\mathrm{BSP}^{\mathrm{dst}})$, which stands for the every specification possible in $\mathrm{BSP}^{\mathrm{dst}}$, but we mean only the typical specifications. In any case, it is not difficult to observe that the typical specifications are bisimilar only if they have isomorphic stochastic delays prefixes (up to α-conversion). This is because it is always possible to find different distributions for different independent stochastic delays. Moreover, the axioms only give transformations from independent to dependent terms and vice versa. Thus, each typical specification can be rewritten in the following form:

$$\sum_{i=1}^{n} a_i.p_i + \sum_{j=1}^{N} \zeta_{X_j}.q_j[\,+\,\epsilon],$$

where $a_i.p_i \not\leftrightarrow a_k.p_k$ for $1 \leq i \neq k \leq n$ and the square brackets around ϵ indicate that it is an optional summand.

Now we can separate action prefixed and stochastic time prefixed summands. The sum of all action prefixed summands of a process p is denoted by p_a and the sum of all stochastic time prefixed summands is denoted by p_s. Thus, we write $p = p_a + p_s[+\epsilon]$.

If $p \leftrightarrow q$ then $p_a \leftrightarrow q_a$ and $p_s \leftrightarrow q_s$. The statement holds because the processes p_a and q_a cannot do any stochastic delays and cannot terminate and vice versa, p_s and q_s cannot do action transitions and they cannot terminate successfully.

We can easily reuse the proof from [11], by example, to show that if $\mathbb{P}(\mathrm{BSP}^{\mathrm{dst}})_{/\leftrightarrow} \models p_a = q_a$ then $\mathrm{BSP}^{\mathrm{dst}} \vdash p_a = q_a$. It remains to be proven that if $\mathbb{P}(\mathrm{BSP}^{\mathrm{dst}})_{/\leftrightarrow} \models p_s = q_s$ then $\mathrm{BSP}^{\mathrm{dst}} \vdash p_s = q_s$. As we mentioned before, in this case, the stochastic delays of p_s and q_s must be syntactically equal. This completes the proof. □

$$\textbf{A1}\ p+q=q+p \quad \textbf{A2}\ (p+q)+r=p+(q+r) \quad \textbf{A3}\ \epsilon+\epsilon=\epsilon \quad \textbf{A4}\ a.p+a.p=a.p$$

$$\textbf{A5}\ p+\delta=p \quad \textbf{A6}\ \sigma_X.p+\sigma_X.q=\sigma_X.(p+q) \quad \textbf{I1}\ \delta=|\delta| \quad \textbf{I2}\ \epsilon=|\epsilon| \quad \textbf{I3}\ a.p=|a.p|$$

$$\textbf{I4}\ \zeta_X.p=|\zeta_X.p| \quad \textbf{I5}\ \zeta_X.p=|\sigma_X.p| \quad \textbf{I6}\ |p+q|=|p|+|q| \text{ if } \mathrm{D}(p)\cap\mathrm{D}(q)=\emptyset$$

$$\textbf{R1}\ \left[\begin{smallmatrix}W\\L\end{smallmatrix}\right]p=\delta \text{ if } \mathrm{D}(p)\not\subseteq W,\ \mathrm{I}(p)\neq\emptyset \quad \textbf{R2}\ \sigma_X.p=\left[\begin{smallmatrix}X\\\emptyset\end{smallmatrix}\right]\sigma_X.p$$

$$\textbf{R3}\ \left[\begin{smallmatrix}W\\L\end{smallmatrix}\right](p+\epsilon)=\left[\begin{smallmatrix}W\\L\end{smallmatrix}\right]p+\epsilon \text{ if } \mathrm{D}(p)\subseteq W,\ \mathrm{I}(p)=\emptyset$$

$$\textbf{R4}\ \left[\begin{smallmatrix}W\\L\end{smallmatrix}\right](p+b.q)=\left[\begin{smallmatrix}W\\L\end{smallmatrix}\right]p+b.q \text{ if } \mathrm{D}(p)\subseteq W,\ \mathrm{I}(p)=\emptyset$$

$$\textbf{R5}\ \left[\begin{smallmatrix}W\\L\end{smallmatrix}\right]\left[\begin{smallmatrix}U\\T\end{smallmatrix}\right]p=\left[\begin{smallmatrix}W\\L\cup T\end{smallmatrix}\right]p \text{ if } \mathrm{D}(p)\subseteq U\subseteq W,\ \mathrm{I}(p)=\emptyset$$

$$\textbf{R6}\ \left[\begin{smallmatrix}W\\L\end{smallmatrix}\right]\sigma_X.p=\left[\begin{smallmatrix}W\\L\end{smallmatrix}\right]\sigma_Y.p \text{ if } X,Y\in W$$

$$\textbf{R7}\ \left[\begin{smallmatrix}W\\L\end{smallmatrix}\right]p+\left[\begin{smallmatrix}U\\T\end{smallmatrix}\right]q=\left[\begin{smallmatrix}W\cup U\\L\cup T\end{smallmatrix}\right](p+q), \text{ if } W\cap U\neq\emptyset,\ W\cap T=L\cap U=\emptyset,$$
$$\mathrm{D}(p)\subseteq W,\ \mathrm{D}(q)\subseteq U,\ \mathrm{I}(p)=\mathrm{I}(q)=\emptyset$$

