Index-aware Model Order Reduction Methods for DAEs

Funded by The Netherlands Organization for Scientific Research (NWO)

A catalogue record is available from the Eindhoven University of Technology Library
ISBN: 978-90-386-3700-6

Index-aware Model Order Reduction Methods for DAEs

PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de
Technische Universiteit Eindhoven, op gezag van de
rector magnificus, prof.dr.ir. C.J. van Duijn, voor een
commissie aangewezen door het College
voor Promoties in het openbaar te verdedigen
op maandag 22 september 2014 om 16.00 uur

door

Nicodemus Banagaaya

geboren te Kampala, Oeganda

Dit proefschrift is goedgekeurd door de promotoren en de samenstelling van de promotiecommissie is als volgt:

voorzitter:	prof. dr. E.H.L. Aarts
1e promotor:	prof. dr. W.H.A. Schilders
2e promotor:	prof. dr. C. Tischendorf (Humboldt University of Berlin, Germany)
copromotor:	dr. M.E. Hochstenbach
leden:	prof. dr. G. Alí (University of Calabria, Italy)
	dr. J. Rommes (Mentor Graphics, France)
	prof. dr. S. Weiland
adviseur:	dr. W. Schoenmaker (Magwel BV, Belgium)

Contents

Nomenclature

Notations	
$\mathbb{R}$	Set of all real numbers
$\mathbb{R}^{m\times n}$	Set of all real matrices of dimension $m \times n$
$\mathbb{R}^{n}$	Set of all real vectors of dimension n
$\mathbb{C}$	Set of all complex numbers
$\mathbb{C}^{-}$	The open left half complex plane
$\dim(\mathbf{V})$	Dimension of a vector space $\mathbf{V}$
I	Identity matrix of the desired order
$\mathbf{A}^{-1}$	Inverse of matrix $\mathbf{A}$
$\mathbf{A}^{\mathrm{T}}$	Transpose of matrix $\mathbf{A}$
$\operatorname{rank}\mathbf{A}$	Rank of matrix $\mathbf{A}$
$\sigma(\mathbf{A})$	Set of all eigenvalues of matrix $\mathbf{A}$
$\det(\mathbf{A})$	Determinant of matrix $\mathbf{A}$
$\sigma(\mathbf{E},\mathbf{A})$	Set of all generalised eigenvalues of matrix pencil $(\mathbf{E},\mathbf{A})$
$\sigma_f(\mathbf{E},\mathbf{A})$	Set of all finite eigenvalues of matrix pencil $(\mathbf{E},\mathbf{A})$
$\sigma_\infty(\mathbf{E},\mathbf{A})$	Set of all infinite eigenvalues of matrix pencil $(\mathbf{E},\mathbf{A})$
$\mathrm{Ker}(\mathbf{T})$	Kernel of transformation or matrix $\mathbf{T}$
$\mathrm{Im}(\mathbf{T})$	Image of transformation or matrix $\mathbf{T}$
$\mathrm{Span}(\mathbf{T})$	Subspace spanned by the columns of matrix $\mathbf{T}$
$\mathcal{K}_r(\mathbf{A},\mathbf{b})$	Order-r Krylov subspace generated by matrix $\mathbf{A}$ of dimension $n \times n$ and a vector $\mathbf{b}$ of dimension n
$\dot{\mathbf{x}}$ or $\mathbf{x}'$	Derivative of $\mathbf{x}$ with respect to time t
$\mathrm{Span}\{\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_n\}$	Vector space spanned by the vectors $\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_n$
$\mathrm{diag}\{\mathbf{d}_1,\mathbf{d}_2,\cdots,\mathbf{d}_n\}$	Diagonal matrix with diagonal elements $\mathbf{d}_1,\mathbf{d}_2,\cdots,\mathbf{d}_n$
Index-μ DAEs	DAEs of tractability index-μ

Acronyms	
MOR	Model Order Reduction
DAEs	Differential algebraic equations
LU	Lower and Upper triangular matrix
SVD	Singular valued decomposition
CFD	Computational fluid dynamics
RLC	Resistor-inductor-capacitor
RC	Resistor-capacitor
MIMO	Multiple-input Multiple-output system
SISO	Single-input Single-output system
IMOR	Index-aware model order reduction
IIMOR	Implicit index-aware model order reduction
ODEs	Ordinary differential algebraic equations
LTI	Linear time invariant
MNA	Modified nodal analysis
PRIMA	Passive Reduced-Order Interconnect Macromodeling Algorithm
SPRIM	Structure-preserving reduced-order interconnect macromodeling
AE	Algebraic elimination

Chapter 1

Introduction

Large scale differential algebraic equations (DAEs) arise in a variety of applications such as modeling of constrained multibody systems, electrical networks, aerospace engineering, chemical processes, computational fluid dynamics (CFD), gas transport networks, see [10, 12, 16, 24, 35, 45]. Such systems have characteristics of leading to state space descriptions of high dimension in which the coefficient of the first order derivative is a singular matrix. In practice, such applications lead to DAEs with very large dimension compared to the number of inputs and the desired outputs. Despite the ever increasing computational power, simulation of these systems in real time on such large scale is very difficult because of the storage requirements and expensive computations. This is an attractive feature to apply model order reduction (MOR). However, if the initial condition is inconsistent or when the smoothness of the input does not correspond to the index of the DAE, currently available MOR techniques may lead to inaccurate reduced-order models, see [1, 2]. These reduced-order models may lead to wrong solutions that do not adequately represent the hidden truly fast modes or are very difficult to solve numerically. In most publications and applications it is assumed that the matrix pencil is regular, and conventional MOR techniques based on either Krylov subspaces or singular value decomposition (SVD) are used to extract dominant behavior of the transfer function [3, 9, 58]. However, it has recently been shown that such approaches may lead to reduced models that are not adequate, as they do not take into account the special behavior due to infinite state variables of the system [45]. The initial condition of the finite state variables can be chosen arbitrary while the initial condition of the infinite state variables have to satisfy certain hidden constraints. Thus the initial condition of the differential algebraic equations must be a consistent initial value.

However, it happens that the conventional MOR methods [3, 9, 58] cannot be applied immediately especially to higher index DAEs because they deal only with a system possessing zero initial condition. Moreover, most conventional MOR methods treat DAEs as ODEs, for example PRIMA method [49] sometimes leads to ordinary differential equations (ODEs) reduced-order models even if the original model is a DAE. This may lead to loss of their mathematical properties. As a consequence, new concepts were needed to provide reliable reduced-order models for DAEs. In the new approach, DAEs must first be decoupled into differential and algebraic parts before applying any MOR technique. This observation has lead to the development of new methods specifically for DAEs, see [17, 18, 25, 32, 45] and to some extent the modification of the existing MOR methods, see [25, 45]. Most of these recently developed methods are application based and some are more general. In [45], they proposed the most successful MOR method for DAEs known as the balanced truncation method for descriptor systems, however it is computationally expensive since it involves solving four Lyapunov equations. Then, most recently the computationally cheaper, model reduction of descriptor systems by interpolatory projection methods was proposed in [25]. Both methods are robust and lead to accurate reduced-order models for DAEs. They both use the spectral projectors to split DAEs into differential and algebraic parts before reduction. However, the Kronecker canonical forms are used to construct the spectral projectors which are well known to be numerically infeasible [1, 42]. Hence, the existing most accurate MOR methods for DAEs are much limited to DAEs with special structures and can not be extended to DAEs with variable coefficients.

In this thesis, a computationally cheaper way of decoupling and reducing DAEs is proposed. This decoupling procedure relies on the framework of special projector and matrix chain for DAEs, enabling a decomposition into separate differential and algebraic parts as introduced by März in [42]. However, the März decoupling procedure leads to much larger decoupled system of dimension $n(\mu+1)$, where n and μ is the dimension and the tractability index of the DAE, respectively. Hence, the März decoupling procedure does not preserve some of the mathematical properties of the DAEs such as dimension and stability. This motivated us to modify the März decoupling procedure using special bases of projectors instead of the full projectors, thus preserving the dimension and stability of DAEs in the decoupled system. Having performed this separation, different reduction methods can be used to each of these parts. For the differential part, one can use the conventional MOR methods while the algebraic part, we have developed new methods since there was no known reduction methods for algebraic systems. This procedure lead to a new MOR method for DAEs which we call the Index-aware MOR method abbreviated as IMOR method [1, 2]. The IMOR method is illustrated in Figure 1.1. This method is very robust and leads to simple reduced-order models for even

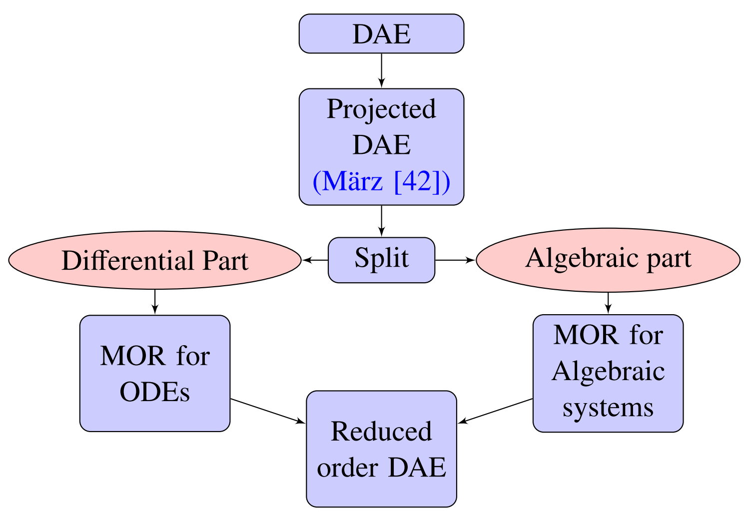

Figure 1.1: IMOR methods procedure

higher index DAEs. However, the IMOR method has an inherited limitation of matrix inversion which makes it computationally very expensive. This lead to the development of its implicit version which we call the implicit IMOR method which is abbreviated as IIMOR method. The implicit IMOR method is computationally cheaper than the IMOR method. However, experiments show that the IMOR method is more accurate, thus one needs to trade off between complexity and accuracy. Using our decoupled systems, we were able to analyse the limitations of the conventional MOR methods. We observed that sometimes conventional MOR methods can lead to accurate reduced-order models even for higher index DAEs, if and only if the consistent initial condition does not depend on the derivatives of the input data. This is equivalent to a DAE having a proper transfer function. The explicit and implicit decoupling procedures are also advantageous for solving DAEs more efficiently numerically since they enable one to use the conventional ODEs integration methods to solve higher index DAEs.

Other, well known tools used to investigate DAEs are the transformation into Kronecker normal form and the decoupling by means of Drazin inverses and spectral projectors. These tools are very accurate but they are numerically infeasible and can not be generalized to variable coefficient linear and nonlinear DAEs [42]. However, our decoupling procedures used in both IMOR and IIMOR methods, relies on the matrix and projector chain approach introduced by März [42] which can also be applied to general variable coefficient equations, see [27]. Hence the IMOR and IIMOR methods can be extended to variable coefficient linear and nonlinear DAEs.

The NWO project

This work is part of the research programme *Model Order Reduction for Differential Algebraic Systems*, which is (partly) financed by the Netherlands Organisation for Scientific Research (NWO).

The aim of this PhD project is to investigate model order reduction techniques for differential algebraic systems. The ultimate goal of the project is to deliver fundamental mathematical knowledge and efficient numerical tools for the next generation of MOR techniques for differential algebraic equations. This thesis addresses the mathematical aspect of the reduction of differential algebraic equations including the limitations of the conventional MOR methods. We have developed reduction methods for DAEs, using the underlying structure of DAEs, with the aim of obtaining robust reduction methods that can also be applied to DAEs with arbitrary index

Chapter 2

Differential Algebraic Equations

In this Chapter, we introduce the differential algebraic equations which we abbreviate as DAEs. DAEs arise in a variety of applications such as modeling of constrained multibody systems, electrical networks, aerospace engineering, chemical processes, computational fluid dynamics (CFD), gas transport networks, see [10, 12, 16, 24, 35, 45]. Therefore their analysis and numerical treatment plays an important role in modern mathematics. In many articles, DAEs are also called singular systems [12], descriptor systems [16, 45, 66], generalized state space systems [45], semi-state systems, degenerated systems, constrained systems, implicit systems but in most literatures they are called DAEs [39, 42, 52]. In this thesis, we shall also call them DAEs.

2.1 What are DAEs?

Consider an explicit ordinary differential equations (ODEs),

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}), \tag{2.1.1}$$

where $\dot{\boldsymbol{x}} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}$ and $\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{f} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. In general the first order ODE can be written in implicit form

$$\boldsymbol{F}(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) = \boldsymbol{0}. \tag{2.1.2}$$

According to [61], if the Jacobian matrix $\frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}}$ is nonsingular then it is possible to solve (2.1.2) for $\dot{\boldsymbol{x}}$ in order to obtain an ODE (2.1.1). However, if $\frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}}$ is singular, this is no longer possible and the solution $\boldsymbol{x}$ has to satisfy certain algebraic constraints. Hence, if $\frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}}$ is singular, then (2.1.2) is referred to as a DAE. In modeling the formulation of pure ODE problems often requires the combination of ; conservation laws (mass and energy balance), constitutive equations (equations of state, pressure drops, heat transfer) and design constraints (desired operations). This means that there are some problems where not all the equations in a differential system involves derivatives, thus we can come up with a special case of DAEs which can be written as,

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{y}), \tag{2.1.3a}$$

$$\boldsymbol{0} = \boldsymbol{g}(t, \boldsymbol{x}, \boldsymbol{y}), \tag{2.1.3b}$$

where $\boldsymbol{x}$-differential variables, $\boldsymbol{y}$-algebraic variables and (2.1.3b) is a constraint equation. Equation (2.1.3) is a special type of DAEs which is commonly called the semi-explicit DAEs.

2.2 Models for DAEs

According to [16], it is well known from modern control theory that two main mathematical representations for dynamical systems are the transfer matrix representation and the state space representation. The former describes only the input-output property of the system, while the latter gives further insight into the structural property of the system.

2.2.1 State space representation

State space representation was developed at the end of the 1950s and the beginning of the 1960s, which has the advantage that it not only provides us with efficient method for control system analysis and synthesis, but also offers us a deeper understanding about the various properties of the systems, see [16]. The state space models of the systems are obtained mainly using the so-called state space variable method [16]. To obtain a state model of a practical system, we need to choose some physical variables such as currents and voltages in an electrical network. Then, by the physical relationships among the variables or by some model identification techniques such as modified nodal analysis in network analysis, a set of equations can be established. Naturally, this set of equations are usually differential and/or algebraic equations, which form a mathematical model of the system. By properly defining a state vector $\boldsymbol{x}(t)$ and an input vector $\boldsymbol{u}(t)$, which are formed by the physical variables of the system, and an output vector $\boldsymbol{y}(t)$, whose elements are properly chosen measurable variables of the system, this set of equations can be arranged into two equations given by

$$\boldsymbol{f}(\dot{\boldsymbol{x}}(t), \boldsymbol{x}(t), \boldsymbol{u}(t), t) = 0, \tag{2.2.1}$$

$$\boldsymbol{g}(\boldsymbol{y}(t), \boldsymbol{x}(t), \boldsymbol{u}(t), t) = 0, \tag{2.2.2}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are vector functions of appropriate dimensions with respect to $\dot{\boldsymbol{x}}(t), \boldsymbol{x}(t), \boldsymbol{y}(t)$, $\boldsymbol{u}(t)$ and t. Equations (2.2.1) and (2.2.2) are the so-called **state equation** and **output equation**, or the observation equation. Equations (2.2.1) and (2.2.2) give the state space representation for a general nonlinear dynamical system. If we consider a special form of (2.2.1)–(2.2.2) :

$$\begin{aligned} \mathbf{E}(t)\,\dot{\boldsymbol{x}}(t) &= \mathbf{F}(\boldsymbol{x}(t), \boldsymbol{u}(t), t) \\ \boldsymbol{y}(t) &= \mathbf{K}(\boldsymbol{x}(t), \boldsymbol{u}(t), t), \end{aligned} \tag{2.2.3}$$

where $t \geq 0$ is the time variable, $\mathbf{F}$ and $\mathbf{K}$ are appropriate dimensional vector functions, $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector, $\boldsymbol{u}(t) \in \mathbb{R}^m$ is the control input vector, $\boldsymbol{y}(t) \in \mathbb{R}^\ell$ is the measured output vector. The matrix $\mathbf{E}(t)$ must be singular for some $t \geq 0$ for our case since we are considering DAEs. Equation (2.2.3) is the general form of the so-called nonlinear DAEs. If we consider the case, when $\mathbf{F}$ and $\mathbf{K}$ are linear functions of vectors $\boldsymbol{x}(t)$ and $\boldsymbol{u}(t)$, the general nonlinear DAEs (2.2.3) simplifies to the following form:

$$\begin{aligned} \mathbf{E}(t)\,\dot{\boldsymbol{x}}(t) &= \mathbf{A}(t)\boldsymbol{x}(t) + \mathbf{B}(t)\boldsymbol{u}(t), \\ \boldsymbol{y}(t) &= \mathbf{C}^{\mathrm{T}}(t)\boldsymbol{x}(t) + \mathbf{D}^{\mathrm{T}}(t)\boldsymbol{u}(t), \end{aligned} \tag{2.2.4}$$

where $\mathbf{E}(t), \mathbf{A}(t) \in \mathbb{R}^{n\times n}$, $\mathbf{C}(t) \in \mathbb{R}^{n\times\ell}$, $\mathbf{D}(t) \in \mathbb{R}^{m\times\ell}$, $\mathbf{B}(t) \in \mathbb{R}^{n\times m}$ are matrix functions of time t, and they are called the coefficients matrices of the system (2.2.4). Equation

(2.2.4) describes the so-called linear time varying DAEs. If the matrix coefficients are constant, i.e., time independent, the system (2.2.4) is called the linear constant coefficient DAEs or linear time invariant (LTI) DAEs which can be written as,

$$\mathbf{E}\,\dot{\boldsymbol{x}}(t) = \mathbf{A}\boldsymbol{x}(t) + \mathbf{B}\boldsymbol{u}(t), \tag{2.2.5a}$$

$$\boldsymbol{y}(t) = \mathbf{C}^{\mathrm{T}}\boldsymbol{x}(t) + \mathbf{D}^{\mathrm{T}}\boldsymbol{u}(t), \tag{2.2.5b}$$

where $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{C} \in \mathbb{R}^{n\times\ell}$, $\mathbf{D} \in \mathbb{R}^{m\times\ell}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ are constant coefficient matrices. For DAEs of the form (2.2.5), there is also a concept of the **dynamical order**, which is defined as the rank of singular matrix $\mathbf{E}$. Equations (2.2.4) and (2.2.5) are the two basic classes of DAEs. From this point, we restrict ourselves on the DAE of the form (2.2.5) unless stated otherwise.