$$\textbf{R8}\ \left[\begin{smallmatrix}W\\L\end{smallmatrix}\right]\sigma_X.p+\left[\begin{smallmatrix}U\\T\end{smallmatrix}\right]\sigma_Y.q=\left[\begin{smallmatrix}W\\L\cup U\cup T\end{smallmatrix}\right]\sigma_X.(p+\left[\begin{smallmatrix}U\\T\end{smallmatrix}\right]\sigma_Y.q)$$
$$\text{if } L\cap U\neq\emptyset,\ L\cap T=W\cap(U\cup T)=\mathrm{R}(p)\cap U=\emptyset,\ X\in W,\ Y\in U$$

$$\textbf{R9}\ \left[\begin{smallmatrix}W\\L\end{smallmatrix}\right]\sigma_X.p+\left[\begin{smallmatrix}U\\T\end{smallmatrix}\right]\sigma_Y.q=$$
$$\left[\begin{smallmatrix}W\\U\cup T\cup L\end{smallmatrix}\right]\sigma_X.(p+\left[\begin{smallmatrix}U\\T\end{smallmatrix}\right]\sigma_Y.q)+\left[\begin{smallmatrix}W\cup U\\L\cup T\end{smallmatrix}\right](\sigma_X.p+\sigma_Y.q)+\left[\begin{smallmatrix}U\\W\cup L\cup T\end{smallmatrix}\right]\sigma_Y.(q+\left[\begin{smallmatrix}W\\L\end{smallmatrix}\right]\sigma_X.p)$$
$$\text{if } W\cap U=L\cap U=T\cap W=\mathrm{R}(p)\cap U=\mathrm{R}(q)\cap W=\emptyset,\ X\in W,\ Y\in U$$

Table 2. Process theory $\mathrm{BSP}^{\mathrm{dst}}$

Normal Forms Along the lines of Example 12 above, every process term can be rewritten in a normal form with completely resolved races as follows.

Theorem 14. *Every process $|p|\in\mathcal{I}$ can be rewritten in the following form:*

$$p=\sum_{i=1}^{m}a_i.p_i+\left|\sum_{j=1}^{n}\left[\begin{smallmatrix}W_j\\L_j\end{smallmatrix}\right]\sigma_{X_j}.q_j\right|[+\epsilon],$$

where $a_i.p_i\not\equiv a_k.p_k$, for $1\leq i,k\leq m$, and $\sum$ is a shorthand for the alternative composition and it is equal to δ if $m=0$, and $X_j\in W_j$, where $\left[\begin{smallmatrix}W_j\\L_j\end{smallmatrix}\right]$ cannot be combined with any $\left[\begin{smallmatrix}W_\ell\\L_\ell\end{smallmatrix}\right]$ using the axioms $R7$, $R8$ and $R9$, for $1\leq j,\ell\leq n$, and $[+\epsilon]$ is an optional termination option.

Proof. The proof is based on a term rewriting system that transforms every term prefixed by an independent stochastic delay $\zeta_X.p$ into a term in the immediate scope of the name resolution operator and a dependent stochastic delay $|\sigma_X.p|$. Afterwards, the name resolution Theorem 9 is used to eliminate all, but the topmost name resolution operator. Axioms $R1$-$R9$ provide the rewriting rules that resolve the name condition. The proof that the rewriting system is strongly terminating is based on the fact that the number of compatible race condition guards decreases with each application of the rules induced by axioms $R7-R9$. The confluence property is obtained as all possible combinations of racing guards are considered and the order of application of the rules is not important. ☐

The initial case study given in [15] pointed out that this normal form carries more information than the underlying generalized semi-Markov process. Moreover, it corresponds to the notion of regional trees that is used to model check stochastic automata [17]. Regional trees are obtained from stochastic automata (e.g., [8]) by explicitly ordering clock samples by their duration to state an outcome of a race, which is symbolically achieved by the race condition guard.

4 Basic Communicating Processes

In this section, we add an ACP-style parallel composition operator to $\mathrm{BSP}^{\mathrm{dst}}$ and obtain the algebra $\mathrm{BCP}^{\mathrm{dst}}$ of basic communication processes with discrete stochastic time. Standardly, the parallel composition allows both for interleaving and communication of immediate actions. In the present setting it should cater for interleaving and synchronization of stochastic delays as well. Immediate actions always take precedence over passage of time in the parallel composition, but do not disable any stochastic delays. As in real-time process algebras, delays are merged in case the processes perform stochastic delays with different duration and combined in case the duration is the same. The race that is induced by the parallel composition has the same probabilistic behavior as the alternative composition discussed before.

We extend the signature with the operator $\|$ and the auxiliary operators $\lfloor\!\lfloor$ and $|$. We reuse $\mathcal{I}$ and $\mathcal{D}$ to represent dependent and independent concurrent processes.

Definition 15. *The signature of* $\mathrm{BCP}^{\mathrm{dst}}$ *contains two constants* δ *and* ϵ*, four unary operator schemes (1)* $a.\_$ *for* $a \in \mathcal{A}$*, (2)* $\sigma_X.\_$ *for* $X \in \mathcal{V}$*, (3)* $\zeta_X.\_$ *for* $X \in \mathcal{V}$ *and (4)* $\left[\begin{smallmatrix} W \\ L \end{smallmatrix}\right]\_$ *for* $W \in (2^{\mathcal{V}} \setminus \{\emptyset\})$ *and* $L \subseteq \mathcal{V}$ *such that* $W \cap L = \emptyset$ *and four binary operators (1)* $\_+\_$*, (2)* $\_\lfloor\!\lfloor\_$*, (3)* $\_|\_$*, (4)* $\_\parallel\_$*. The syntax of* $\mathrm{BCP}^{\mathrm{dst}}$ *is given as follows:*

$$D ::= P \mid \sigma_X.D \mid \left[\begin{smallmatrix} W \\ L \end{smallmatrix}\right] D \mid D + D \mid D \parallel D \mid D \lfloor\!\lfloor D \mid (D \mid D)$$
$$P ::= \delta \mid \epsilon \mid |D| \mid \zeta_X.P \mid P + P \mid P \parallel P \mid P \lfloor\!\lfloor P \mid (P \mid P)$$

where $a \in \mathcal{A}$*,* $X \in \mathcal{V}$*,* $W \in (2^{\mathcal{V}} \setminus \emptyset)$ *and* $L \subseteq \mathcal{V}$ *such that* $W \cap L = \emptyset$*, and* $H \subseteq \mathcal{A}$*. The precedence of the operators is given by the following ordering:* $a.\_$*,* $\sigma_X.\_$*,* $\zeta_X.\_$*,* $\left[\begin{smallmatrix} W \\ L \end{smallmatrix}\right]\_$*,* $\_\parallel\_$*,* $\_\lfloor\!\lfloor\_$*,* $\_\mid\_$*,* $\_+\_$*,* $|\_|$*. Again, we write* $\mathcal{I}$ *for independent* $\mathrm{BCP}^{\mathrm{dst}}$*-terms and* $\mathcal{D}$ *for dependent process terms and use* p*,* q *and* r *to range over all terms.*

The parallel composition $p \parallel q$ imposes a race condition in the same way as the alternative composition, whereas the actions are synchronized according to the synchronization function γ. The race conditions extend to the auxiliary operators $p \lfloor\!\lfloor q$ and $p \mid q$ as for the other compositions. The semantics of independent terms is given again via stochastic transition schemes. The definitions of $\mathrm{D}(\_)$ and $\mathrm{I}(\_)$ are extended straightforwardly to apply to the new operators by putting $\mathrm{D}(p \diamond q) = \mathrm{D}(p) \cup \mathrm{D}(q)$ and $\mathrm{I}(p \diamond q) = \mathrm{I}(p) \cup \mathrm{I}(q)$ for $\diamond = \parallel, \lfloor\!\lfloor, |$.

We give the operational semantics of the new operators in Table 4. Again we assume the uniqueness of variables names and, for the sake of compactness of presentation we put $\diamond = \|, \lfloor\!\lfloor, |$ for the common rules.

$$\textbf{32}\ \frac{\langle p,\alpha\rangle\downarrow, \langle q,\alpha\rangle\downarrow}{\langle p \parallel q,\alpha\rangle\downarrow} \qquad \textbf{33}\ \frac{\langle p,\alpha\rangle \xrightarrow{a} \langle p',\alpha\rangle}{\langle p \parallel q,\alpha\rangle \xrightarrow{a} \langle p' \parallel q,\alpha\rangle} \qquad \textbf{34}\ \frac{\langle q,\alpha\rangle \xrightarrow{a} \langle q',\alpha\rangle}{\langle p \parallel q,\alpha\rangle \xrightarrow{a} \langle p \parallel q',\alpha\rangle}$$

$$\textbf{35}\ \frac{\langle p,\alpha\rangle \xrightarrow{a} \langle p',\alpha\rangle,\ \langle q,\alpha\rangle \xrightarrow{b} \langle q',\alpha\rangle,\ \gamma(a,b)=c}{\langle p \parallel q,\alpha\rangle \xrightarrow{c} \langle p' \parallel q',\alpha\rangle}$$

$$\textbf{36}\ \frac{\langle p,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \not\longmapsto,\ \langle q,\alpha\rangle\downarrow}{\langle p \parallel q,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle} \qquad \textbf{37}\ \frac{\langle p,\alpha\rangle \not\longmapsto,\ \langle p,\alpha\rangle\downarrow,\ \langle q,\alpha\rangle \overset{T}{\longmapsto} \langle q',\alpha'\rangle}{\langle p \parallel q,\alpha\rangle \overset{T}{\longmapsto} \langle q',\alpha'\rangle}$$