2.2.2 Transfer matrix representation

In this Section, we discuss the transfer matrix representation. This representation is derived from the state space representation using the Laplace transform . The transfer matrix representation is commonly used to validate reduced-order models in the model order reduction community and is commonly called the transfer function.

Definition 2.2.1 (Laplace transform [58]) *The Laplace transform of a function $f(t)$ in the time domain is the function $F(s)$ in the frequency domain and it is defined as,* $\mathcal{L}\{f(t)\} = F(s) := \int_0^{\infty} e^{-st} f(t)\,\mathrm{d}t$, *where $s = \sigma + j\omega \in \mathbb{C}$, with $\sigma, \omega \in \mathbb{R}$.*

We shall restrict ourselves on the transfer matrix representation of the LTI DAEs (2.2.5) and also assume $s = j\omega$, i.e $\sigma = 0$. Taking the Laplace transform of (2.2.5) and simplifying, we obtain

$$\boldsymbol{Y}(s) = \left[\mathbf{C}^{\mathrm{T}}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}^{\mathrm{T}}\right]\mathbf{U}(s) + \mathbf{C}^{\mathrm{T}}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}\boldsymbol{x}(0), \tag{2.2.6}$$

where $\mathbf{U}(s)$ and $\mathbf{Y}(s)$ are the Laplace transforms of $\mathbf{u}(t)$ and $\boldsymbol{y}(t)$, respectively. The rational matrix-valued function

$$\mathbf{H}(s) = \mathbf{C}^{\mathrm{T}}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}^{\mathrm{T}} \in \mathbb{R}^{\ell\times m}, \tag{2.2.7}$$

is called the transfer matrix representation of (2.2.5) or transfer function. Then, $\mathbf{H}(s)$ gives the relation between the Laplace transforms of the input $\mathbf{u}(t)$ and the output $\boldsymbol{y}(t)$. In other words, $\mathbf{H}(s)$ describes the input-output behavior of (2.2.5) in the frequency domain.

Definition 2.2.2 ([62]) *The transfer function* $\mathbf{H}(s)$ *is called proper if* $\lim_{s\to\infty} \mathbf{H}(s) < \infty$, *and improper otherwise. if* $\lim_{s\to\infty} \mathbf{H}(s) = 0$, *then* $\mathbf{H}(s)$ *is called strictly proper.*

Almost all conventional MOR methods assume vanishing initial condition, i.e., $\mathbf{E}\boldsymbol{x}(0) = 0$, which leads to $\boldsymbol{Y}(s) = \mathbf{H}(s)\mathbf{U}(s)$. We need to ask ourselves wether, we can always describe the transfer matrix representation or transfer function of an entire DAE dynamical system as for the case of ODE systems, i.e., Is it always possible to assume $\mathbf{E}\boldsymbol{x}(0) = 0$ in (2.2.6) to obtain $\boldsymbol{Y}(s) = \mathbf{H}(s)\mathbf{U}(s)$? This question is answered in Section 2.3.4 after gathering enough knowledge about DAEs.

2.3 Linear constant coefficient DAEs

In this Section, we discuss the analysis of LTI DAEs. For simplicity, the coefficient matrix $\mathbf{D}$ in (2.2.5) is assumed to be zero matrix unless specified. Thus, (2.2.5) simplifies to:

$$\mathbf{E}\,\dot{\boldsymbol{x}}(t) = \mathbf{A}\,\boldsymbol{x}(t) + \mathbf{B}\,\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{2.3.1a}$$

$$\boldsymbol{y}(t) = \mathbf{C}^{\mathrm{T}}\boldsymbol{x}(t), \tag{2.3.1b}$$

where $\mathbf{E} \in \mathbb{R}^{n\times n}$ is singular, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{n\times \ell}$, the input vector $\boldsymbol{u}(t) \in \mathbb{R}^m$, output vector $\boldsymbol{y}(t) \in \mathbb{R}^\ell$ and $\boldsymbol{x}_0 \in \mathbb{R}^n$ is the initial value.

2.3.1 Solvability of DAEs

Here, we are interested in the solutions of the homogenous system obtained by setting $\boldsymbol{u}(t) = 0$, then (2.3.1a) becomes

$$\mathbf{E}\dot{\boldsymbol{x}}(t) - \mathbf{A}\boldsymbol{x}(t) = \mathbf{0}. \tag{2.3.2}$$

As for the case of ODEs using the guess solution $\boldsymbol{x}_*(t) = \boldsymbol{x}_0 e^{\lambda_* t}$. Substituting the guess solution into (2.3.2) leads to $e^{\lambda_* t}(\lambda_* \mathbf{E} - \mathbf{A})\boldsymbol{x}_0 = \mathbf{0}$. Hence according to [34], $\boldsymbol{x}_*(t)$ is a nontrivial solution of the DAE (2.3.2) if λ_* is a zero of polynomial $\mathcal{P}(\lambda) := \det(\lambda\mathbf{E} - \mathbf{A})$, $\lambda \in \mathbb{C}$ and $\boldsymbol{x}_0 \neq \mathbf{0}$ satisfies $(\lambda_* \mathbf{E} - \mathbf{A})\boldsymbol{x}_0 = \mathbf{0}$. λ and $\boldsymbol{x}_0$ are called the generalized eigenvalues and eigenvectors, respectively. Thus, we say that the DAE (2.3.1a) is solvable provided the matrix pencil $\lambda\mathbf{E} - \mathbf{A}$ is regular, see [34]. We note that $\lambda\mathbf{E} - \mathbf{A}$ can also be written as $(\mathbf{E}, \mathbf{A})$ which is called the matrix pencil or matrix pair.

Definition 2.3.1 ([34, 52]) *A matrix pair* $(\mathbf{E}, \mathbf{A})$ *is called regular if the polynomial* $\mathcal{P}(\lambda) = \det(\lambda\mathbf{E} - \mathbf{A})$ *is not identically zero otherwise singular.*

A pair $(\mathbf{E}, \mathbf{A})$ with nonsingular $\mathbf{E}$ is always regular, and its polynomial $\mathcal{P}(\lambda)$ is of degree n. In case of singular matrices $\mathbf{E}$, the polynomial degree is lower. According to [33], regularity of a matrix pair is closely related to the solution behavior of the corresponding DAE. In particular, regularity is necessary and sufficient for the property that for every sufficiently smooth inhomogeneity $\mathbf{u}$ the DAE is solvable and the solution is unique for every consistent initial value. This is well understood, if we consider Weierstraß-Kronecker canonical form of a given DAE as discussed in Section 2.3.3.

Definition 2.3.2 ([62]) *A pair* $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ *is said to be a generalized eigenvalue* $\lambda = \frac{\alpha}{\beta}$ *of the matrix pencil* $\lambda\mathbf{E} - \mathbf{A}$ *if* $\det(\alpha\beta\mathbf{E} - \mathbf{A}) = 0$. *If* $\beta \neq 0$ *, then the pair* (α, β) *represents a finite eigenvalue* $\lambda = \frac{\alpha}{\beta}$ *of the matrix pencil* $\lambda\mathbf{E} - \mathbf{A}$. *But if* $\beta = 0$, *the pair* $(\alpha, 0)$ *represents an infinite eigenvalue of* $\lambda\mathbf{E} - \mathbf{A}$. *Clearly, the pencil* $\lambda\mathbf{E} - \mathbf{A}$ *has an eigenvalue at infinity if and only if the matrix* $\mathbf{E}$ *is singular.*

The set of all finite eigenvalues of the matrix pencil $(\mathbf{E}, \mathbf{A})$ is denoted by $\sigma_f(\mathbf{E}, \mathbf{A})$ while the infinite spectrum of the matrix pencil $(\mathbf{E}, \mathbf{A})$ is denoted by $\sigma_\infty(\mathbf{E}, \mathbf{A})$. Thus, the set of all generalized eigenvalues (finite and infinite) of the matrix pencil $(\mathbf{E}, \mathbf{A})$ is called the spectrum of $(\mathbf{E}, \mathbf{A})$ and donated by $\sigma(\mathbf{E}, \mathbf{A}) = \sigma_f(\mathbf{E}, \mathbf{A}) \cup \sigma_\infty(\mathbf{E}, \mathbf{A})$. We note that if $\mathbf{E}$ is nonsingular, then $\sigma(\mathbf{E}, \mathbf{A}) = \sigma_f(\mathbf{E}, \mathbf{A})$ which is equal to the spectrum of $\mathbf{E}^{-1}\mathbf{A}$. This also means that if $\mathbf{E}$ is nonsingular, the homogeneous equation (2.3.2) represents an implicit regular ODE and its fundamental solution system forms an n-dimensional subspace in C^1. But what happens if $\mathbf{E}$ is singular, this is closely related to the notion of the regular matrix pencil $(\mathbf{E}, \mathbf{A})$ [34] as discussed in Section 2.3.3.

2.3.2 Stability of DAEs

According to [16], in practice a practical system should be stable otherwise, it may not work properly or may even be destroyed in practical use. Like the ODE systems case, when studying stability of DAEs, we also need to only consider the homogenous system (2.3.2).

Definition 2.3.3 ([62]) *The DAE* (2.3.1) *is called asymptotically stable if* $\lim_{t\to\infty} \boldsymbol{x}(t) = 0$ *for all solutions* $\boldsymbol{x}(t)$ *of the homogenous system* $\mathbf{E}\dot{\boldsymbol{x}}(t) = \mathbf{A}\boldsymbol{x}(t)$.

This leads us to the following theorem that collects equivalent conditions for system (2.3.1) to be asymptotically stable.

Theorem 2.3.1 ([12, 62]) *Consider a DAE* (2.3.1) *with regular matrix pencil* $\lambda\mathbf{E} - \mathbf{A}$. *The following statements are equivalent.*

1. *System* (2.3.1) *is asymptotically stable.*
2. *All finite eigenvalues of the matrix pencil* $\lambda\mathbf{E} - \mathbf{A}$ *lie in the open left half complex plane, i.e.,* $\sigma(\mathbf{E}, \mathbf{A}) \subset \mathbb{C}^-$, *where* $\mathbb{C}^- = \{s | s \in \mathbb{C}, \mathrm{Re}(s) < 0\}$ *represents the open left half complex plane.*

According to [62], the matrix pencil $\lambda\mathbf{E} - \mathbf{A}$ is called c-stable if it is regular and all the finite eigenvalues of $\lambda\mathbf{E} - \mathbf{A}$ have negative real part.

We can note that, in view of the above theorem, the infinite eigenvalues of the matrix pencil $(\mathbf{E}, \mathbf{A})$ have no effect on stability of DAEs of the form (2.3.1), since the infinite eigenvalues of $\lambda\mathbf{E} - \mathbf{A}$ do not affect the behavior of the homogenous system at infinity [62].

2.3.3 Weierstraß-Kronecker canonical form

In this Section, we present the Weierstraß-Kronecker canonical form. This is the most commonly used tool to understand the DAE structure of constant coefficients linear DAEs [16, 33, 42]. Scaling (2.3.1) by nonsingular matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$ and the state variable $\boldsymbol{x}$ according to $\boldsymbol{x} = \mathbf{Q}\tilde{\boldsymbol{x}}$ with a nonsingular matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$, we obtain

$$\tilde{\mathbf{E}}\,\dot{\tilde{\boldsymbol{x}}}(t) = \tilde{\mathbf{A}}\tilde{\boldsymbol{x}}(t) + \tilde{\mathbf{B}}\boldsymbol{u}(t), \tag{2.3.3a}$$

$$\boldsymbol{y}(t) = \tilde{\mathbf{C}}^{\mathrm{T}}\tilde{\boldsymbol{x}}(t), \tag{2.3.3b}$$

where $\tilde{\mathbf{E}} = \mathbf{PEQ}$, $\tilde{\mathbf{A}} = \mathbf{PAQ}$, $\tilde{\mathbf{B}} = \mathbf{PB}$ and $\tilde{\mathbf{C}} = \mathbf{Q}^T\mathbf{C}$, which is again a DAE with constant coefficients. According to [33], the relation $\boldsymbol{x} = \mathbf{Q}\tilde{\boldsymbol{x}}$ gives a one-to-one correspondence between the corresponding solution sets. This means that we can consider the transformed problem (2.3.3) instead of (2.3.1) in order to understand the underlying structure of constant coefficients linear DAEs. This leads to the following definition of equivalence [33].

Definition 2.3.4 ([33]) *Two matrix pairs* $(\mathbf{E}_i, \mathbf{A}_i)$, $\mathbf{E}_i, \mathbf{A}_i \in \mathbb{C}^{m\times n}$ *are called (strongly) equivalent if there exits nonsingular matrices* $\mathbf{P} \in \mathbb{R}^{m\times m}$ *and* $\mathbf{Q} \in \mathbb{R}^{n\times n}$ *such that*

$$\mathbf{E}_2 = \mathbf{P}\mathbf{E}_1\mathbf{Q}, \quad \mathbf{A}_2 = \mathbf{P}\mathbf{A}_1\mathbf{Q}. \tag{2.3.4}$$

If this is the case, we can write $(\mathbf{E}_1, \mathbf{A}_1) \sim (\mathbf{E}_2, \mathbf{A}_2)$.

As already suggested by the definition, relation (2.3.4) fixes an equivalence relation [33]. Thus, this relation poses reflexivity, transitivity and symmetry.

Lemma 2.3.1 ([33]) *The relation introduced in Definition* (2.3.4) *is an equivalence relation.*

Proof 2.3.1 ([33]) *We must show that the relation is reflexive, symmetric, and transitive.*
Reflexivity: *We have* $(\mathbf{E}, \mathbf{A}) \sim (\mathbf{E}, \mathbf{A})$ *by* $\mathbf{P} = \mathrm{I}_m$ *and* $\mathbf{Q} = \mathrm{I}_n$.
Symmetry: *From* $(\mathbf{E}_1, \mathbf{A}_1) \sim (\mathbf{E}_2, \mathbf{A}_2)$. *It follows that* $\mathbf{E}_2 = \mathbf{P}\mathbf{E}_1\mathbf{Q}$ *and* $\mathbf{A}_2 = \mathbf{P}\mathbf{A}_1\mathbf{Q}$ *with nonsingular matrices* $\mathbf{P}$ *and* $\mathbf{Q}$. *Hence, we have* $\mathbf{E}_1 = \mathbf{P}^{-1}\mathbf{E}_2\mathbf{Q}^{-1}$, $\mathbf{A}_1 = \mathbf{P}^{-1}\mathbf{A}_2\mathbf{Q}^{-1}$ *implying that* $(\mathbf{E}_2, \mathbf{A}_2) \sim (\mathbf{E}_1, \mathbf{A}_1)$.
Transitivity: *From* $(\mathbf{E}_1, \mathbf{A}_1) \sim (\mathbf{E}_2, \mathbf{A}_2)$ *and* $(\mathbf{E}_2, \mathbf{A}_2) \sim (\mathbf{E}_3, \mathbf{A}_3)$ *it follows that*
$\mathbf{E}_2 = \mathbf{P}_1\mathbf{E}_1\mathbf{Q}_1, \mathbf{A}_2 = \mathbf{P}_1\mathbf{A}_1\mathbf{Q}_1$ *and* $\mathbf{E}_3 = \mathbf{P}_2\mathbf{E}_2\mathbf{Q}_2, \mathbf{A}_3 = \mathbf{P}_2\mathbf{A}_2\mathbf{Q}_2$ *with nonsingular matrices* $\mathbf{P}_i, \mathbf{Q}_i, i = 1, 2$.

Having defined an equivalence relation, the standard procedure then is to look for a canonical form, i.e., to look for a matrix pair which is equivalent to a given matrix pair and which has a simple form from which we can directly read off the properties and invariants of the corresponding DAE [33]. In our case, such a canonical form is represented by the so-called Weierstraß-Kronecker canonical form. Here, we briefly discussed about Weierstraß-Kronecker canonical form but more details can be found in [16, 33, 52]. A special case which we want to discuss here in more detail and for which we want to derive the associated part of the Weierstraß-Kronecker canonical form is that of the so-called regular matrix pairs.