$$\textbf{38, 49, 60}\ \frac{\langle p,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\longmapsto} \langle q',\alpha''\rangle,\ X \in S}{\langle p \diamond q,\alpha\rangle \overset{S}{\longmapsto} \langle p' \diamond q,\alpha'\{\alpha'_X/\mathrm{R}(q)\}\rangle}$$

$$\textbf{39, 50, 61}\ \frac{\langle p,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\longmapsto} \langle q',\alpha''\rangle,\ Y \in T}{\langle p \diamond q,\alpha\rangle \overset{T}{\longmapsto} \langle p \diamond q',\alpha''\{\alpha''_Y/\mathrm{R}(p)\}\rangle}$$

$$\textbf{40, 51, 62}\ \frac{\langle p,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\longmapsto} \langle q',\alpha''\rangle}{\langle p \diamond q,\alpha\rangle \overset{S\cup T}{\longmapsto} \langle p' \diamond q',\alpha\{\alpha'/\mathrm{R}(p)\}\{\alpha''/\mathrm{R}(q)\}\rangle}$$

$$\textbf{41, 52, 63}\ \frac{\langle p,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\mapstochar\dashrightarrow} \langle q',\alpha''\rangle,\ S\cap \mathrm{R}(q)=\emptyset,\ X \in S}{\langle p \diamond q,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle p' \diamond q,\alpha'\{\alpha'_X/\mathrm{R}(q)\}\rangle}$$

$$\textbf{42, 53, 64}\ \frac{\langle p,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\mapstochar\dashrightarrow} \langle q',\alpha''\rangle,\ \mathrm{R}(p)\cap T=\emptyset,\ Y \in T}{\langle p \diamond q,\alpha\rangle \overset{T}{\mapstochar\dashrightarrow} \langle p' \diamond q,\alpha''\{\alpha''_Y/\mathrm{R}(p)\}\rangle}$$

$$\textbf{43, 54, 65}\ \frac{\langle p,\alpha\rangle \overset{S}{\mapstochar\dashrightarrow} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\mapstochar\dashrightarrow} \langle q',\alpha''\rangle,\ S\cap (\mathrm{R}(q)\setminus T)=\emptyset,\ (\mathrm{R}(p)\setminus S)\cap T=\emptyset}{\langle p \diamond q,\alpha\rangle \overset{S\cup T}{\mapstochar\dashrightarrow} \langle p' \diamond q',\alpha\{\alpha'/\mathrm{R}(p)\}\{\alpha''/\mathrm{R}(q)\}\rangle}$$

$$\textbf{44, 55, 66}\ \frac{\langle p,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\mapstochar\dashrightarrow} \langle q',\alpha''\rangle,\ X \in S}{\langle p \diamond q,\alpha\rangle \overset{S}{\longmapsto} \langle p' \diamond q,\alpha'\{\alpha'_X/\mathrm{R}(q)\}\rangle}$$

$$\textbf{45, 56, 67}\ \frac{\langle p,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\mapstochar\dashrightarrow} \langle q',\alpha''\rangle,\ Y \in T}{\langle p \diamond q,\alpha\rangle \overset{T}{\mapstochar\dashrightarrow} \langle p' \diamond q,\alpha''\{\alpha''_Y/\mathrm{R}(p)\}\rangle}$$

$$\textbf{46, 57, 68}\ \frac{\langle p,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \overset{T}{\mapstochar\dashrightarrow} \langle q',\alpha''\rangle}{\langle p \diamond q,\alpha\rangle \overset{S\cup T}{\mapstochar\dashrightarrow} \langle p' \diamond q',\alpha\{\alpha'/\mathrm{R}(p)\}\{\alpha''/\mathrm{R}(q)\}\rangle}$$

$$\textbf{47}\ \frac{\langle p,\alpha\rangle \xrightarrow{a} \langle p',\alpha\rangle}{\langle p \lfloor\!\lfloor q,\alpha\rangle \xrightarrow{a} \langle p' \parallel q,\alpha\rangle} \qquad \textbf{48}\ \frac{\langle p,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle,\ \langle q,\alpha\rangle \not\longmapsto,\ \langle q,\alpha\rangle\downarrow}{\langle p \lfloor\!\lfloor q,\alpha\rangle \overset{S}{\longmapsto} \langle p',\alpha'\rangle}$$

$$\textbf{58}\ \frac{\langle p,\alpha\rangle\downarrow, \langle q,\alpha\rangle\downarrow}{\langle p \mid q,\alpha\rangle\downarrow} \qquad \textbf{59}\ \frac{\langle p,\alpha\rangle \xrightarrow{a} \langle p',\alpha\rangle,\ \langle q,\alpha\rangle \xrightarrow{b} \langle q',\alpha\rangle,\ \gamma(a,b)=c}{\langle p \mid q,\alpha\rangle \xrightarrow{c} \langle p' \parallel q',\alpha\rangle}$$

Table 3. Structural operational semantics for $\mathrm{BCP^{dst}}$

We briefly discuss the new rules. The rules 32-35 are the standard for the termination options and action transitions with synchronization. Rules 36 and 37 show that if one process terminates, the other one proceeds with passage of time. Rules 38-40, 41-43 and 44-46 enable independent, dependent and mixed races, respectively, as for the alternative composition. Rule 47 is the standard for action transitions of the left operand of the left merge. Rule 48 enables passage of time if the right summand terminates. Rules $49-57$ enable independent, dependent and mixed races in the left merge analogous to the parallel composition. The termination options and synchronized action transitions for the synchronization operator are given by the rules 58 and 59, respectively. Finally, rules $60-68$ enable independent, dependent and mixed races for the synchronization operator.

We note that the bisimulation remains unaltered. We straightforwardly extend $\mathrm{cf}_r(\_)$ and $\mathrm{mdr}_r(\_,\_)$, where $r\colon \mathcal{V} \to \mathcal{V}$, for $\diamond = \|, \lfloor\!\lfloor, |$ as follows:

$$\begin{array}{l}
\mathrm{cf}_r(p \diamond q, p' \diamond q') \text{ if } \mathrm{cf}_r(p,p'), \mathrm{cf}_r(q,q') \\
\mathrm{mdr}_{r''}(p' \diamond q', p \diamond q) \text{ if } \mathrm{cf}_{r''}(p \diamond q, p' \diamond q'),\ r'' \in [\mathrm{R}(p \diamond q)) \to \mathrm{R}(p' \diamond q')], \\
\qquad (\mathrm{A}(p' \diamond q') \setminus \mathrm{R}(p' \diamond q')) \cap (\mathrm{A}(p \diamond q) \setminus \mathrm{R}(p \diamond q)) = \emptyset, \\
\qquad \mathrm{mdr}_r(p', p),\ \mathrm{mdr}_{r'}(q', q),\ r'' = r'\{r/\mathrm{A}(p)\}
\end{array}$$

Also, the α-conversion conditions remain unaltered as well as the name resolution, which now straightforwardly extends to the other compositions as for the other operations.

The typical specifications are also extended with the new operators, as well as the term algebra and the term model. The equational theory for the typical specifications is given in Table 4. We comment some of the axioms. The merger of the scope of the name resolution operators is given by the axioms $P1$, $L1$ and $S1$. The standard expansion axiom for the parallel composition is $P2$. The left merge expansion where the left operand is prefixed by a dependent stochastic delay is given only for the normal form of the right operand as the races must be resolved. The synchronization operator behaves similarly to the alternative composition and the race conditions are handled in a similar fashion as given by the axioms $S10$-$S12$. The soundness and completeness property are proven as before with the most complicated being axiom $L6$, which is basically a 'mini' expansion law for the left merge.

Theorem 16. *The equational theory in Table 2 is sound and ground-complete for the typical specifications of* $\mathrm{BCP}^{\mathrm{dst}}$. □

Expansion theorem By using the normal forms, the outcomes of a race become explicit. This enables us to readily state an expansion theorem in the same vein of real-time process theories, cf. [12, 11], a result lacking in previous work [14].

P1 $|p| \parallel |q| = |p \parallel q|$ if $\mathrm{D}(p) \cap \mathrm{D}(q) = \emptyset$ **P2** $p \parallel q = p \mathbin{\lfloor\!\lfloor} q + q \mathbin{\lfloor\!\lfloor} p + p \mid q$ if $\mathrm{I}(p) = \emptyset$

L1 $|p| \mathbin{\lfloor\!\lfloor} |q| = |p \mathbin{\lfloor\!\lfloor} q|$ if $\mathrm{D}(p) \cap \mathrm{D}(q) = \emptyset$ **L2** $\delta \mathbin{\lfloor\!\lfloor} p = \delta$ **L3** $\epsilon \mathbin{\lfloor\!\lfloor} p = \delta$

L4 $a.p \mathbin{\lfloor\!\lfloor} q = a.(p \parallel q)$ **L5** $(p + q) \mathbin{\lfloor\!\lfloor} r = p \mathbin{\lfloor\!\lfloor} r + q \mathbin{\lfloor\!\lfloor} r$

L6 $\sigma_X.p \mathbin{\lfloor\!\lfloor} q = \sigma_X.p \mathbin{\lfloor\!\lfloor} \big(\sum_{i=1}^{m} a_i.p_i + \sum_{j=1}^{M} \genfrac{[}{]}{0pt}{}{W_j}{L_j} \sigma_{X_j}.q_j\ [+\epsilon]\big) =$

$\sum_{X \in W_j} \genfrac{[}{]}{0pt}{}{W_j \cup \{X\}}{L_j} \sigma_X.(p \mathbin{\lfloor\!\lfloor} q_j) + \sum_{X \in L_j} \genfrac{[}{]}{0pt}{}{W_j}{L_j \cup \{X\}} \sigma_{X_j}.(\sigma_X.p \mathbin{\lfloor\!\lfloor} q_j) +$