Lemma 2.3.2 ([33]) *Every matrix pair which is strongly equivalent to a regular matrix pair is regular.*

Proof 2.3.2 ([33]) *We only need to discuss square matrices. Let* $\mathbf{E}_2 = \mathbf{P}\mathbf{E}_1\mathbf{Q}$ *and* $\mathbf{A}_2 = \mathbf{P}\mathbf{A}_1\mathbf{Q}$ *with nonsingular matrices* $\mathbf{P}$ *and* $\mathbf{Q}$*. Using Definition 2.3.1 for regular matrix pairs, we have*

$$
\begin{aligned}
\mathcal{P}_2(\lambda) &= \det(\lambda\mathbf{E}_2 - \mathbf{A}_2) = \det(\lambda\mathbf{P}\mathbf{E}_1\mathbf{Q} - \mathbf{P}\mathbf{A}_1\mathbf{Q}),\\
&= \det(\mathbf{P})\det(\lambda\mathbf{E}_1 - \mathbf{A}_1)\det(\mathbf{Q}) = c\mathcal{P}_1(\lambda), \quad \textit{with } c \neq 0.
\end{aligned}
$$

Theorem 2.3.2 (Weierstraß-Kronecker canonical form [33,52]) *Let* $(\mathbf{E},\mathbf{A})$ *be a regular matrix pencil. Then, we have* $(\mathbf{E},\mathbf{A}) \sim \left(\begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{N}\end{pmatrix}, \begin{pmatrix}\mathbf{J} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}\right)$*, where* $\mathbf{J} \in \mathbb{R}^{k\times k}$ *for some nonnegative* $k \leq n$*, is a matrix in Jordan canonical form and* $\mathbf{N} \in \mathbb{R}^{(n-k)\times(n-k)}$ *is a nilpotent matrix with index* $\mu \leq n-k$ *also in Jordan canonical form. Moreover, it is allowed that one or the other block is not present.*

The proof of Theorem 2.3.2 can be found in [33]. The regular matrix pencil $\lambda\mathbf{E} - \mathbf{A}$ can be transformed into $\lambda\tilde{\mathbf{E}} - \tilde{\mathbf{A}}$, where

$$
\tilde{\mathbf{E}} = \mathbf{P}\mathbf{E}\mathbf{Q} = \begin{pmatrix}\mathbf{I}_{n_f} & \mathbf{0}\\ \mathbf{0} & \mathbf{N}\end{pmatrix}, \quad \tilde{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{Q} = \begin{pmatrix}\mathbf{J} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{n_\infty}\end{pmatrix}, \tag{2.3.5}
$$

by the use of suitable nonsingular matrices $\mathbf{P},\mathbf{Q} \in \mathbb{R}^{n\times n}$, where the block matrix $\mathbf{J} \in \mathbb{R}^{n_f\times n_f}$ corresponds to the finite eigenvalues and has the form [62]

$$
\mathbf{J} = \operatorname{diag}(\mathbf{J}_{1,1}, \mathbf{J}_{1,2}, \cdots, \mathbf{J}_{1,m_1}, \mathbf{J}_{2,1}, \cdots, \mathbf{J}_{2,m_2}, \cdots, \mathbf{J}_{k,1}, \cdots, \mathbf{J}_{k,m_k}),
$$

where $\mathbf{J}_{j,q} = \begin{pmatrix}\lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1\\ & & & \lambda_j\end{pmatrix}$ is the Jordan block of order $n_{j,q}$ with $\sum_{j=1}^{k}\sum_{q=1}^{m_j} n_{j,q} = n_f$ and λ_j is a finite eigenvalues of the matrix pencil $\lambda\mathbf{E} - \mathbf{A}$. According to [62], the number m_j

is called the *geometric multiplicity* of λ_j, the number $a_j = \sum_{q=1}^{m_j} n_{j,q}$ is called the *algebraic multiplicity* of λ_j and n_f is the dimension of the left and right deflating subspaces of $\lambda\mathbf{E} - \mathbf{A}$ corresponding to the finite eigenvalues. The definition of deflating subspaces of a matrix pencil can be found in [62]. The block matrix $\mathbf{N}$ in (2.3.5) corresponds to the eigenvalues at infinity of the pencil $\lambda\mathbf{E} - \mathbf{A}$ and has the form $\mathbf{N} = \mathrm{diag}(\mathbf{N}_{n_1}, \cdots, \mathbf{N}_{n_t})$, where $\mathbf{N}_{n_j} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$ is a nilpotent Jordan block of order n_j. The size of the largest nilpotent block, denoted by μ, is called the index of the matrix pencil $\lambda\mathbf{E} - \mathbf{A}$ or the index of the DAE (2.3.1a). This index concept is commonly called the Kronecker index [16,33,42,52,62]. We can clearly observe that $\mathbf{N}^{\mu-1} \neq 0$ and $\mathbf{N}^{\mu} = 0$. If the matrix $\mathbf{E}$ is nonsingular, then matrix pencil $\lambda\mathbf{E} - \mathbf{A}$ is of index zero ($\mu = 0$). According to [62], the matrix pencil $\lambda\mathbf{E} - \mathbf{A}$ is of index one if and only if it has exactly $\mathbf{n}_f = \mathrm{rank}(\mathbf{E})$ finite eigenvalues. We note that it is possible to have $n_f = 0$, meaning $\tilde{\mathbf{E}} = \mathbf{N}$, $\tilde{\mathbf{A}} = \mathbf{I}$ this implies that the spectrum of the matrix pencil $\lambda\mathbf{E} - \mathbf{A}$ has only infinite spectrum, i.e., $\sigma(\mathbf{E}, \mathbf{A}) = \sigma_\infty(\mathbf{E}, \mathbf{A})$. Also if $\mathbf{E}$ is nonsingular then $n_f = n$ which yields $\tilde{\mathbf{E}} = \mathbf{I}$, $\tilde{\mathbf{A}} = \mathbf{J}$, this implies that the spectrum of the matrix pencil $\sigma(\mathbf{E}, \mathbf{A}) = \sigma_f(\mathbf{E}, \mathbf{A})$ has only finite spectrum. Assuming the matrix pencil $\lambda\mathbf{E} - \mathbf{A}$ has both the finite and infinite spectrum, then the matrices $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ can be partitioned in blocks corresponding to the partitions of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{A}}$ given by $\tilde{\mathbf{B}} = \begin{pmatrix} \tilde{\mathbf{B}}_1^{\mathrm{T}} & \tilde{\mathbf{B}}_2^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$ and $\tilde{\mathbf{C}} = \begin{pmatrix} \tilde{\mathbf{C}}_1^{\mathrm{T}} & \tilde{\mathbf{C}}_2^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$. Under the coordinate transformation $\tilde{\boldsymbol{x}} = \mathbf{Q}^{-1}\boldsymbol{x} = (\tilde{\boldsymbol{x}}_1^{\mathrm{T}}(t), \tilde{\boldsymbol{x}}_2^{\mathrm{T}}(t))^{\mathrm{T}}$, system (2.3.3a) can be written as Weierstraß-Kronecker canonical form which leads to an equivalent decoupled system

$$\dot{\tilde{\boldsymbol{x}}}_1(t) = \mathbf{J}\tilde{\boldsymbol{x}}_1(t) + \tilde{\mathbf{B}}_1\boldsymbol{u}(t), \tag{2.3.6a}$$

$$\mathbf{N}\dot{\tilde{\boldsymbol{x}}}_2(t) = \tilde{\boldsymbol{x}}_2(t) + \tilde{\mathbf{B}}_2\boldsymbol{u}(t). \tag{2.3.6b}$$

We observe that (2.3.6a) represent a standard explicit ODE and without loss of generality the solution of (2.3.6b) can be written as

$$\tilde{\boldsymbol{x}}_2(t) = -\sum_{i=0}^{\mu-1} \mathbf{N}^i\tilde{\mathbf{B}}_2\boldsymbol{u}^{(i)}(t), \tag{2.3.7}$$

since $\mathbf{N}$ is a nilpotent matrix with index-μ, where $\boldsymbol{u}^{(i)}(t) = \frac{\mathrm{d}^i}{\mathrm{d}t}\boldsymbol{u}(t)$ provided $\mathbf{u}(t)$ is smooth enough, that is, at least $\mu - 1$ times differentiable. Equation (2.3.7) shows the dependence of the solution $\boldsymbol{x}(t)$ of (2.3.1a) on the derivatives of the input function $\mathbf{u}(t)$. We can observe that the higher the index-μ, the more differentiations are involved. It is only in the index-1 case where, we have $\mathbf{N} = \mathbf{0}$, hence $\tilde{\boldsymbol{x}}_2(t) = -\tilde{\mathbf{B}}_2\boldsymbol{u}(t)$, and no derivatives are involved. According to [34, 42], since numerical differentiations in these circumstances may cause considerably trouble numerically, it is very important to know the index-μ of the DAE (2.3.1a) as well as details on the structure responsible for a higher index ($\mu > 1$) when modeling and simulating with DAEs in practice. From (2.3.6), we can observe that the number of finite eigenvalues, n_f and infinite eigenvalues, n_∞ are equal to the number of differential and algebraic equations, respectively, in a given DAE. Thus, for the case of index-1 DAEs the number of differential equations is equal to the rank of singular matrix $\mathbf{E}$. We also note that the solutions $\tilde{\boldsymbol{x}}_1$ and $\tilde{\boldsymbol{x}}_2$ which corresponds to the differential and algebraic part are commonly known as the slow and fast solutions, respectively [12, 16, 33, 62]. We can also observe that the general solution of the homogeneous DAE (2.3.2), if matrix pencil $(\mathbf{E}, \mathbf{A})$ is regular, is of the form $\boldsymbol{x}(t) = \mathbf{Q}\begin{pmatrix} e^{-t\mathbf{J}} \\ \mathbf{0} \end{pmatrix}\tilde{\boldsymbol{x}}_1(0)$, $\tilde{\boldsymbol{x}}_1(0) \in \mathbb{R}^k$, this means that the solution space has dimension k according to [34]. We can easily prove that the differential part of (2.3.6) inherits the stability of DAEs of the form (2.3.1), that is, $\sigma_f(\mathbf{E}, \mathbf{A}) = \sigma(\mathbf{J})$.

Index concept of DAEs

An index of a DAE is commonly defined as the measure of the difficulties arising in the theoretical and numerical treatment of a given DAE. According to [33], the motivation to introduce an index is to classify different types of DAEs with respect to the difficulty to solve them analytically as well as numerically. Sometimes the index of a DAE is defined as a measure of how much the DAE deviates from an ODE. In the previous Section, we have defined index-μ as the nilpotency index of a nilpotent matrix $\mathbf{N}$. This index is also known as the Kronecker index of a DAE. They are many other index concepts that exist, see [8, 52, 61], but in this thesis we shall restrict ourselves to only three, i.e., Kronecker index, differentiation index and tractability index. We note that all these index concepts coincide for the case of linear DAEs with constant matrices. If we differentiate (2.3.7) with respect to t, we obtain: $\dot{\tilde{\boldsymbol{x}}}_2(t) = -\sum_{i=0}^{\mu-1} \mathbf{N}^i\tilde{\mathbf{B}}_2\boldsymbol{u}^{(i+1)}(t)$. This means that exactly μ differentiations are needed to transform (2.3.6) into a system of explicit ordinary differential equations. Hence, the Kronecker and the differentiation index coincide for LTI DAEs. This type of index is called the differentiation index and it is defined as in Definition 2.3.5. According to [33], the differentiation index was introduced to determine how far the DAE is away from an ODE, for which the analysis and numerical techniques are well established.

Definition 2.3.5 ([61]) *The nonlinear DAE, $\boldsymbol{F}(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) = \boldsymbol{0}$, has differentiation index γ, if γ is the minimal number of differentiations*

$$\boldsymbol{F}(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) = \boldsymbol{0},\ \frac{d}{dt}\Big(\boldsymbol{F}(t, \boldsymbol{x}, \dot{\boldsymbol{x}})\Big) = \boldsymbol{0}, \cdots, \frac{d^{\gamma}}{dt}\Big(\boldsymbol{F}(t, \boldsymbol{x}, \dot{\boldsymbol{x}})\Big) = \boldsymbol{0}, \tag{2.3.8}$$

in order to extract an explicit ordinary differentiation system $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$ using only algebraic manipulations.

This can be illustrated in the example below.

Example 2.3.1 Consider a semi-explicit DAE of the form,

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}), \\ \boldsymbol{0} &= \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}). \end{aligned} \tag{2.3.9}$$

Using chain rule on the constraint equation:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}), \\ \dot{\mathbf{y}} &= -\mathbf{g}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y})^{-1}\Big[\mathbf{g}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})\Big]. \end{aligned}$$

The DAE has a differentiation index $\gamma = 1$ provided $\det(\mathbf{g}_{\boldsymbol{y}}) \neq 0$.

In the next example, we compare the differentiation index and the Kronecker index.

Example 2.3.2 Consider a DAE of the form (2.3.1) with system matrices,

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{2.3.11}$$

The matrix pencil $(\mathbf{E}, \mathbf{A})$ is regular since $\det(\lambda\mathbf{E} - \mathbf{A}) = \lambda^2 + \lambda + 1 \neq 0$. Using Theorem 2.3.2, we can choose nonsingular matrices, $\mathbf{Q} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ such that $(\mathbf{E}, \mathbf{A}) \sim \left(\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{pmatrix}, \begin{pmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \right)$, where $\mathbf{J} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus this DAE is of Kronecker index-1, since $\mathbf{N}$ is of nilpotent index one. Next, we compute the differentiation index. This can be done as follows: We need to first rewrite system (2.3.11) in the semi-explicit form (2.3.9) where $\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_3 \end{pmatrix}$, $\boldsymbol{y} = \begin{pmatrix} \boldsymbol{x}_2 \\ \boldsymbol{x}_4 \end{pmatrix}$, $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = \begin{pmatrix} \boldsymbol{x}_2 \\ \boldsymbol{x}_1 \end{pmatrix}$ and $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) = \begin{pmatrix} -\boldsymbol{x}_1 + \boldsymbol{x}_4 \\ \boldsymbol{x}_2 + \boldsymbol{x}_3 + \boldsymbol{x}_4 \end{pmatrix}$. Thus, $\mathbf{g}_{\boldsymbol{y}} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Since $\det(\mathbf{g}_{\boldsymbol{y}}) = -1 \neq 0$, then DAE (2.3.11) has differentiation index-1. Hence the Kronecker index and differentiation index coincide, that is, $\mu = \gamma = 1$.

Consistent initial condition of DAEs

From system (2.3.6), we observe that (2.3.6a) is a linear differential equation which can easily be solved when an arbitrary initial condition $\tilde{\boldsymbol{x}}_1(0)$ is applied and its analytic solution can be written as:

$$\tilde{\boldsymbol{x}}_1(t) = \tilde{\boldsymbol{x}}_1(0)e^{t\mathbf{J}} + e^{t\mathbf{J}} \int_0^{\mathrm{T}} e^{-\tau\mathbf{J}} \tilde{\mathbf{B}}_1 \boldsymbol{u}(\tau)\, \mathrm{d}\tau. \tag{2.3.12}$$

We observe that the solution (2.3.12) of (2.3.6a) is always unique for any choice of the initial value $\tilde{\boldsymbol{x}}_1(0)$ while the initial value of (2.3.6b) has to satisfy the hidden constraint,

$$\tilde{\boldsymbol{x}}_2(0) = -\sum_{i=0}^{\mu-1} \mathbf{N}^i \tilde{\mathbf{B}}_2 \boldsymbol{u}^{(i)}(0). \tag{2.3.13}$$

We can observe that, we have no enough freedom to arbitrary choose the initial values $\tilde{\boldsymbol{x}}_2(0)$. For example, if $\mu = 1$, we have to choose the initial value such that
$\tilde{\boldsymbol{x}}_2(0) = -\tilde{\mathbf{B}}_2\boldsymbol{u}(0)$. For $\mu > 1$, equation (2.3.13) is a differentiation problem, thus the initial value $\tilde{\boldsymbol{x}}_2(0)$ is fixed, and the input function $\boldsymbol{u}(t)$ has to be at least $\mu - 1$ times differentiable, i.e., $\boldsymbol{u}(t) \in C^{\mu-1}$. The initial value problems for (2.3.1) lead to unique classical solutions if the initial value $\boldsymbol{x}(0) = \boldsymbol{x}_0$ is consistent, that is

$$\boldsymbol{x}(0) = \mathbf{Q}\tilde{\mathbf{x}}(0) = \mathbf{Q}\left(\tilde{\boldsymbol{x}}_1(0)^{\mathrm{T}}\ \tilde{\boldsymbol{x}}_2(0)^{\mathrm{T}}\right)^{\mathrm{T}}, \tag{2.3.14}$$

where $\tilde{\boldsymbol{x}}_1(0)$ is a free parameter while $\tilde{\boldsymbol{x}}_2(0)$ has to satisfy (2.3.13). Thus $\boldsymbol{x}(0)$ must be a consistent initial condition of the DAE (2.3.1a). We note that, if the initial condition $\boldsymbol{x}_0$ is inconsistent or the input function $\boldsymbol{u}(t)$ is not sufficiently smooth, then the solution of a DAE (2.3.1a) may have impulsive modes, see [12, 62].