$\sum_{X \notin W_j \cup L_j} \Big(\genfrac{[}{]}{0pt}{}{X}{W_j \cup L_j} \sigma_X.(p \mathbin{\lfloor\!\lfloor} \genfrac{[}{]}{0pt}{}{W_j}{L_j} q_j) + \genfrac{[}{]}{0pt}{}{W_j \cup \{X\}}{L_j} \sigma_X.(p \mathbin{\lfloor\!\lfloor} q_j) + \genfrac{[}{]}{0pt}{}{W_j}{L_j \cup \{X\}} \sigma_{X_j}.(\sigma_X.p \mathbin{\lfloor\!\lfloor} q_j) \Big)$

if $\sum_{i=1}^{m} a_i.p_i + \sum_{j=1}^{M} \genfrac{[}{]}{0pt}{}{W_j}{L_j} \sigma_{X_j}.q_j\ [+\epsilon]$ is the normal form of q

S1 $|p| \mid |q| = |p \mid q|$ if $\mathrm{D}(p) \cap \mathrm{D}(q) = \emptyset$ **S2** $(p + q) \mid r = p \mid r + q \mid r$ if $\mathrm{I}(r) = \emptyset$

S3 $p \mid q = q \mid p$ **S4** $\delta \mid p = \delta$ **S5** $\epsilon \mid \epsilon = \epsilon$ **S6** $a.p \mid \epsilon = \delta$ **S7** $a.p \mid \sigma_X.q = \delta$

S8 $a.p \mid b.q = c.(p \parallel q)$ if $\gamma(a, b) = c$ **S9** $a.p \mid b.q = \delta$ if $\gamma(a, b)$ not defined

S10 $\genfrac{[}{]}{0pt}{}{W}{L} \sigma_X.p \mid \genfrac{[}{]}{0pt}{}{W'}{L'} \sigma_Y.q = \genfrac{[}{]}{0pt}{}{W \cup W'}{L \cup L'} \sigma_X.(p \mid q)$
if $W \cap W' \neq \emptyset,\ W \cap L' = L \cap W' = \emptyset,\ X \in W,\ Y \in W',\ \mathrm{I}(p) = \mathrm{I}(q) = \emptyset$

S11 $\genfrac{[}{]}{0pt}{}{W}{L} \sigma_X.p \mid \genfrac{[}{]}{0pt}{}{W'}{L'} \sigma_Y.q = \genfrac{[}{]}{0pt}{}{W}{L \cup W' \cup L'} \sigma_X.(p \mid \genfrac{[}{]}{0pt}{}{W'}{L'} \sigma_Y.q)$
if $L \cap W' \neq \emptyset,\ L \cap L' = W \cap (L \cup W' \cup L') = \mathrm{R}(p) \cap W' = \emptyset,\ X \in W,\ Y \in W'$

S12 $\genfrac{[}{]}{0pt}{}{W}{L} \sigma_X.p \mid \genfrac{[}{]}{0pt}{}{W'}{L'} \sigma_Y.q =$
$\genfrac{[}{]}{0pt}{}{W}{W' \cup L' \cup L} \sigma_X.(p \mid \genfrac{[}{]}{0pt}{}{W'}{L'} \sigma_Y.q) + \genfrac{[}{]}{0pt}{}{W \cup W'}{L \cup L'} (\sigma_X.p \mid \sigma_Y.q) + \genfrac{[}{]}{0pt}{}{W'}{W \cup L \cup L'} \sigma_Y.(q \mid \genfrac{[}{]}{0pt}{}{W}{L} \sigma_X.p)$
if $W \cap W' = L \cap W' = L' \cap W = \mathrm{R}(p) \cap W' = \mathrm{R}(q) \cap W = \emptyset,\ X \in W,\ Y \in W'$

Table 4. Process theory $\mathrm{BCP}^{\mathrm{dst}}$

Theorem 17. *For processes* $x = \sum_{i=1}^{m} a_i.p_i + \sum_{j=1}^{M} \left[{W_j \atop L_j}\right] \sigma_{X_j}.q_j \;\; [+\epsilon]$ *and* $y = \sum_{k=1}^{n} b_k.r_k + \sum_{\ell=1}^{N} \left[{U_\ell \atop T_\ell}\right] \sigma_{Y_\ell}.s_\ell \; [+\epsilon]$, *it holds that* $x \parallel y =$