2.3.4 Transfer matrix representation of the Kronecker form

Using (2.3.6) and the decomposed output equation, the control problem (2.3.1) can also be written in equivalent form

$$\dot{\tilde{\boldsymbol{x}}}_1(t) = \mathbf{J}\tilde{\boldsymbol{x}}_1(t) + \tilde{\mathbf{B}}_1\boldsymbol{u}(t), \tag{2.3.15a}$$

$$\tilde{\boldsymbol{x}}_2(t) = -\sum_{i=0}^{\mu-1}\mathbf{N}^i\tilde{\mathbf{B}}_2\boldsymbol{u}^{(i)}(t), \tag{2.3.15b}$$

$$\boldsymbol{y}(t) = \tilde{\mathbf{C}}_1^{\mathrm{T}}\tilde{\boldsymbol{x}}_1(t) + \tilde{\mathbf{C}}_2^{\mathrm{T}}\tilde{\boldsymbol{x}}_2(t). \tag{2.3.15c}$$

Taking the Laplace transform of (2.3.15) and using the fact that

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - \sum_{k=1}^{n} s^{k-1}f^{(n-k)}(0),$$

and simplifying we obtain,

$$\begin{aligned}\boldsymbol{Y}(s) = {} & \tilde{\mathbf{C}}_1^{\mathrm{T}}(s\mathrm{I}-\mathbf{J})^{-1}\tilde{\mathbf{B}}_1\mathbf{U}(s) + \tilde{\mathbf{C}}_1^{\mathrm{T}}(s\mathrm{I}-\mathbf{J})^{-1}\tilde{\boldsymbol{x}}_1(0) + \\ & -\tilde{\mathbf{C}}_2^{\mathrm{T}}\sum_{i=0}^{\mu-1}\mathbf{N}^i\tilde{\mathbf{B}}_2[s^i\mathbf{U}(s) - \sum_{k=1}^{i}s^{k-1}\mathbf{u}^{(i-k)}(0)].\end{aligned} \tag{2.3.16}$$

Recall from Section 2.2.2, in order to obtain the transfer matrix representation we need to assume vanishing initial data $\tilde{\boldsymbol{x}}(0)$ for the case of ODEs. Here comes the answer to our question wether it always possible to set initial data to zero also for case of DAEs. This is discussed as follows. We have already discussed that for the case of DAEs, we always need to apply consistent initial data $\tilde{\boldsymbol{x}}(0) = \left(\tilde{\boldsymbol{x}}_1(0), \tilde{\boldsymbol{x}}_2(0)\right)^{\mathrm{T}}$, where $\tilde{\boldsymbol{x}}_1(0)$ can be chosen arbitrary, thus we can set $\tilde{\boldsymbol{x}}_1(0) = 0$ while $\tilde{\boldsymbol{x}}_2(0)$ has to satisfy some constraint equation (2.3.13) which depends on the smoothness of the input vector $\boldsymbol{u}(t)$. Setting $\tilde{\boldsymbol{x}}_1(0) = 0$, simplifies (2.3.16) to,

$$\boldsymbol{Y}(s) = [\mathbf{H}_1(s) + \mathbf{H}_2(s)]\mathbf{U}(s) + \tilde{\mathbf{C}}_2^{\mathrm{T}} \sum_{i=0}^{\mu-1} \mathbf{N}^i \tilde{\mathbf{B}}_2 \sum_{k=1}^{i} s^{k-1} \mathbf{u}^{(i-k)}(0), \tag{2.3.17}$$

where $\mathbf{H}_1(s) = \tilde{\mathbf{C}}_1^{\mathrm{T}}(s\mathrm{I} - \mathbf{J})^{-1}\tilde{\mathbf{B}}_1$ and $\mathbf{H}_2(s) = -\tilde{\mathbf{C}}_2^{\mathrm{T}} \sum_{i=0}^{\mu-1} \mathbf{N}^i \tilde{\mathbf{B}}_2 s^i$.

Lemma 2.3.3 ([16]) *Two systems with matrix coefficients* $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$ *and* $(\tilde{\mathbf{E}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$ *whose matrix pairs* $(\mathbf{E}, \mathbf{A})$ *and* $(\tilde{\mathbf{E}}, \tilde{\mathbf{A}})$*, respectively, are (strongly) equivalent. Their transfer functions must coincide.*

Proof 2.3.3 *Let* $\tilde{\mathbf{E}} = \mathbf{PEQ}$, $\tilde{\mathbf{A}} = \mathbf{PAQ}$, $\tilde{\mathbf{B}} = \mathbf{PB}$ *and* $\tilde{\mathbf{C}} = \mathbf{Q}^{\mathrm{T}}\mathbf{C}$ *with nonsingular* $\mathbf{P}$ *and* $\mathbf{Q}$. *Then,*

$$\begin{aligned}
\tilde{\mathbf{H}}(s) &= \tilde{\mathbf{C}}^{\mathrm{T}}(s\tilde{\mathbf{E}} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{B}} = \mathbf{C}^{\mathrm{T}}\mathbf{Q}(s\mathbf{PEQ} - \mathbf{PAQ})^{-1}\mathbf{PB}, \\
&= \mathbf{C}^{\mathrm{T}}\mathbf{QQ}^{-1}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{P}^{-1}\mathbf{PB}, \\
&= \mathbf{C}^{\mathrm{T}}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} = \mathbf{H}(s).
\end{aligned}$$

From the above Lemma, $\tilde{\mathbf{H}}(s) = \mathbf{H}_1(s) + \mathbf{H}_2(s)$ coincides with the conventional definition of the transfer function $\mathbf{H}(s) = \mathbf{C}^{\mathrm{T}}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$, since (2.3.1) and (2.3.15) are equivalent systems, i.e., $\tilde{\mathbf{H}}(s) = \mathbf{H}(s)$. Thus, the input-output function (2.3.17) can be written as $\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s) + \mathcal{P}(s)$, where $\mathcal{P}(s) := \tilde{\mathbf{C}}_2^{\mathrm{T}} \sum_{i=0}^{\mu-1} \mathbf{N}^i \tilde{\mathbf{B}}_2 \sum_{k=1}^{i} s^{k-1} \mathbf{u}^{(i-k)}(0)$. This means that assuming $\mathbf{E}\boldsymbol{x}(0) = 0$ implies $\tilde{\boldsymbol{x}}_1(0) = 0$ and $\mathbf{N}\tilde{\boldsymbol{x}}_2(0) = 0$ as a result the hidden polynomial $\mathcal{P}(s)$ is forced to zero. But, we can observe that this polynomial contains some parts of the DAE which might be vital. Thus assuming $\mathbf{E}\boldsymbol{x}(0) = 0$ may lead to loss of important information of the DAE especially for higher index ($\mu > 1$) DAEs. For the case of index-1 DAE, the nilpotent matrix $\mathbf{N} = 0$, thus $\mathbf{N}\tilde{\boldsymbol{x}}_2(0) = 0$ always and the polynomial $\mathcal{P}(s)$ does not exist. Thus, for index-1 DAE assuming $\mathbf{E}\boldsymbol{x}(0) = 0$ has no effect on the conventional MOR methods. However for higher index DAEs, the nilpotent $\mathbf{N} \neq \mathbf{0}$, thus $\mathbf{N}\tilde{\boldsymbol{x}}_2(0) \neq \mathbf{0}$ and the polynomial $\mathcal{P}(s)$ exists and $\mathcal{P}(s) \neq 0$, apart from some special cases when $\tilde{\mathbf{B}}_2 = 0$. We can observe that (2.3.17) can also be written in decomposed form as

$$\boldsymbol{Y}(s) = \boldsymbol{Y}_1(s) + \boldsymbol{Y}_2(s),$$

where $\boldsymbol{Y}_1(s) = \mathbf{H}_1(s)\mathbf{U}(s)$ and $\boldsymbol{Y}_2(s) = \mathbf{H}_2(s)\mathbf{U}(s) + \mathcal{P}(s)$ represent the input-output function of the differential and algebraic parts, respectively. We note that the $\mathbf{H}_1(s)$ and $\mathbf{H}_2(s)$ are commonly called the strictly proper and the polynomial parts of $\mathbf{H}(s)$, respectively, see [25, 62]. We observe that the input-output relation of the differential part is given by $\boldsymbol{Y}_1(s) = \mathbf{H}_1(s)\mathbf{U}(s)$, where $\mathbf{H}_1(s)$ is its transfer matrix representation and it is independent of the index-μ of the DAE (2.3.1a) while input-output relation of the algebraic part is given by $\boldsymbol{Y}_2(s) = \mathbf{H}_2(s)\mathbf{U}(s) + \mathcal{P}(s)$ which depends on the index-μ of the DAE (2.3.1a). This explains why the conventional MOR techniques based on the assumption that $\mathbf{E}\boldsymbol{x}(0) = 0$ can only be used on index-1 DAEs but become cumbersome for higher index DAEs. This is illustrated with some numerical examples in Section 3.2.1. Hence, the best way to apply model order reduction on DAEs is to first split the control DAE (2.3.1) into differential and algebraic parts. Then, apply reduction on the two parts separately. According to [39, 52], transforming (2.3.1) into a Kronecker canonical form is just in theory, but practical implementation may be difficult or impossible. This is due to the fact that computing the Kronecker canonical form in finite precision arithmetic is, in general, an ill-conditioned problem in the sense that small changes in the data may extremely change the Kronecker canonical form [62]. Proper formulations and projector methods attempt to overcome these drawbacks, allowing additionally for an extension of the results to the time varying context [52]. These techniques provide an index characterization in terms of the original problem description. This motivated us to use the projector and matrix chain approach in order to decompose DAEs into differential and algebraic parts as introduced by März in [42]. This is discussed in Chapter 4.

2.4 Real-life applications of DAEs

DAEs appear in many fields as mentioned earlier. In this Section, we present some of the applications of DAEs in the real-world. After modeling these applications, they lead to DAEs of the form (2.3.1a) with singular matrix $\mathbf{E}$, since some of the rows are always zeros as illustrated in examples below.

2.4.1 Electrical network problems

Many electrical circuit systems can be described by DAEs. This is due to the fact that, the most commonly used method in electrical circuit networks design is the modified nodal analysis (MNA). This approach leads a DAE when modeling a network involving resistor networks such as RLC network, i.e., Resistor-Inductor-Capacitor network, RC network, i.e., Resistor-Capacitor network, RL network, i.e., Resistor-Inductor network and so on, see [28, 43, 64]. For illustration, we consider only RLC and RC networks.

(i) **RLC network.** Consider a linear RLC electric network, that is, a network which connects linear capacitors, inductors and resistors, and current sources, $\boldsymbol{v}(t) \in \mathbb{R}^{n_V}$ and $\boldsymbol{\imath}(t) \in \mathbb{R}^{n_I}$. The unknowns which describe the network are the node potentials $\boldsymbol{e}(t) \in \mathbb{R}^n$, and the currents through inductors $\boldsymbol{\jmath}_L(t) \in \mathbb{R}^{n_L}$.

Following the formalism of modified nodal analysis [28, 43], we introduce: the incidence matrices $\boldsymbol{A}_C \in \mathbb{R}^{n\times n_C}$, $\boldsymbol{A}_L \in \mathbb{R}^{n\times n_L}$ and $\boldsymbol{A}_R \in \mathbb{R}^{n\times n_G}$, which describe the branch-node relationships for capacitors, inductors and resistors; the incidence matrices $\boldsymbol{A}_V \in \mathbb{R}^{n\times n_V}$ and $\boldsymbol{A}_I \in \mathbb{R}^{n\times n_I}$, which describe this relationship for voltage and current sources, respectively. Then, the DAE model for the RLC network with unknown $\boldsymbol{x} = (\boldsymbol{e}, \boldsymbol{\jmath}_L, \boldsymbol{\jmath}_V)^{\mathrm{T}}$ is given by [5]

$$\begin{pmatrix} \boldsymbol{A}_C\boldsymbol{C}\boldsymbol{A}_C^{\mathrm{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \frac{d\boldsymbol{x}}{dt} = \begin{pmatrix} -\boldsymbol{A}_R\boldsymbol{\mathcal{G}}\boldsymbol{A}_R^{\mathrm{T}} & -\boldsymbol{A}_L & -\boldsymbol{A}_V \\ \boldsymbol{A}_L^{\mathrm{T}} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{A}_V^{\mathrm{T}} & \mathbf{0} & \mathbf{0} \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} -\boldsymbol{A}_I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathrm{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{\imath} \\ \boldsymbol{v} \end{pmatrix}. \tag{2.4.1}$$

where $\boldsymbol{C} \in \mathbb{R}^{n_C,n_C}$, $\mathbf{L} \in \mathbb{R}^{n_L,n_L}$ and $\boldsymbol{\mathcal{G}} \in \mathbb{R}^{n_G,n_G}$ are the capacitance, inductance and conductance matrices, respectively which are usually assumed to be symmetric and positive definite.

(ii) **RC network.** The RC model can be derived from that of the RLC model (2.4.1) by simply eliminating the inductor currents $\boldsymbol{\jmath}_L$. Then, the DAE model of the RC network with unknown $\boldsymbol{x} = (\boldsymbol{e}, \boldsymbol{\jmath}_V)^{\mathrm{T}}$ is given by

$$\begin{pmatrix} \boldsymbol{A}_C\boldsymbol{C}\boldsymbol{A}_C^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \frac{d\boldsymbol{x}}{dt} = \begin{pmatrix} -\boldsymbol{A}_R\boldsymbol{G}\boldsymbol{A}_R^{\mathrm{T}} & -\boldsymbol{A}_V \\ \boldsymbol{A}_V^{\mathrm{T}} & \mathbf{0} \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} -\boldsymbol{A}_I & \mathbf{0} \\ \mathbf{0} & -\mathrm{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{\imath} \\ \boldsymbol{v} \end{pmatrix}. \tag{2.4.2}$$

We can observe that (2.4.1) and (2.4.2) are DAEs of the form (2.3.1a).

2.4.2 Computational fluid dynamics problems

(i) **Supersonic Inlet flow example.** This example originates from [35]. Consider an unsteady flow through a supersonic diffuser as shown in Figure 2.1. The diffuser operates at a nominal Mach number of 2.2, however it is subject to perturbations in the incoming flow, which may be due to atmospheric variations. In nominal operation, there is a strong shock downstream of the diffuser throat, as can be seen from the Mach contours plotted in Figure 2.1. Incoming disturbances can cause the shock to move forward towards the throat. When the shock sits at the throat, the inlet is unstable, since any disturbance that moves the shock slightly upstream will cause it to move forward rapidly, leading to unstart of the inlet. This is extremely undesirable, since unstart results in a large loss of thrust. In order to prevent unstart from occurring, one option is to actively control the position of the shock. This control may be effected through flow bleeding upstream of the diffuser throat. In order to derive effective active control strategies, it is imperative to have low-order models which accurately capture the relevant dynamics. Figure 2.2

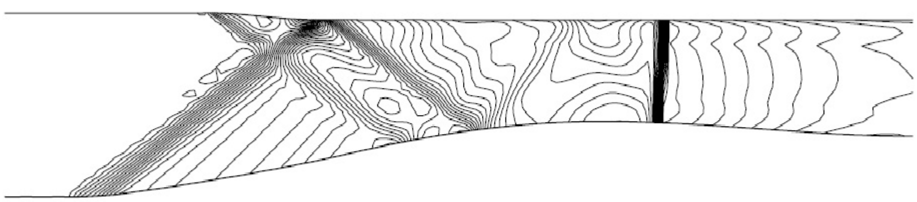

Figure 2.1: Steady-state mach contours inside diffuser.

presents the schematic of the actuation mechanism. Incoming flow with possible disturbances enters the inlet and is sensed using pressure sensors. The controller then adjusts the bleed upstream of the throat in order to control the position of the shock and to prevent it from moving upstream. In simulations, it is difficult to automatically determine the shock location. The average Mach number at the diffuser throat provides an appropriate surrogate that can be easily computed. There are several transfer functions of interest in this problem. The shock position will be controlled by monitoring the average Mach number at the diffuser throat. The reduced-order model must capture the dynamics of this output in response to two inputs: the incoming flow disturbance and the bleed actuation. In addition, total pressure measurements at the diffuser wall are used for sensing. The response of this output to the two inputs must also be captured. This problem is modeled using

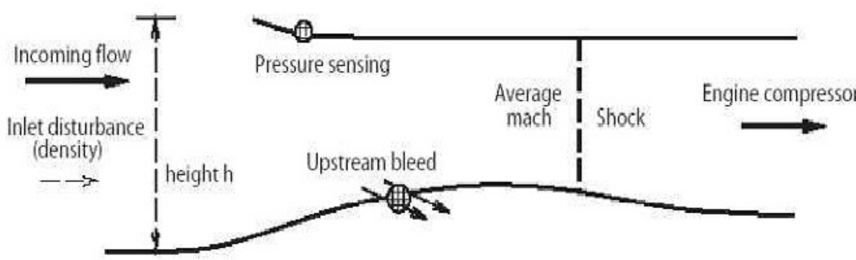

Figure 2.2: Supersonic diffuser active flow control problem setup

an unsteady, two-dimensional flow of an inviscid, compressible fluid which is governed by the Euler equations. The two-dimensional integral Euler equations are linearized about the steady state solution to obtain a semi-explicit DAE of index-1 of the form (2.3.1) with system matrices

$$\mathbf{E} = \begin{pmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \; \mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}, \; \mathbf{B} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix}, \; \mathbf{C} = \begin{pmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{pmatrix}, \; \boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix}, \tag{2.4.3}$$

where $\mathbf{E}_{11} \in \mathbb{R}^{n_1 \times n_1}$ and $\mathbf{A}_{21}\mathbf{E}_{11}^{-1}\mathbf{E}_{12} - \mathbf{A}_{22} \in \mathbb{R}^{n_2 \times n_2}$ are nonsingular matrices due to index-1 property and $n = n_1 + n_2$ is the dimension of the DAE.