$$\begin{aligned}
&\textstyle\sum_{i=1}^{m} a_i.(p_i \parallel y) + \sum_{k=1}^{n} b_k.(x \parallel r_k) + \sum_{\gamma(a_i,b_k)\ def.} \gamma(a_i,b_k).(p_i \parallel r_k)[+\epsilon] + \\
&\textstyle\sum_{W_j\cap U_\ell\neq\emptyset, W_j\cap T_\ell = L_j\cap U_\ell=\emptyset} \left[{W_j\cup U_\ell \atop L_j\cup T_\ell}\right]\sigma_{X_j}.(q_j \parallel s_\ell) + \\
&\textstyle\sum_{L_j\cap U_\ell\neq\emptyset, L_j\cap T_\ell = W_j\cap(U_\ell\cup T_\ell)=\mathrm{R}(q_j)\cap U_\ell=\emptyset} \left[{W_j \atop L_j\cup U_\ell\cup T_\ell}\right]\sigma_{X_j}.(q_j \parallel \left[{U_\ell \atop T_\ell}\right]\sigma_{Y_\ell}.s_\ell) + \\
&\textstyle\sum_{W_j\cap T_\ell\neq\emptyset, L_j\cap T_\ell = U_\ell\cap(W_j\cup L_j)=W_j\cap\mathrm{R}(s_\ell)=\emptyset} \left[{U_\ell \atop W_j\cup L_j\cup T_\ell}\right]\sigma_{Y_\ell}.(\left[{W_j \atop L_j}\right]\sigma_{X_j}.q_j \parallel s_\ell) + \\
&\textstyle\sum_{W_j\cap U_\ell = W_j\cap T_\ell = L_j\cap U_\ell=\mathrm{R}(q_j)\cap U_\ell = W_j\cap\mathrm{R}(s_k)=\emptyset} \Big(\left[{W_j \atop L_j\cup U_\ell\cup T_\ell}\right]\sigma_{X_j}.(q_j \parallel \left[{U_\ell \atop T_\ell}\right]\sigma_{Y_\ell}.s_\ell) + \\
&\qquad \textstyle\left[{W_j\cup U_\ell \atop L_j\cup T_\ell}\right]\sigma_{X_j}.(q_j \parallel s_\ell) + \left[{U_\ell \atop W_j\cup L_j\cup T_\ell}\right]\sigma_{Y_\ell}.(\left[{W_j \atop L_j}\right]\sigma_{X_j}.q_j \parallel s_\ell)\Big)\square.
\end{aligned}$$

The first three summands of the expansion are the standard ones for untimed theory. As usual, the termination option of $x \parallel y$ is enabled only if both x and y have termination options. The fourth summand synchronizes races with common winners analogous to axiom $R7$ for the alternative composition in Table 2. The fifth and the sixth summand treat the races when the winners come both from the left and the right summand, respectively, as corresponding to axiom $R8$. The seventh summand gives the expansion of unresolved races as in axiom $R9$.

For comparison, we present the expansion of the parallel composition with clocks in residual lifetime semantics [4]. The treatment of expansion for clocks with spent lifetimes and start-termination semantics is similar [5, 9]. Here, the processes are given by $x = \mathsf{set}\ C\ \mathsf{in}\ x'$ and $y = \mathsf{set}\ D\ \mathsf{in}\ y'$, for $x' = \sum_{i=1}^{m}(\mathsf{when}\ C_i \mapsto a_i; p_i)$ and $y' = \sum_{j=1}^{n}(\mathsf{when}\ D_j \mapsto b_j; q_j)$, where the operator set sets the clocks and 'a; ' is the action prefix operator. Then it holds that $x \parallel_A y =$

$$\begin{aligned}
&\mathsf{set}\ (C\cup D)\ \mathsf{in}\ \Big(\textstyle\sum_{a_i\notin A} \mathsf{when}\ C_i \mapsto a_i; (p_i \parallel_A y') + \\
&\textstyle\sum_{b_j\notin A} \mathsf{when}\ D_j \mapsto b_j; (x' \parallel_A q_j) + \sum_{a_i=b_j\in A} \mathsf{when}\ (C_i\cup D_j) \mapsto a_i; (p_i \parallel_A q_j)\Big).
\end{aligned}$$

Note that the above expansion actually involves setting the joint sets of clocks (i.e., the starting activities). However, the typical expansion for real-time processes is given on completed delays (cf. [11]), as in our Theorem 17. We argue that this stresses the similarity of the notions of a stochastic and real-timed delay.

5 Real-Time

Relative standard timed delays of duration $t > 0$ are introduced in the current setting by means of Dirac (or degenerated) random variables X_t, where $P(X_t = t) = 1$ for $X_t \in \mathcal{V}_{\mathrm{deg}}$ (cf. [14]). Again, we distinguish between independent and dependent delays because of the race condition. For example, the degenerated stochastic delay Y_t in $\left|\left[{X \atop Y_t}\right]\sigma_X.\sigma_{Y_t}.\delta\right|$ is dependent on a stochastic delay. So, its residual distribution is no longer a real-time delay. We note that the degenerated stochastic delays comply with the race condition. However, such inclusion of real-time in our stochastic process algebra has a side effect, viz. the stochastic transition schemes may contain non-accessible transitions. For example, the transition $\langle \zeta_{X_t}.p + \zeta_{X_{t+s}}.q, \alpha_\emptyset \rangle \xmapsto{X_{t+s}} \langle \zeta_{X_t}.p + q, \alpha_\emptyset\{X_{t+s}/\{X_t\}\}\rangle$ will never be observed in the concrete model and, similarly, the only transition with non-zero probability of $\zeta_{X_t}.p + \zeta_{Y_t}.q$ is the joint stochastic delay transition labeled by $\{X_t, Y_t\}$. One way to avoid such zero-probability transitions is to introduce real-time as a separate timed delay transition in the stochastic transition schemes. In any case, the treatment of real-time proposed here gives rise to several issues of interest when considering stochastic time and real-time.

Bisimulation In general, stochastic bisimulation is a one-step bisimulation, i.e., it takes into consideration only one stochastic delay transition at a time. In contrast, the timed bisimulation typically employs time additivity [12], i.e., merging of subsequent timed delays into one delay to compare processes that delay in time. To the best of our knowledge, with the exception of [18], all stochastic process theories consider a one-step-like stochastic bisimulation: in [2] the actions are coupled with the stochastic clocks, in [4] there is an alternation between clocks and action transitions, whereas in [3, 5] the merging is impeded by the combination of pre-selection policy and start-termination semantics. Although originally introduced with one-step-like stochastic bisimulation [6], an attempt is made in [18] to present weak stochastic bisimulation that merges subsequent stochastic delays. Unfortunately, such an approach is not compositional as the merging of stochastic delays does not support the race condition. A simple counter-example is the process $\zeta_X.\zeta_Y.\delta$. Although the process has the same stochastic properties as the process $\zeta_Z.\delta$, provided $F_Z = F_{X+Y}$, these processes are not bisimilar in any composition with any other process that is capable of performing a stochastic delay. For example, $\zeta_X.\zeta_Y.\delta + \zeta_U.\delta$ is not bisimilar to $\zeta_Z.\delta + \zeta_U.\delta$ because the race of X and U induces a different probability choice on the winner compared to the race between Z and U.

We conclude that the states in the stochastic transition schemes between the stochastic delays transitions are relevant. The same holds for the degenerated stochastic delays. Thus, the concept of time additivity is no longer valid, although it is a standard property of real-time process algebra [12]. However, the race condition together with the one-step-like stochastic bisimulation induce, in fact, a finer notion than (arbitrary) time additivity as depicted in Fig. 4. We propose to refer to this concept as as 'context-sensitive interpolation'.

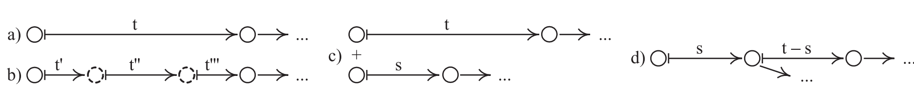

Fig. 4. Arbitrary interpolation of the timed delay t in a) by additivity is given in b), where $t = t' + t'' + t'''$; Context-sensitive interpolation of the timed delay t in c) in the context of the alternative composition with the timed delay s is given in d)

Additivity vs. Interpolation Time additivity allows the observer to distinguish a partial passage of time of a real-time delay before a change of state is induced. Therefore, it treats a timed delay as the aggregate of every possible shorter delay (cf. Fig. 4a and Fig. 4b). However, this approach is not valid in the context of stochastic time as the observer can statistically measure the 'branching' probabilities for the winners of the race in the interpolated states. Now, the view of strong interpolation is that a real-time delay can be interpolated only in the context of a composition performing a shorter delay in order to conform to the race condition. In that way, we ensure that the intermediate state actually exists, as illustrated in Fig. 4c and Fig. 4d. Axioms that describe strong interpolation for the Dirac stochastic delays in an alternative composition are:

A7 $\zeta_{X_t}.p + \zeta_{Y_t}.q = \zeta_{X_t}.(p+q)$ **A8** $\zeta_{X_t}.p + \zeta_{Y_{t+s}}.q = \zeta_{X_t}.p + \zeta_{Y_t}.\zeta_{Y_s}.q$

At first sight this may seem too restrictive, in view of the time additivity law $\sigma^t.p = \sigma^{t'}.\sigma^{t''}.p$ where $t' + t'' = t$. However, strong interpolation does exactly what time additivity is typically used for: merging of delays with the same duration by taking the shortest/minimal delay. Moreover, as argued, the interpolation fits naturally in the expansion of the parallel composition, which makes it a promising candidate for a finer notion of time additivity in general purpose real-time process algebra.

6 Conclusions and Future Work

A stochastic process algebra that comprises real-time has been proposed. Its so-called typical processes involving complete race conditions come equipped with a sound and complete axiomatization exploiting normal forms in which the race condition is explicitly resolved. This enables an expansion of the parallel composition in the style of real-time process theories. Finally, stochastic bisimulations and standard timed bisimulation are compared resulting in a proposal for a finer notion of standard time additivity, called context-sensitive interpolation, induced by the race condition and depending on the context.

As future work, we continue our axiomatization efforts to completely describe all possible specifications. Current investigations point out that we need a priority operator that disables the weak choice, also required by the maximal progress operator [11]. We also schedule further study of real-time process theories that implement context-sensitive interpolation and one-step-like timed bisimulation. At this point, we expect that such theories can also accommodate for verification and analysis of processes with timed delays. Also, we find it worthwhile to investigate deeper into the relation between the race condition and the pre-selection policies, which might pave the way for merging subsequent stochastic delays. Our final goal is the analysis of contemporary Internet protocols involving time-outs as well as generally distributed delays.

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