(ii) **Semidiscretized Stokes equation**. In this Section, we present the semidiscretized Stokes equation originating from [45]. Consider the instationary Stokes equation describing the flow of an incompressible fluid

$$\begin{aligned} \frac{\partial \upsilon}{\partial t} &= \Delta\upsilon - \nabla \boldsymbol{p} + \boldsymbol{f}, \quad (\zeta, t) \in \Omega \times (0, \mathrm{T}) \\ 0 &= \mathrm{div}\upsilon, \end{aligned} \tag{2.4.4}$$

with appropriate initial condition and boundary condition. Here $\upsilon(\zeta, t) \in \mathbb{R}^d$ is the velocity vector (d = 2 or 3 is the dimension of the spatial domain), $\boldsymbol{p}(\zeta, t) \in \mathbb{R}$ is the pressure, $\boldsymbol{f}(\zeta, t) \in \mathbb{R}^d$ is the vector of external forces, $\Omega \in \mathbb{R}^d$ is a bounded open domain and $\mathrm{T} > 0$ is the endpoint of the time interval. The spatial discretization of the Stokes equation (2.4.4) by either the finite difference or finite element methods on a uniform staggered grid leads to a DAE of the form (2.3.1) with system matrices;

$$\mathbf{E} = \begin{pmatrix} \mathbf{E}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \; \mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^{\mathrm{T}} & \mathbf{0} \end{pmatrix}, \; \mathbf{B} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix}, \; \mathbf{C} = \begin{pmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{pmatrix}, \; \boldsymbol{x} = \begin{pmatrix} \upsilon_h \\ \boldsymbol{p}_h \end{pmatrix}, \tag{2.4.5}$$

where $\upsilon_h \in \mathbb{R}^{n_1}$ and $\boldsymbol{p}_h \in \mathbb{R}^{n_2}$ are the semidiscretized vectors of velocity and pressure, respectively, see [45]. The matrix $\mathbf{E}_{11} \in \mathbb{R}^{n_1 \times n_1}$ is a nonsingular matrix, but for this case $\mathbf{E}_{11} = \mathrm{I}$, $\mathbf{A}_{11} \in \mathbb{R}^{n_1 \times n_1}$ is the discrete Laplace operator, $-\mathbf{A}_{12} \in \mathbb{R}^{n_1 \times n_2}$ and $-\mathbf{A}_{12}^{\mathrm{T}} \in \mathbb{R}^{n_2 \times n_1}$ are, the discrete gradient and divergence operators, respectively. Due to the non-uniqueness of the pressure, the matrix $\mathbf{A}_{12}$ has a rank defect one. In this case, instead of $\mathbf{A}_{12}$ we can take a full column rank matrix obtained from $\mathbf{A}_{12}$ by discarding the last column. Therefore, in the following we will assume without loss of generality that $\mathbf{A}_{12}$ has full column rank. In this case system with matrix coefficients (2.4.5) is of index-2. The matrices $\mathbf{B}_1 \in \mathbb{R}^{n_1 \times m}$, $\mathbf{B}_2 \in \mathbb{R}^{n_2 \times m}$ and the control input $\boldsymbol{u}(t) \in \mathbb{R}^m$ are the resulting from the boundary condition and external forces, the output $\boldsymbol{y}(t) \in \mathbb{R}^{\ell}$ is the vector of interest. The order $n = n_1 + n_2$ of system (2.4.5) depends on the level of refinement of the discretization and is usually very large, whereas the number m of inputs and the number ℓ of outputs are typically small.

2.4.3 Constrained mechanical problems

This example originates from [45]. We consider the holonomically constrained damped mass-spring system as illustrated in Figure 2.3. The ith mass of weight m_i is connected to the $(i+1)st$ mass by a spring and a damper with constants k_i and d_i, respectively, and also to the ground by a spring and a damper with constants k_i and δ_i, respectively. Additionally, the first mass is connected to the last one by a rigid bar and it is influenced by the control $\boldsymbol{u}(t)$. The vibration of this system is described by a DAE of the form (2.3.1) with system matrices

$$\mathbf{E} = \begin{pmatrix} \mathrm{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \; \mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathrm{I} & \mathbf{0} \\ \mathbf{K} & \mathbf{D} & -\mathbf{G}^{\mathrm{T}} \\ \mathbf{G} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \; \mathbf{B} = \begin{pmatrix} \mathbf{0} \\ \mathbf{B}_2 \\ \mathbf{0} \end{pmatrix}, \; \mathbf{C} = \begin{pmatrix} \mathbf{C}_1 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \; \boldsymbol{x}(t) = \begin{pmatrix} \boldsymbol{p}(t) \\ \boldsymbol{v}(t) \\ \lambda(t) \end{pmatrix}, \tag{2.4.6}$$

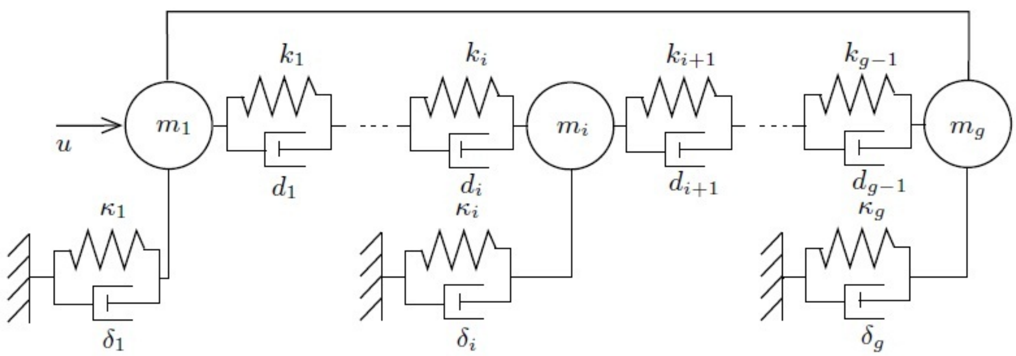

Figure 2.3: A damped mass-spring system with a holonomic constraint.

where $\boldsymbol{p}(t) \in \mathbb{R}^g$ is the position vector, $\boldsymbol{v}(t) \in \mathbb{R}^g$ is the velocity vector, $\boldsymbol{\lambda}(t) \in \mathbb{R}$ is the Lagrange multiplier, $\mathbf{M} = \mathrm{diag}(m_1, \cdots, m_g)$ is the mass matrix, $\mathbf{D}$ and $\mathbf{K}$ are the tridiagonal damping and stiffness matrices, $\mathbf{G} = [1, 0, \cdots, 0, -1] \in \mathbb{R}^{1\times g}$ is the constraint matrix, $\mathbf{B}_2 = \boldsymbol{e}_1$ and $\mathbf{C}_1 = [\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_{g-1}]^{\mathrm{T}}$. Here $\boldsymbol{e}_i$ denotes the ith column of the identity matrix I_g. Thus the system is of dimension $n = 2g + 1$. According [45], system (2.4.6) is of index-3 since $\mathbf{G}$ is a full row rank.

Chapter 3

Model Order Reduction

In this Chapter, we introduce the Model Order Reduction (MOR) or Model Reduction. In simple words, model order reduction can be defined as a mathematical theory to replace a given mathematical model of a control system or a control process by a model that is much smaller than the original model, but still describes at least approximately certain aspects of the system or process, see [16]. This is normally achieved by preserving the input-output relationship commonly knows as the transfer function. These system of equations can sometimes be ODEs or DAEs. However model order reduction, was mainly developed for ODEs, this is the reason why many methods reduce mainly linear ODE systems and very few methods can reduce either algebraic equations or DAEs, see [3, 9, 48, 58]. Hence, model order reduction techniques for DAEs are a lot less developed and less well understood than the ODE ones. MOR techniques for ODEs are often applied to DAEs [19, 49], which may lead to inaccurate reduced-order models or reduced-order models which are very difficult to solve numerically, see [1, 2]. This is also illustrated in Example 3.2.1 and 3.2.2.

3.1 Introduction

Consider a LTI system

$$\mathbf{E}\dot{\boldsymbol{x}}(t) = \mathbf{A}\boldsymbol{x}(t) + \mathbf{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{3.1.1a}$$

$$\boldsymbol{y}(t) = \mathbf{C}^{\mathrm{T}}\boldsymbol{x}(t), \tag{3.1.1b}$$

where $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{n\times \ell}$, the input vector $\boldsymbol{u}(t) \in \mathbb{R}^m$ and output vector $\boldsymbol{y}(t) \in \mathbb{R}^\ell$ of the system. $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector and $\boldsymbol{x}_0$ is the initial value. The number of state variables n is called the order of system or the state-space dimension. m and ℓ are the number of inputs and outputs, respectively. If $\mathrm{I} = \mathbf{E}$, then (3.1.1) is a standard state space system. Otherwise, (3.1.1) is a descriptor system or generalized state space system, see [44]. According to [45], modeling of complex physical and technical process such as fluid flow, very large system integrated (VLSI) chip design or mechanical systems simulation, leads to descriptor systems of very large order n, while the number m of inputs and the number ℓ of outputs are typically small compared to n. Despite the ever increasing computational power, simulation of these systems in real- time for such large scale is very difficult because of the storage requirements and expensive computations. This is an attractive feature to apply model order reduction. This is done by replacing (3.1.1) by a reduced-order model

$$\begin{aligned} \mathbf{E}_r\dot{\boldsymbol{x}}_r(t) &= \mathbf{A}_r\boldsymbol{x}_r(t) + \mathbf{B}_r\mathbf{u}(t) \\ \boldsymbol{y}_r(t) &= \mathbf{C}_r\boldsymbol{x}_r(t), \end{aligned} \tag{3.1.2}$$

where $\mathbf{E}_r, \mathbf{A}_r \in \mathbb{R}^{r\times r}$, $\mathbf{B}_r \in \mathbb{R}^{r\times m}$, $\mathbf{C}_r \in \mathbb{R}^{r\times \ell}$ and the reduced dimension $r \ll n$. In model order reduction, we require the reduced-order model (3.1.2) to preserve mathematical properties of the original model (3.1.1) such as regularity, stability and passivity. It is also desired for the approximation error $\|\boldsymbol{y} - \boldsymbol{y}_r\|$ to be small, in a suitable norm and the computation of the reduced-order system should be numerically reliable and more efficient than the original model. For higher index DAEs, we may also need to preserve the index. The biggest challenge in model order reduction is to measure the quality of the reduced-order models, the commonly used concept is the concept of the use of the transfer function $\mathbf{H}(s)$. In the next Section, we shall see that the transfer function being accurate does not mean that the output solutions are also accurate for the case of higher index DAEs. Hence, we require both the approximation error $\|\mathbf{H} - \mathbf{H}_r\|$ and $\|\boldsymbol{y} - \boldsymbol{y}_r\|$ to be small in the suitable norm.

3.2 Conventional MOR methods

In this Section, we discuss about the conventional or traditional MOR methods. By conventional MOR methods, we mean those reduction techniques originally developed to reduce ODEs. In [45], conventional MOR methods are called model reduction approaches for standard state space systems. These approaches include balanced truncation [3, 63], moment matching approximation [19, 49, 58], singular perturbation approximation [37] and optimal Hankel norm approximation [3]. Model order reduction approaches can be divided into two basic methods: Krylov subspace based methods or moment-matching methods such as PRIMA [49], SPRIM [19] and the singular value decomposition (SVD) based methods such as the balanced truncation method [45]. Overview of these methods can be found in [3, 9, 48, 58] and also for some special applications in [29, 38, 65]. Traditionally these methods were developed for ODE dynamical systems in standard state space system, i.e., $\mathbf{E} = \mathrm{I}$. It is just recent that MOR methods have been developed to reduce systems in descriptor form or generalized state space system with $\mathbf{E}$ nonsingular or singular, see [19, 29, 38, 45, 49]. However, little effort has been made to develop MOR methods specifically for DAEs, i.e., $\mathbf{E}$ is singular. What is currently been done is to just replace I with $\mathbf{E}$ in the conventional MOR method especially for the Krylov subspace based methods, see [19, 49], but this is not always lead to good reduced-order model especially with DAEs with index great than one [2]. For the case of the SVD based methods especially the balanced truncation method, this problem has already been noticed and solved, see [45], however the computations are too expensive and much restricted on DAEs with special structures. In this thesis, we focus more on the Krylov subspace based methods. This is discussed as follows. MOR techniques based on Krylov subspace methods also known as moment matching techniques aim at generating a reduced-order model which preserves a reasonable number of moments of the transfer function of the original model. This is done by using projection methods. There is a large variety of projection methods such as the Lanczos-type and the Arnoldi processes, but we shall restrict ourselves on Arnoldi process commonly known as PRIMA method [49]. PRIMA method's main features are provably passive reduced-order models and a Padé-type approximation property [19]. It employs a block version of the Arnoldi process. This is done as follows. We choose an arbitrary $s_0 \in \mathbb{C}$ as the expansion point such that the matrix pencil $s_0\mathbf{E}-\mathbf{A}$ is regular. In practice, s_0 is chosen such that it is in some sense close to the frequency range of interest. The frequency range of interest is usually a subset of the imaginary axis in the complex s-plane. In order to use the Krylov subspace techniques, we need to rewrite the transfer $\mathbf{H}(s)$ of the original system (3.1.1) using the identity: $\mathbf{H}(s) = \mathbf{C}^{\mathrm{T}}(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{B} = \mathbf{C}^{\mathrm{T}}\Big[\mathrm{I}+(s-s_0)\mathbf{M}\Big]^{-1}\mathbf{R}$, where $\mathbf{M} = \mathbf{M}(s_0) = (s_0\mathbf{E}-\mathbf{A})^{-1}\mathbf{E}$ and $\mathbf{R} = \mathbf{R}(s_0) = (s_0\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}$. The function $\mathbf{H}(s)$ admits the Taylor expansion

$$\mathbf{H}(s) = \mathbf{h}^{(0)} + \mathbf{h}^{(1)}(s-s_0) + \mathbf{h}^{(2)}(s-s_0)^2 + \cdots + \mathbf{h}^{(j)}(s-s_0)^j + \cdots$$

about s_0. The coefficients $\mathbf{h}^{(j)}$, $j = 0, 1, \ldots$, are called the moments of the transfer function of system (3.1.1) about the expansion point s_0. These moments can be constructed as follows. Using the Neumann expansion [23]: $(\mathrm{I}-\eta\mathbf{G})^{-1} = \sum_{k=0}^{\infty}(\eta\mathbf{G})^k$, where $\mathbf{G}$ is a square matrix and $\eta > 0$. Then we have

$$\mathbf{H}(s) = \sum_{k=0}^{\infty}\mathbf{h}^{(k)}(s-s_0)^k, \tag{3.2.1}$$

where $\mathbf{h}^{(k)} = (-1)^k\mathbf{C}^{\mathrm{T}}\mathbf{M}^k\mathbf{R}$, $k = 0, 1, \ldots$ defines the moments of the transfer function around the expansion point s_0. We then consider the order-r Krylov subspace generated by $\mathbf{M}$ and $\mathbf{R}$ given by $\mathcal{K}_r(\mathbf{M},\mathbf{R}) = \mathrm{span}\{\mathbf{R},\mathbf{MR},\ldots,\mathbf{M}^{r-1}\mathbf{R}\}$, $r \leq n$, and denote by $\mathbf{V} \in \mathbb{R}^{n\times rm}$ the matrix of an orthonormal basis for $\mathcal{K}_r(\mathbf{M},\mathbf{R})$, so that $\mathbf{V}^{\mathrm{T}}\mathbf{V} = \mathrm{I}_{rm}$. Then we seek an approximate solution of the form $\mathbf{x} = \mathbf{V}\mathbf{x}_r$ that is,

$$\mathbf{V}^{\mathrm{T}}\mathbf{E}\mathbf{V}\dot{\mathbf{x}}_r(t) = \mathbf{V}^{\mathrm{T}}\mathbf{A}\mathbf{V}\mathbf{x}_r(t) + \mathbf{V}^{\mathrm{T}}\mathbf{B}\mathbf{u}(t), \quad \mathbf{x}_r(0) = \mathbf{V}^{\mathrm{T}}\mathbf{x}_0 \tag{3.2.2a}$$

$$\mathbf{y}_r(t) = \mathbf{C}^{\mathrm{T}}\mathbf{V}\mathbf{x}_r(t). \tag{3.2.2b}$$

From (3.2.2), the matrices of the reduced-order model (3.1.2) are given by

$$\mathbf{E}_r = \mathbf{V}^{\mathrm{T}}\mathbf{E}\mathbf{V},\ \mathbf{A}_r = \mathbf{V}^{\mathrm{T}}\mathbf{A}\mathbf{V},\ \mathbf{B}_r = \mathbf{V}^{\mathrm{T}}\mathbf{B} \quad \text{and} \quad \mathbf{C}_r = \mathbf{V}^{\mathrm{T}}\mathbf{C},$$

and its transfer function is given by $\mathbf{H_r}(s) = {\mathbf{C}_r}^{\mathrm{T}}(s\mathbf{E}_r-\mathbf{A}_r)^{-1}\mathbf{B}_r$. Then, we have

$$\mathbf{H}_r(s) = \sum_{k=0}^{r}(-1)^k{\mathbf{C}_r}^{\mathrm{T}}\mathbf{M}_r^k\mathbf{R}_r(s-s_0)^k,$$

where $\mathbf{M}_r = (s_0\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{E}_r$ and $\mathbf{R}_r = (s_0\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{B}_r$. Then, the k-th moment, $\mathbf{h}_r^{(k)}$, of the transfer function for the reduced model is given by $\mathbf{h}_r^{(k)} = (-1)^k\mathbf{C}_r^{\mathrm{T}}\mathbf{M}_r^k\mathbf{R}_r$. According to [45], the moment matching approximation problem for the DAE (3.1.1) consists in determining a rational matrix-valued function $\mathbf{H}_r(s)$ at s_0 has the form

$$\mathbf{H}_r(s) = \mathbf{h}_r^{(0)} + \mathbf{h}_r^{(1)}(s - s_0) + \mathbf{h}_r^{(2)}(s - s_0)^2 + \cdots + \mathbf{h}_r^{(k)}(s - s_0)^k + \cdots$$

where the moments $\mathbf{h}_r^{(k)}$ satisfy the moment matching conditions $\mathbf{h}_r^{(k)} = \mathbf{h}^{(k)}$, $k = 0, 1, 2, \ldots, r - 1$. If $s_0 = \infty$, the $\mathbf{h}_r^{(k)}$ are the Markov parameters of (3.1.1) and the corresponding approximation problem is known as partial realization [21]. For s_0, the approximation problem reduces to the padé approximation problem [30]. For an arbitrary complex number $s_0 \neq 0$, the moment matching approximation of the problem of rational interpolation or shifted Padé approximation that has been considered in [23]. Apart from a single interpolation point one can construct a reduced-order system with the transfer function $\mathbf{H}_r(s)$ that matches $\mathbf{H}(s)$ at multiple points $\{s_0, s_1, \cdots, s_k\}$. Such as approximation is called a multi-point Padé approximation or a rational interpolant [30].

3.2.1 Limitation of conventional MOR methods

In this Section, we discuss the limitations of conventional MOR methods. We note that for some special cases conventional MOR methods can be used to reduce higher index DAEs and lead to accurate reduced-order models. But in general, conventional MOR methods may not always lead to accurate reduced-order models, so one has to be very careful. Using conventional MOR methods may lead to reduced-order models which lead to wrong solutions or they are very difficult to solve numerically especially those with index higher than one. This is illustrated in Example 3.2.1 and 3.2.2.

Example 3.2.1 This example originates from [2]. Consider an index-2 DAE of the form (3.1.1) with system matrices

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -4 & 2 & -1 & 1 & 0.5 \\ 1 & -1 & 1 & 0 & -0.5 \\ -1 & 1 & 0 & 1 & 0 \\ 1.25 & 2.25 & 0 & -4 & 1 \\ -0.5 & -0.5 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

This system is solvable since the polynomial $\det(\lambda\mathbf{E} - \mathbf{A}) = 2\lambda + 3$ does not vanish identically and in addition, we assume that input function $\mathbf{u}$ is differentiable in the desired time interval and $\boldsymbol{x}(0)$ is a consistent initial condition. In this example we consider two different cases of control input matrix $\mathbf{B}$ with input data $\mathbf{u}(t) = \cos(t)$.

(i) If $\mathbf{B} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}$, then the consistent initial condition is given by

$$\boldsymbol{x}(0) = \begin{pmatrix} 3 & 1 & -4 & 2 & -1 \end{pmatrix}^{\mathrm{T}} \boldsymbol{x}_2(0) + \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}^{\mathrm{T}} \mathbf{u}(0),$$

where $\boldsymbol{x}_2(0)$ can be chosen arbitrary. We then apply the PRIMA method [49] on the DAE. Using $s_0 = 0$ as the expansion point. We were able to obtain a reduced-order model of dimension 3. We observed that the reduced-order model is an ODE. We compared the transfer function of the original model with that of the reduced-order model. We observed that the transfer functions coincide with a very small approximation error as shown in Figure 3.1. Then, we numerically solved the reduced-order model and the

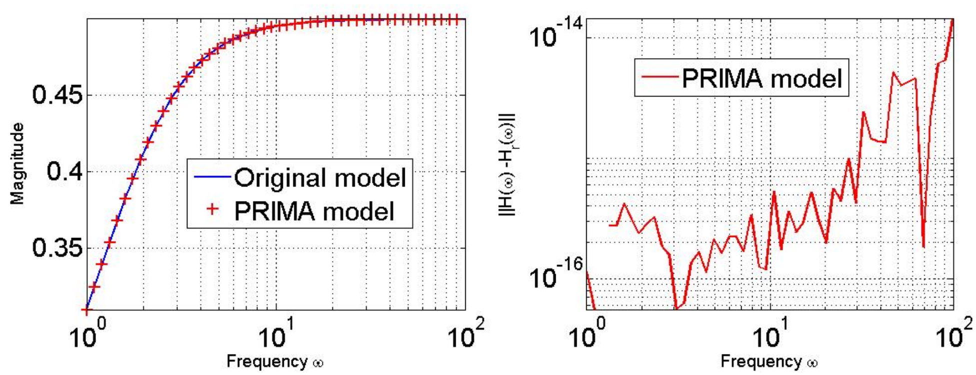

Figure 3.1: Comparison of the transfer function and error.

original DAE model using the Matlab software implicit ODE solver ode15s. We observed that the solution of the original model coincides with that of the reduced-order model (PRIMA model) as shown in Figure 3.2. Thus, the PRIMA model is a good reduced-order model for the original model since the reduced-order model leads to accurate solutions with ease.

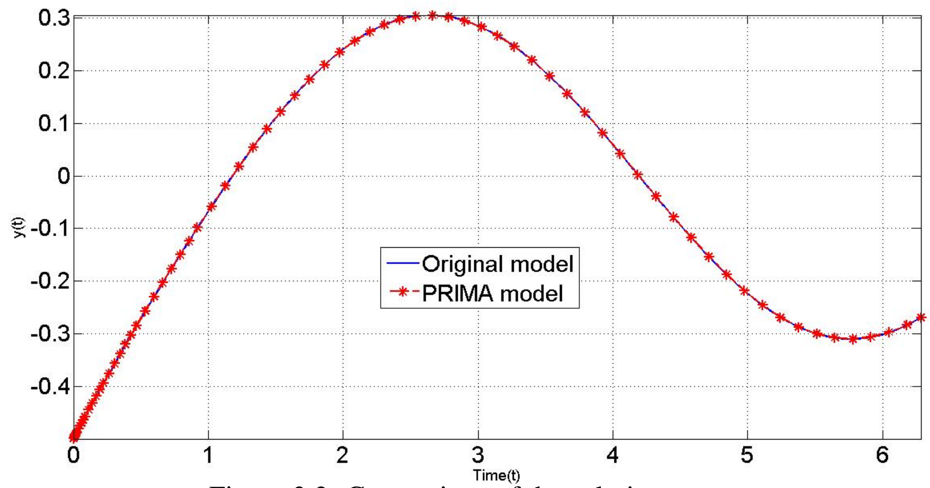

Figure 3.2: Comparison of the solutions

(ii) If $\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \end{pmatrix}^{\mathrm{T}}$, then the consistent initial condition is given by

$$\boldsymbol{x}(0) = \begin{pmatrix} 3 & 1 & -4 & 2 & -1 \end{pmatrix}^{\mathrm{T}} \boldsymbol{x}_2(0) + \begin{pmatrix} 2 & 0 & -\frac{5}{4} & 2 & \frac{7}{2} \end{pmatrix}^{\mathrm{T}} \mathbf{u}(0) + \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}} \mathbf{u}'(0),$$

where $\boldsymbol{x}_2(0)$ can be chosen arbitrary, for our case we chose $\boldsymbol{x}_2(0) = -0.5$. Using the same expansion point as before we obtain a reduced-order model of dimension 3. Still the PRIMA method leads to a ODE reduced-order model and also for this case the transfer function of the original model coincides with that of the reduced-order model with very small approximation error as shown in Figure 3.3. We also numerically solved

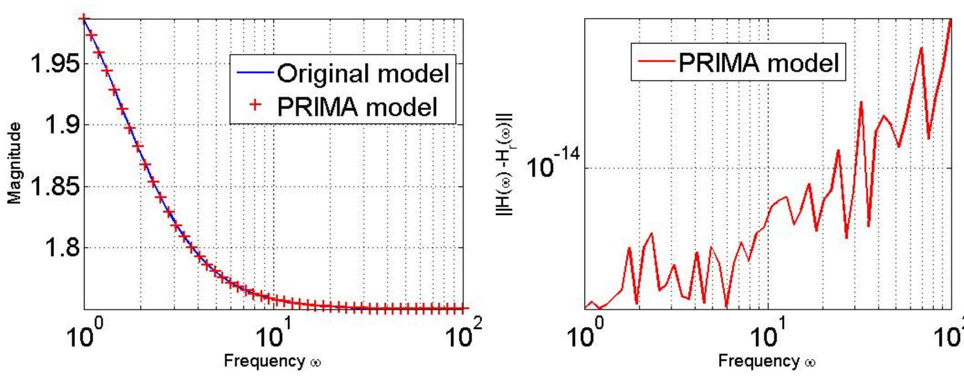

Figure 3.3: Comparison of the transfer function and error.

the reduced-order model using the Matlab software implicit ODE solver ode15s. We observed that the reduced-order model leads to a good solution, provided the absolute error tolerance is greater than 10^{-2} (AbsTol $> 10^{-2}$) otherwise the implicit ODE solver fails after a few time-steps. For more details of the choice of AbsTol the reader should refer to the MATLAB documentation. In Figure 3.4, we compare the solutions of the reduced-order model and the original model at different choices of absolute error tolerances. We can observe that the solutions of the PRIMA reduced-order model coincides with that of the original model when the absolute error tolerance is greater than 10^{-2} in this given time interval. We note that different Matlab software implicit ODE solvers can have different limits but all fails if you use very small absolute error tolerance. However, if one uses the backward Euler method this difficulty is not visible. Hence for this case one should always use the low order implicit integration techniques instead of the the higher order techniques.

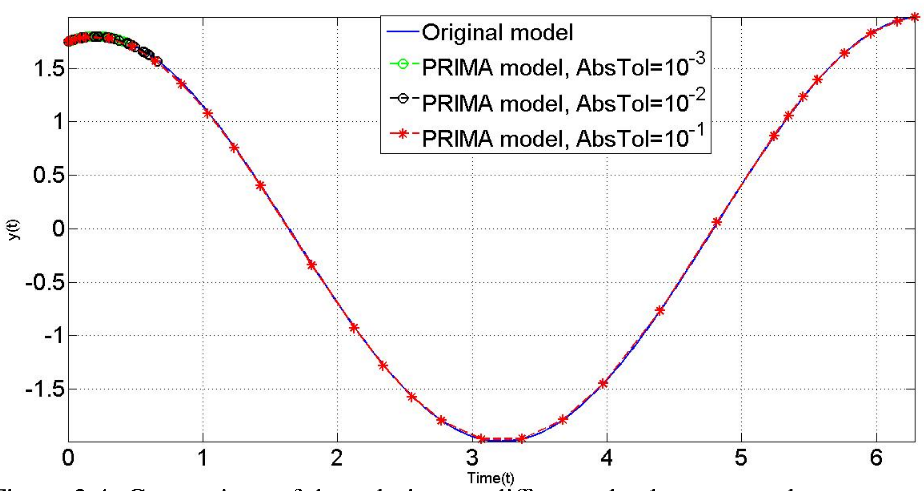

Figure 3.4: Comparison of the solutions at different absolute error tolerances

Example 3.2.1 illustrates that using conventional MOR methods on higher index DAEs may lead to less accurate reduced-order models. This is due to the fact that the consistent initial condition $\boldsymbol{x}_0$ in this example depends on $\mathbf{u}$ and its derivative, while in the former it only depends on $\mathbf{u}$. In the previous Chapter, we have already discussed that conventional MOR methods always assume that $\mathbf{E}\boldsymbol{x}(0) = 0$, but this assumption is not valid for DAEs, since we do not have enough freedom to choose the initial condition because of the hidden constraints. Making this assumption, implies that some mathematical information of the original DAE is not inherited in the reduced-order system. However, they are some special cases where assuming $\mathbf{E}\boldsymbol{x}(0) = 0$ does not affect the conventional methods. One of the special cases is if the consistent initial condition $\boldsymbol{x}(0)$ of the DAEs only depends on $\mathbf{u}$, as illustrated in Example 3.2.1(i). In the previous example, we have discussed that the difficult of solving the reduced order model from conventional MOR methods can be avoided by using lower order implicit integration techniques such as the backward Euler method. However, this remedy only works for some special cases. For worst case scenario the reduced-order model may be unsolvable if one applies conventional MOR methods on higher index DAEs as illustrated in the next example.

Example 3.2.2 In this example, we use the generator model originating from [20], as described in Figure 3.5. In this model: The input function is the angle ϕ_{in} on the left

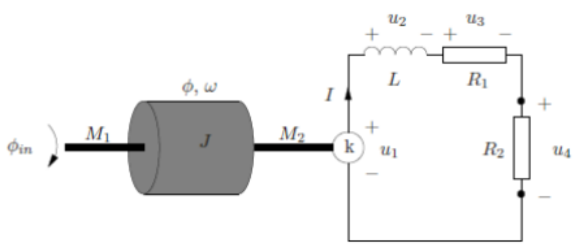

Figure 3.5: A model of a generator

axis. This axis is connected to a rotating mass with inertia J which is rotated at an angle ϕ and rotates with the angular velocity ω. The torque acting on the left side of the mass is M_1 and the torque on the right side is M_2. The mass is then connected to a second axis which is connected to the actual generator. The variables describing the second axis and the electrical quantities are then assumed to depend on each other according to $M_2 = k\mathrm{I}$ and $\mathbf{u}_1 = k\omega$ for some constant k. The rest of the electrical circuit consists of two resistors and one inductor. The measured output is the voltage $\mathbf{u}_4$. This model leads to an index-3 DAE of the form (3.1.1) with system matrices

$$\mathbf{E} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&J&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&L&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0&0&0&-1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 0&-1&0&0&k&0&0&0&0\\ 0&0&k&0&0&-1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&R_1&0&0&-1&0\\ 0&0&0&0&R_2&0&0&0&-1\\ 0&0&0&0&0&-1&1&1&1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\1 \end{pmatrix}, \tag{3.2.3}$$

and $\boldsymbol{x} = \begin{pmatrix} M_1 & M_2 & \omega & \phi & \mathrm{I} & \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{pmatrix}^{\mathrm{T}}$, $\mathbf{u} = \phi_{in}$. This system is solvable since $\det(\lambda\mathbf{E} - \mathbf{A}) = -R_1 - R_2 - \lambda L$ does not vanish identically. Letting $J = 1, k = -1, R_1 = 1$, $R_2 = 1, L = 1$ and using $s_0 = 0$ as the expansion point, we are able to construct the orthonormal basis matrix $\mathbf{V}_r$ using the PRIMA method. We then used this $\mathbf{V}_r$ to construct PRIMA reduced-order model of (3.2.3) given by

$$\mathbf{E}_r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.2774 & 0.4615 & -0.1155 & 0.0665 \\ -0.0595 & -0.2227 & 0.0557 & -0.0321 \\ -0.2637 & -0.1197 & 0.0299 & -0.0172 \end{pmatrix}, \ \mathbf{A}_r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.2774 & 0.1538 & -0.7175 & 0 \\ -0.8326 & 0.0330 & 0.2944 & -0.0278 \\ 0.4795 & 0.1463 & 0.0962 & -0.0483 \end{pmatrix},$$
$$\mathbf{B}_r = \begin{pmatrix} 0 & 0.2774 & 0.8326 & -0.4795 \end{pmatrix}^{\mathrm{T}}, \quad \text{and} \quad \mathbf{C}_r = \begin{pmatrix} 0 & 0.2774 & 0.0595 & 0.2637 \end{pmatrix}^{\mathrm{T}}. \tag{3.2.4}$$

We can see that the original model is reduced to dimension 4. The next step is to check the validity of the derived reduced-order model. Unfortunately the reduced-order model leads to a singular matrix pencil since $\det(\lambda \mathbf{E}_r - \mathbf{A}_r) = 0$. Thus, the reduced-order model is unsolvable even if the original system is solvable. Hence, we cannot use the conventional MOR methods to reduce the DAE (3.2.3). If we look closely at the Krylov sequence $\mathcal{V} = \{\mathbf{R}, \mathbf{MR}, \mathbf{M}^2\mathbf{R}, \cdots, \mathbf{M}^8\mathbf{R}\}$, where $\mathbf{M} = -\mathbf{A}^{-1}\mathbf{E}$ and $\mathbf{R} = -\mathbf{A}^{-1}\mathbf{B}$ generated by the PRIMA method. We observe that the sequence can be written as $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_1\}$, where $\mathcal{V}_1 = \{\mathbf{R}, \mathbf{MR}, \mathbf{M}^2\mathbf{R}\}$ and
$\mathcal{V}_2 = \{\mathbf{M}^3\mathbf{R}, \frac{1}{2}\mathbf{M}^3\mathbf{R}, \frac{1}{2^2}\mathbf{M}^3\mathbf{R}, \frac{1}{2^3}\mathbf{M}^3\mathbf{R}, \frac{1}{2^4}\mathbf{M}^3\mathbf{R}, \frac{1}{2^5}\mathbf{M}^3\mathbf{R}\}$ is a geometric sequence with common ratio $\frac{1}{2}$ and $\mathbf{M}^3\mathbf{R}$ as the starting term. Thus, the Krylov subspace $\mathcal{K}_9(\mathbf{M}, \mathbf{R})$ has a maximum dimension $4<9$. But this information does not tell us why the PRIMA method lead to an unsolvable reduced-order model. Hence more research is still need in this direction.

From the above examples, we have seen that not always we can reduce DAEs using conventional MOR methods and lead to accurate reduced-order models. So one has to be very careful when applying conventional MOR methods on DAEs. We note that this limitation is not just on PRIMA method but also other MOR methods developed specifically for ODEs such as the interpolatory model reduction, see [25]. This observation has lead to the development of new MOR methods specifically for DAEs, see [17, 18, 25, 32, 45] and to some extent the modification of the existing MOR methods for ODEs, see [25,45]. Most of these recently developed methods are application based and some are more general such as [25, 45]. In the next section, we briefly discuss some of these methods.

3.3 Recent MOR methods for DAEs

In this Section, we discuss the recently developed MOR methods specifically for DAEs. All these methods find the way of manipulating the index of the DAEs. As we have already discussed in the previous Chapter that the index of the DAE is the source of difficult of both reducing and solving DAEs. Earlier developed methods focused on index reduction than preserving the index of the DAE, but this is very dangerous since it may lead to loss of the important mathematics properties of the DAE. Other earlier methods developed were application specific, see [17, 18] and cannot be used on general DAEs. The main characteristics of these methods is to extract differential equations from original DAEs and then apply the conventional MOR methods.

3.3.1 Kron reduction method

There are systematic ways to extract several sets of ODEs from the original DAEs. The algebraic variables are excluded from the DAE, e.g., using Kron reduction [14]. According to [13], the Kron reduction can be demonstrated as follows. Consider a linear DAE (3.1.1a), where the variable $\boldsymbol{x}$ is partitioned into the state variables $\boldsymbol{x}_1$ and algebraic variables, $\boldsymbol{x}_2$. Then (3.1.1a) can be rewritten as:

$$\begin{pmatrix} \mathbf{E}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix}' = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix} \boldsymbol{u}(t). \tag{3.3.1}$$

Assuming $\mathbf{A}_{22}$ and $\mathbf{E}_{11}$is nonsingular, we can then eliminate the algebraic variables leading to a differential equation given by

$$\mathbf{E}_{11}\boldsymbol{x}_1' = \left[\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right]\boldsymbol{x}_1 + \left[\mathbf{B}_1 - \mathbf{A}_{22}\mathbf{B}_2\right]\boldsymbol{u}(t).$$

Then the algebraic variables $\boldsymbol{x}_2$ can be obtained from the state variables $\boldsymbol{x}_1$ and the input function $\boldsymbol{u}(t)$ using $\boldsymbol{x}_2 = -\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\boldsymbol{x}_1 - \mathbf{A}_{22}^{-1}\mathbf{B}_2\boldsymbol{u}(t)$. According to [13], the above step is called the Kron reduction. The main idea is transforming a DAE into an ODE. This procedure can be viewed as an index reduction procedure. In the recent publications [17, 18] the same approach has been used to reduce the index of the DAEs from power systems. If we also partition $\mathbf{C}$ as $\mathbf{C} = \left(\mathbf{C}_1^{\mathrm{T}}\ \mathbf{C}_2^{\mathrm{T}}\right)^{\mathrm{T}}$, we can further transform a DAE into an ODE given by

$$\hat{\boldsymbol{x}}' = \hat{\mathbf{A}}\hat{\boldsymbol{x}} + \hat{\mathbf{B}}\mathbf{u}(t), \tag{3.3.2a}$$

$$\boldsymbol{y} = \hat{\mathbf{C}}^{\mathrm{T}}\hat{\boldsymbol{x}} + \hat{\mathbf{D}}^{\mathrm{T}}\mathbf{u}(t), \tag{3.3.2b}$$

where $\hat{\mathbf{A}} = \mathbf{E}_{11}^{-1}\left[\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right]$, $\hat{\mathbf{B}} = \mathbf{E}_{11}^{-1}\left[\mathbf{B}_{1} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{B}_{2}\right]$, $\hat{\mathbf{C}}^{\mathrm{T}} = \mathbf{C}_1^{\mathrm{T}} - \mathbf{C}_2^{\mathrm{T}}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}$ and $\hat{\mathbf{D}}^{\mathrm{T}} = -\mathbf{C}_2\mathbf{A}_{22}^{-1}\mathbf{B}_2$ and $\hat{\boldsymbol{x}} = \boldsymbol{x}_1$. Then the conventional MOR methods can be used to further reduce system (3.3.2). We can observe that algebraic part is not reduced, it is rather just hidden. Unfortunately it is not always possible to transform (3.1.1a) into (3.3.1), hence this approach is much restricted on index-1 DAEs. Moreover, the numerical solutions of the index-reduced problems will most likely suffer from the so called "drift off" effect, see [26]. Hence this approach cannot be applied on general DAEs.

3.3.2 Balanced truncation method for DAEs

Balanced truncation MOR method is one of the conventional MOR methods which have been extended or modified to be able to reduce DAEs, see [45]. If we assume $\mathbf{E} = \mathrm{I}$ the balanced truncation method makes use of two Lyapunov equations,

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^{\mathrm{T}} = -\mathbf{B}\mathbf{B}^{\mathrm{T}}, \quad \mathbf{A}^{\mathrm{T}}\mathcal{Q} + \mathcal{Q}\mathbf{A} = -\mathbf{C}\mathbf{C}^{\mathrm{T}}. \tag{3.3.3}$$

The solutions $\mathcal{P} \in \mathbb{R}^{n\times n}$ and $\mathcal{Q} \in \mathbb{R}^{n\times n}$ of these equations are called the controllability and observability Gramians, respectively. The balanced truncation method consists of transforming the state space system into a balanced form whose controllability and observability Gramians become diagonal and equal, together with a truncation of those states that are both difficult to reach and observe. This method is one of few well known method which fulfills almost all goals of model order reduction, moreover even the existence of a priori computable error bound that allows an adaptive choice of the state space dimension r of the reduced model depending on how accurate the approximation is needed. The main drawback of the balanced truncation used to be that the two matrix Lyapunov equations (3.3.3) have to be solved, followed by an SVD and that both steps are computationally very expensive, since they require $\mathcal{O}(n^2)$ storage and $\mathcal{O}(n^3)$ flops [3]. However, recently a low rank approximations to the solutions of Lyapunov equations make the balanced truncation model reduction approach attractive to large scale systems [45]. In [45], they extended the balanced truncation method to descriptor system or DAEs, i.e., **E** is singular, using spectral projectors. This is done as follows: From (2.3.5), there exist nonsingular matrices **P** and **Q** such that

$$\mathbf{E} = \mathbf{P}^{-1}\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{pmatrix}\mathbf{Q}^{-1}, \quad \mathbf{A} = \mathbf{P}^{-1}\begin{pmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}\mathbf{Q}^{-1}, \tag{3.3.4}$$

where matrices **J** and **N** are defined as in Theorem 2.3.2. In [45], they used these nonsingular matrices to construct the spectral projectors given by

$$\mathbf{P}_r = \mathbf{Q}\begin{pmatrix} \mathrm{I}_{n_f} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{Q}^{-1} \quad \text{and} \quad \mathbf{P}_l = \mathbf{P}\begin{pmatrix} \mathrm{I}_{n_f} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{P}^{-1}, \tag{3.3.5}$$

onto the right and left deflating subspaces, respectively of the matrix pencil $\lambda\mathbf{E} - \mathbf{A}$ corresponding to the finite eigenvalues. It has been proven in [62] that the proper controllability and observability Gramians are unique symmetric, positive definite solutions of the projected generalized continuous-time Lyapunov equations

$$\mathbf{E}\mathcal{G}_{pc}\mathbf{A}^{\mathrm{T}} + \mathbf{A}\mathcal{G}_{pc}\mathbf{E}^{\mathrm{T}} = -\mathbf{P}_l\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{P}_l^{\mathrm{T}}, \quad \mathcal{G}_{pc} = \mathbf{P}_r\mathcal{G}_{pc}\mathbf{P}_r^{\mathrm{T}}, \tag{3.3.6}$$

$$\mathbf{E}^{\mathrm{T}}\mathcal{G}_{po}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathcal{G}_{po}\mathbf{E} = -\mathbf{P}_r\mathbf{C}\mathbf{C}^{\mathrm{T}}\mathbf{P}_r^{\mathrm{T}}, \quad \mathcal{G}_{po} = \mathbf{P}_l^{\mathrm{T}}\mathcal{G}_{po}\mathbf{P}_l. \tag{3.3.7}$$

Also, the improper controllability and observability Gramians are unique symmetric, positive definite solutions of the projected generalized discrete-time algebraic Lyapunov equations

$$\mathbf{A}\mathcal{G}_{ic}\mathbf{A}^{\mathrm{T}} - \mathbf{E}\mathcal{G}_{ic}\mathbf{E}^{\mathrm{T}} = (\mathrm{I} - \mathbf{P}_l)\mathbf{B}\mathbf{B}^{\mathrm{T}}(\mathrm{I} - \mathbf{P}_l)^{\mathrm{T}}, \quad \mathbf{P}_r\mathcal{G}_{ic}\mathbf{P}_r^{\mathrm{T}} = 0, \tag{3.3.8}$$

$$\mathbf{A}^{\mathrm{T}}\mathcal{G}_{io}\mathbf{A} - \mathbf{E}^{\mathrm{T}}\mathcal{G}_{io}\mathbf{E} = (\mathrm{I} - \mathbf{P}_r)^{\mathrm{T}}\mathbf{C}\mathbf{C}^{\mathrm{T}}(\mathrm{I} - \mathbf{P}_r), \quad \mathbf{P}_l^{\mathrm{T}}\mathcal{G}_{io}\mathbf{P}_l = 0. \tag{3.3.9}$$

Similarly as in standard state space systems or ODEs, the controllability and observability Gramians can be used to define Hankel singular values for the descriptor system (3.1.1) that are of great importance in model reduction via balanced truncation, see [45] for more details about this method. From this point, we can observe that for the case of DAEs one needs to solve four Lyapunov equations (3.3.6), (3.3.7), (3.3.8) and (3.3.9). This means that the computational effort has now doubled and moreover in Chapter 2, we have already discussed that computing the Kronecker canonical form in finite precision arithmetic is, in general, an ill-conditioned problem. Hence numerical computational of spectral projectors (3.3.5) may not be feasible for general DAEs. However, for some structured problems arising in circuit simulation, multibody systems and computational fluid dynamics, these projectors can be constructed in an explicit form that significantly reduces the computation complexity of the balanced truncation method for DAEs. Consider the matrices $\mathcal{G}_{pc}\mathbf{E}^{\mathrm{T}}\mathcal{G}_{po}\mathbf{E}$ and $\mathcal{G}_{ic}\mathbf{A}^{\mathrm{T}}\mathcal{G}_{io}\mathbf{A}$. According to [45], these matrices play the same role for DAEs as the product of controllability and observability Gramians for standard state space systems. Since the proper and improper controllability and observability Gramians are symmetric and positive semidefinite, there exist Cholesky factorizations

$$\mathcal{G}_{pc} = \mathbf{R}_p\mathbf{R}_p^{\mathrm{T}}, \quad \mathcal{G}_{po} = \mathbf{L}_p\mathbf{L}_p^{\mathrm{T}}, \quad \mathcal{G}_{ic} = \mathbf{R}_i\mathbf{R}_i^{\mathrm{T}}, \quad \mathcal{G}_{io} = \mathbf{L}_i\mathbf{L}_i^{\mathrm{T}}, \tag{3.3.10}$$

where the matrices $\mathbf{R}_p$, $\mathbf{L}_p$, $\mathbf{R}_i$, $\mathbf{L}_i \in \mathbb{R}^{n\times n}$ are Cholesky factors of the Gramians. In this case the proper Hankel singular values of system (3.1.1) can be computed as the n_f largest singular values of the matrix $\mathbf{L}_p^{\mathrm{T}}\mathbf{E}\mathbf{R}_p$, and the improper Hankel singular values of (3.1.1) are the n_∞ largest singular values of the matrix $\mathbf{L}_i^{\mathrm{T}}\mathbf{A}\mathbf{R}_i$. The square roots of the largest n_f eigenvalues of the matrix $\mathcal{G}_{pc}\mathbf{E}^{\mathrm{T}}\mathcal{G}_{po}\mathbf{E}$ denoted by ζ_j, are called the proper Hankel singular values of the continuous-time DAE (3.1.1). The square roots of the largest n_∞ eigenvalues of the matrix $\mathcal{G}_{ic}\mathbf{A}^{\mathrm{T}}\mathcal{G}_{io}\mathbf{A}$, denoted by θ_j, are called the improper Hankel singular values of the system (3.1.1). n_f and n_∞ are the dimensions of the deflating subspaces of the matrix pencil $\lambda\mathbf{E} - \mathbf{A}$ corresponding to the finite and infinite eigenvalues, respectively. Assume that the proper and improper Hankel singular values are order decreasingly, i.e., $\zeta_1 \geq \zeta_2 \geq \cdots \geq \zeta_{n_f} \geq 0$ and $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_{n_\infty} \geq 0$.

Definition 3.3.1 ([45]) *A realization* $\left[\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}\right]$ *of the transfer function* $\mathbf{H}(s)$ *is called balanced if*

$$\mathcal{G}_{pc} = \mathcal{G}_{po} = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{G}_{ic} = \mathcal{G}_{io} = \begin{pmatrix} 0 & 0 \\ 0 & \Theta \end{pmatrix},$$

where $\Lambda = \mathrm{diag}(\zeta_1, \ldots, \zeta_{n_f})$ *and* $\Theta = \mathrm{diag}(\theta_1, \ldots, \theta_{n_\infty})$.

According to [45], for a minimal realization $\left[\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}\right]$ with a c-stable matrix pencil $\lambda\mathbf{E} - \mathbf{A}$, it is possible to find nonsingular transformation matrices $\mathbf{W}_b$ and $\mathbf{T}_b$ such that the transformed realization $\left[\mathbf{W}_b^{\mathrm{T}}\mathbf{E}\mathbf{T}_b, \mathbf{W}_b^{\mathrm{T}}\mathbf{A}\mathbf{T}_b, \mathbf{W}_b^{\mathrm{T}}\mathbf{B}, \mathbf{C}^{\mathrm{T}}\mathbf{T}_b\right]$ is balanced. These matrices are given by

$$\mathbf{W}_b = \left[\mathbf{L}_p\mathbf{U}_p\Sigma^{-1/2}, \quad \mathbf{L}_i\mathbf{U}_i\Theta^{-1/2}\right], \ \mathbf{T}_b = \left[\mathbf{R}_p\mathbf{V}_p\Sigma^{-1/2}, \quad \mathbf{R}_i\mathbf{V}_i\Theta^{-1/2}\right]. \tag{3.3.11}$$

Observe, however, as for the ODEs, the balancing transformation for DAEs is not unique [45]. It should also be noted that for the matrices $\mathbf{W}_p$ and $\mathbf{T}_b$ as in (3.3.11), we have

$$\begin{aligned} \mathbf{E}_b &= \mathbf{W}_b^{\mathrm{T}}\mathbf{E}\mathbf{T}_b = \begin{pmatrix} \mathrm{I}_{n_f} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_2 \end{pmatrix}, \quad \mathbf{A}_b = \mathbf{W}_b^{\mathrm{T}}\mathbf{A}\mathbf{T}_b = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_{n_\infty} \end{pmatrix}, \\ \mathbf{B}_b &= \mathbf{W}_b^{\mathrm{T}}\mathbf{B} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix}, \quad \mathbf{C}_b = \mathbf{T}_b^{\mathrm{T}}\mathbf{C} = \begin{pmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{pmatrix}, \end{aligned} \tag{3.3.12}$$

where the matrix $\mathbf{E}_2 = \Theta^{-1/2}\mathbf{U}_i^{\mathrm{T}}\mathbf{L}_i^{\mathrm{T}}\mathbf{E}\mathbf{R}_i\mathbf{V}_i\Theta^{-1/2}$ is nilpotent and the matrix $\mathbf{A}_1 = \Sigma^{-1/2}\mathbf{U}_p^{\mathrm{T}}\mathbf{L}_p^{\mathrm{T}}\mathbf{A}\mathbf{R}_p\mathbf{V}_p\Sigma^{-1/2}$ is nonsingular. Thus, the pencil $\lambda\mathbf{E}_b - \mathbf{A}_b$ of a balanced DAE is in a form that resembles the Weierstraß-Kronecker canonical form discussed in Section 2.3.3. We can observe that the balanced DAE (3.3.12) can be decoupled into differential and algebraic parts. These two parts can then be reduced separately. For the balanced system (3.3.12), the differential states related to the small proper Hankel singular values Σ are difficult to reach and to observe at the same time. The truncation of these states essentially does not change system properties [45] and reduces the order of the differential part. Unfortunately, this does not hold for the improper Hankel singular values. If we truncate the algebraic states that correspond to the small non-zero improper Hankel singular values, then the pencil of the reduced-order system may get finite eigenvalues in the closed right half-plane, see [36]. In this case the approximation may be inaccurate. According to [45], reducing the order of the algebraic subsystem of system (3.3.12) is equivalent to the balanced model reduction of the discrete-time system

$$\begin{aligned} \xi_{k+1} &= \mathbf{E}_2\xi_k + \mathbf{B}_2\mathbf{u}_k, \\ y_{2,k} &= \mathbf{C}_2\xi_k. \end{aligned}$$

The Hankel singular values of this system are just the improper Hankel singular values of (3.1.1), see [45]. Since we truncate only the states corresponding to the zero improper Hankel values, the polynomial part of the transfer function $\mathbf{H}(s)$ of the reduced and original model are equal and the index of the system is equal to the degree of the polynomial part plus one. In this case the error system is strictly proper, and we have the following $\mathbb{H}_\infty-$ norm error bound [45]

$$\|\mathbf{H}(s) - \mathbf{H}_r(s)\|_{\mathbb{H}_\infty} \leq 2(\zeta_{\ell_f+1} + \cdots + \zeta_{n_f}).$$

Existence of the error bound is an important property of the balanced truncation model reduction approach for DAEs. It makes this approach preferable compared, for instance, to moment matching techniques [45]. However the balanced truncation model reduction approach for DAEs is computationally very expensive and it relies on Weierstraß-Kronecker canonical form to construct the spectral projectors (3.3.5). This limits its application to DAEs with special structure. Hence alternative procedures are required to decouple the DAE more efficiently.

3.3.3 Interpolatory projection method for DAEs

In this Section, we discuss the interpolatory projection method which was extended in order to reduce DAEs of the form (3.1.1) accurately. This extension is proposed in [25]. Consider the reduced-order model (3.1.2) with system matrices

$$\mathbf{E}_r = \mathbf{W}^{\mathrm{T}}\mathbf{E}\mathbf{V},\ \mathbf{A}_r = \mathbf{W}^{\mathrm{T}}\mathbf{A}\mathbf{V},\ \mathbf{B}_r = \mathbf{W}^{\mathrm{T}}\mathbf{B} \quad \text{and} \quad \mathbf{C}_r = \mathbf{V}^{\mathrm{T}}\mathbf{C}, \tag{3.3.13}$$

where the $n \times r$ projection matrices $\mathbf{V}$ and $\mathbf{W}$ determine the subspaces of interest and can be computed in many different ways depending on the model order reduction method. According to [25], in the projection-based interpolatory model reduction methods, the choice of $\mathbf{V}$ and $\mathbf{W}$ enforces certain tangential interpolation of the transfer function $\mathbf{H}(s)$. In [25], they stress that extending interpolatory model order reduction from standard state space systems with $\mathbf{E} = \mathrm{I}$ to descriptor systems (DAEs) with singular $\mathbf{E}$ is not as simple as replacing I by $\mathbf{E}$. This is illustrated in [25], by example showing that the naive approach may lead to a poor approximation with an unbounded error $\mathbf{H}(s) - \mathbf{H}_r(s)$ although the classical interpolatory subspace conditions are satisfied. According to [25], the reason is simple, even though $\mathbf{E}$ is singular, $\mathbf{E}_r$ may genetically be a nonsingular matrix . Then the transfer function $\mathbf{H}_r(s)$ of the reduced-order model (3.3.1) is proper, although $\mathbf{H}(s)$ might be improper. In this case $\mathbf{H}(s)$ can be decomposed as $\mathbf{H}(s) = \mathbf{H}_{sp}(s) + \mathcal{P}(s)$, where $\mathbf{H}_{sp}(s)$ and $\mathcal{P}(s)$ denote the strictly proper and the polynomial parts of $\mathbf{H}(s)$, respectively. Hence, special care needs to be taken in order to match the polynomial part of $\mathbf{P}(s)$. This agrees with our observation in Section 2.3.4.

In [25], they modified the classical interpolatory subspace condition in order to enforce bounded error using spectral projectors, see [25] for more details. In their modified interpolatory subspace condition they ensured that the polynomial part of $\mathbf{H}_r(s)$ has to match $\mathbf{P}(s)$ exactly. Based on the literature from [25], the interpolatory projection methods for descriptor systems which we call DAEs in this thesis is briefly discussed as follows. In order to have bounded $\mathcal{H}_2$ and $\mathcal{H}_\infty$ errors, the polynomial part of $\mathbf{H}_r(s)$ has to match the polynomial part of $\mathbf{H}(s)$ exactly, see [25]. They enforced that the transfer function $\mathbf{H}_r(s)$ of the reduced order model (3.3.13) to have the decomposition $\mathbf{H}_r(s) = \mathbf{H}_{sp_r}(s) + \mathcal{P}_r(s)$ with $\mathcal{P}_r(s) = \mathcal{P}(s)$. This implies that the error transfer function does not contain a polynomial part, i.e.,

$$\mathbf{H}_{err}(s) = \mathbf{H}(s) - \mathbf{H}_r(s) = \mathbf{H}_{sp}(s) - \mathbf{H}_{sp_r}(s)$$

is strictly proper meaning $\lim\limits_{s\to\infty} \mathbf{H}_{err}(s) = 0$. Clearly if $\mathbf{H}_{sp_r}(s)$ interpolates $\mathbf{H}_{sp}(s)$, we can be able to enforce that $\mathbf{H}_r(s)$ interpolates $\mathbf{H}(s)$. According to [25], the spectral projectors $\mathbf{P}_l$ and $\mathbf{P}_r$ as defined in (3.3.5) plays a vital role in interpolatory-based model reduction. This lead to the following theorem that provides the projection matrices $\mathbf{W}$ and $\mathbf{V}$ satisfying subspace conditions such that the reduced-order model $\mathbf{H}_r(s)$ obtained by projections as in (3.3.13) will not only satisfy the interpolation conditions but also match the polynomial part of $\mathbf{H}(s)$.

Theorem 3.3.1 ([25]) *Given a full-order model* $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$, *define* $\mathbf{P}_l$ *and* $\mathbf{P}_r$ *to be the spectral projectors onto the left and right deflating subspaces of the matrix pencil* $\lambda\mathbf{E} - \mathbf{A}$ *corresponding to the finite eigenvalues. Let the columns of* $\mathbf{W}_\infty$ *and* $\mathbf{V}_\infty$ *span the left and right deflating subspaces of* $\lambda\mathbf{E} - \mathbf{A}$ *corresponding to the eigenvalue at infinity. Let* $\sigma, \mu \in \mathbb{C}$ *be interpolation points such that* $s\mathbf{E} - \mathbf{A}$ *and* $s\mathbf{E}_r - \mathbf{A}_r$ *are nonsingular for* $s = \sigma, \mu$ *and let* $\mathbf{b} \in \mathbb{C}^m$ *and* $\mathbf{c} \in \mathbb{C}^\ell$. *Define* $\mathbf{V}_f$ *and* $\mathbf{W}_f$ *such that*

$$\begin{aligned}\mathrm{Im}(\mathbf{V}_f) &= \mathrm{span}\{((\sigma\mathbf{E} - \mathbf{A})^{-1}\mathbf{E})^{j-1}(\sigma\mathbf{E} - \mathbf{A})^{-1}\mathbf{P}_l\mathbf{B}\mathbf{b},\ j = 1, \cdots, N\},\\ \mathrm{Im}(\mathbf{W}_f) &= \mathrm{span}\{((\mu\mathbf{E} - \mathbf{A})^{-1}\mathbf{E})^{j-1}(\mu\mathbf{E} - \mathbf{A})^{-1}\mathbf{P}_r^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{c},\ j = 1, \cdots, M\}.\end{aligned}$$

Then, with the choice of $\mathbf{W} = \left[\mathbf{W}_f, \mathbf{W}_\infty\right]$ *and* $\mathbf{V} = \left[\mathbf{V}_f, \mathbf{V}_\infty\right]$, *the reduced-order model* $\mathbf{H}_r(s) = \mathbf{C}_r(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{B} + \mathbf{D}_r$ *obtained via projection as in* (3.3.13) *satisfies*

1. $\mathcal{P}_r(s) = \mathcal{P}(s)$,
2. $\mathbf{H}^{(\ell)}(\sigma)\mathbf{b} = \mathbf{H}_r^{(\ell)}(\sigma)\mathbf{b}$ *for* $\ell = 0, 1, \cdots, N-1$,
3. $\mathbf{c}^{\mathrm{T}}\mathbf{H}^{(\ell)}(\mu) = \mathbf{c}^{\mathrm{T}}\mathbf{H}_r^{(\ell)}(\sigma)$ *for* $\ell = 0, 1, \cdots, M-1$.

If $\sigma = \mu$, *we have, additionally,* $\mathbf{c}^{\mathrm{T}}\mathbf{H}^{(\ell)}(\mu)\mathbf{b} = \mathbf{c}^{\mathrm{T}}\mathbf{H}_r^{(\ell)}(\sigma)\mathbf{b}$ *for* $\ell = 0, \cdots, M + N + 1$.

The proof of this theorem can be found in [25]. In this proof, they were able to show that the matrices of the reduced-order model (3.3.13) have the form

$$\mathbf{E}_r = \begin{pmatrix} \mathbf{W}_f^{\mathrm{T}}\mathbf{E}\mathbf{V}_f & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_\infty^{\mathrm{T}}\mathbf{E}\mathbf{V}_\infty \end{pmatrix}, \quad \mathbf{A}_r = \begin{pmatrix} \mathbf{W}_f^{\mathrm{T}}\mathbf{A}\mathbf{V}_f & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_\infty^{\mathrm{T}}\mathbf{A}\mathbf{V}_\infty \end{pmatrix},$$

$$\mathbf{B}_r = \begin{pmatrix} \mathbf{W}_f^{\mathrm{T}}\mathbf{B} \\ \mathbf{W}_\infty^{\mathrm{T}}\mathbf{B} \end{pmatrix}, \quad \mathbf{C}_r = \begin{pmatrix} \mathbf{W}_f^{\mathrm{T}}\mathbf{C} & \mathbf{W}_\infty^{\mathrm{T}}\mathbf{C} \end{pmatrix}. \tag{3.3.14}$$

From (3.3.14), we can observe that the reduced-order model is decoupled into differential and algebraic part. Also, from Theorem 3.3.1 we can observe that the interpolatory projection method is only applied on the differential part and the algebraic part is unreduced. This makes sense since in [25] , they emphasize the polynomial part of $\mathbf{H}(s)$ and $\mathbf{H}_r(s)$ to be exact. The model reduction approach also involves the explicit computation of the spectral projectors (3.3.5), which could be numerically infeasible for general large-scale problems. Hence this limits it application to general DAEs.

3.4 MOR methods for algebraic systems

Currently, there is no yet known published MOR method specifically for algebraic systems. However there is a lot of progress made in reducing algebraic systems from electrical networks especially the resistor networks, see [15, 59]. The underlying method used is the Kron reduction method [32]. This method is used in [15, 59], to reduce resistor networks. The basic idea of the Kron reduction method for algebraic systems can be discussed as follows. We note that most of the literature presented in this Section is from [59]. Consider a linear resistor electric network, that is, a network which connects linear resistors and current sources, $\boldsymbol{\imath}(t) \in \mathbb{R}^{n_I}$. The unknowns which describe the network are the node potentials $\boldsymbol{e}(t) \in \mathbb{R}^n$. Following the formalism of modified nodal analysis (MNA), we introduce: the incidence matrix $\boldsymbol{A}_R \in \mathbb{R}^{n \times n_G}$, which describe the branch-node relationships for resistors; the incidence matrix $\boldsymbol{A}_I \in \mathbb{R}^{n \times n_I}$, which describe this relationship for current sources. Then the model for a resistor network for the unknown $\boldsymbol{v}$ is given by

$$\boldsymbol{A}_R \boldsymbol{G} \boldsymbol{A}_R^{\mathrm{T}} \boldsymbol{v} = -\boldsymbol{A}_I \boldsymbol{\imath}, \tag{3.4.1}$$

where $\boldsymbol{G}$ is the conductance matrix. For convenience (3.4.1) can be written as

$$\mathcal{G}\boldsymbol{v} = \boldsymbol{i}_{in}, \tag{3.4.2}$$

where $\mathcal{G} = \boldsymbol{A}_R \boldsymbol{G} \boldsymbol{A}_R^{\mathrm{T}}$ is symmetric positive semidefinite matrix and $\boldsymbol{i}_{in} = -\boldsymbol{A}_I \boldsymbol{\imath}$ are in injected node currents. Since currents can only be injected in external nodes, and not in internal nodes of the network, system (3.4.2) can be reordered to obtain the following partitioned structure:

$$\begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} \\ \mathcal{G}_{12}^{\mathrm{T}} & \mathcal{G}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_e \\ \boldsymbol{v}_{in} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b} \\ 0 \end{pmatrix} \boldsymbol{\imath}, \tag{3.4.3}$$

where $\boldsymbol{v}_e \in \mathbb{R}^{n_e}$ and $\boldsymbol{v}_{in} \in \mathbb{R}^{n_i}$ are the voltages of external and internal nodes, respectively, and $\boldsymbol{b} \in \mathbb{R}^{n_e \times n_I}$ is the incidence matrix for the current injections. The next step is to reduce (3.4.3). The most trivial reduction is to eliminate all the internal nodes which are not connected to the external currents which leads to a reduced linear system which is given by

$$\mathcal{G}_r \boldsymbol{v}_r = \boldsymbol{b}\boldsymbol{\imath}, \tag{3.4.4}$$

where $\mathcal{G}_r = \mathcal{G}_{11} - \mathcal{G}_{12}\mathcal{G}_{22}^{-1}\mathcal{G}_{12}^{\mathrm{T}} \in \mathbb{R}^{n_e \times n_e}$ and $\boldsymbol{v}_r = \boldsymbol{v}_e$. We note that $\mathcal{G}_r$ is the Schur compliment of $\mathcal{G}_{22}$. Thus the system is reduced from n to n_e. This is illustrated in the next example.

Example 3.4.1 Consider a resistor network with incidence matrices

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{A}_I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.4.5}$$

and the conductance matrix given by $\boldsymbol{G} = \mathrm{diag}\big(\boldsymbol{G}_1, \boldsymbol{G}_2, \boldsymbol{G}_3, \boldsymbol{G}_4, \boldsymbol{G}_5, \boldsymbol{G}_6\big)$. Thus substituting (3.4.5) and the conductance matrix into (3.4.1). We obtain a linear system of order 6

given by

$$\begin{pmatrix} G_1 & -G_1 & 0 & 0 & 0 & 0 \\ -G_1 & G_1+G_2+G_4 & -G_2 & 0 & -G_4 & 0 \\ 0 & -G_2 & G_2+G_3+G_5 & -G_3 & -G_5 & 0 \\ 0 & 0 & -G_3 & G_3 & 0 & 0 \\ 0 & -G_4 & -G_5 & 0 & G_4+G_5+G_6 & -G_6 \\ 0 & 0 & 0 & 0 & -G_6 & G_6 \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \\ \boldsymbol{v}_4 \\ \boldsymbol{v}_5 \\ \boldsymbol{v}_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\imath}_1 \\ \boldsymbol{\imath}_2 \\ \boldsymbol{\imath}_3 \end{pmatrix}. \quad (3.4.6)$$

After reordering the above system, we obtain a system in a partitioned structure of the form (3.4.3) given by

$$\begin{pmatrix} G_1 & 0 & 0 & \vdots & -G_1 & 0 & 0 \\ 0 & G_3 & 0 & \vdots & 0 & -G_3 & 0 \\ 0 & 0 & G_6 & \vdots & 0 & 0 & -G_6 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -G_1 & 0 & 0 & \vdots & G_1+G_2+G_4 & -G_2 & -G_4 \\ 0 & -G_3 & 0 & \vdots & -G_2 & G_2+G_3+G_5 & -G_5 \\ 0 & 0 & -G_6 & \vdots & -G_4 & -G_5 & G_4+G_5+G_6 \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_4 \\ \boldsymbol{v}_5 \\ \cdots \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_6 \\ \boldsymbol{v}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\imath}_1 \\ \boldsymbol{\imath}_2 \\ \boldsymbol{\imath}_3 \end{pmatrix}.$$

We can observe that the voltages of the external and internal nodes are given by $\boldsymbol{v}_e = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_4 & \boldsymbol{v}_5 \end{pmatrix}^{\mathrm{T}}$, $\boldsymbol{v}_{in} = \begin{pmatrix} \boldsymbol{v}_2 & \boldsymbol{v}_6 & \boldsymbol{v}_3 \end{pmatrix}^{\mathrm{T}}$ and the incidence matrix for current injections given by $\boldsymbol{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. For convenience, we can set all the resistors to $\mathbf{G}_i = 1$, $i = 1, 2, \cdots, 6$. Thus, we can now eliminate all the internal nodes which are not connected to the external currents which leads to a reduced-order system

$$\begin{pmatrix} 0.50 & -0.25 & -0.25 \\ -0.25 & 0.50 & -0.25 \\ -0.25 & -0.25 & 0.50 \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_4 \\ \boldsymbol{v}_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \boldsymbol{\imath}, \quad (3.4.7)$$

Hence the original system (3.4.6) is reduced to a reduced-order system (3.4.7) of dimension 3 using Kron reduction method.

This technique is also known as reduction by Elimination Internal Nodes in [59]. In practice, since the number n_e of terminals is usually much smaller than the number n_i of internal unknowns, in terms of unknowns this leads in general to a huge reduction. According to [59], in many cases, however, elimination of all internal nodes leads to a dramatic increase in the number of resistors and is hence not advisable. However, if you construct an efficient way of finding these specific internal nodes, that cause the most fill-in can greatly improve this reduction procedure. In [59], they were able to solve this problem by using the graph and matrix reordering algorithms that can be applied to very large scale networks. Two algorithms: fastR and reduceR, for efficient computations with large resistors networks were developed, see [59] for more details. However these algorithms are much restricted on the resistor networks, hence reduction MOR methods for general algebraic systems still need to be developed.

Chapter 4

Decoupling of DAEs using special projectors

In Section 2.3, we discussed the decoupling of DAEs into differential and algebraic parts. This was done by transforming the DAE into a Weierstraß-Kronecker canonical form. However, this form is numerically infeasible, thus it can not be used in practice. Other tools that can be used to decoupled DAEs are the Drazin inverses and spectral projectors. According to [42], these tools are much restricted on linear constant DAEs and there are no sufficiently good ideas on appropriate generalizations for variable coefficient linear DAEs and nonlinear ones, respectively. This motivated März to decouple DAEs in a different way using special projector and matrix chain [22]. Fortunately, the matrix and projector chain approach applies also in the case of general variable coefficient equations, see [27]. According to [42], there is some first experience to use these decoupling via linearizations for lower index nonlinear problems, in particular. These projectors are approved to be a useful tool, e.g., for stating local solvability, asymptotical stability, see [40]. Actually, some of the most important questions in discussing DAEs seem to be whether the DAE induces a vector field on a manifold and how the state manifold can be described in terms of the original DAE (cf. [50, 51]). Also from this point of view the canonical projector chain has proved its value, see [41]. In [42], these matrix and projector chain were extended to linear constant coefficient DAEs. In this Chapter, we discuss the März decoupling procedure for linear constant coefficient DAEs based on the content in her paper [42].

4.1 März decoupling method

In this Section, instead of using the Weierstraß-Kronecker canonical form, we use projector and matrix chain to decouple DAEs into differential and algebraic parts. This is done iteratively, based on the literature from [42] as follows. Consider linear constant coefficient DAEs of the form

$$\mathbf{E}\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u},\ \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{4.1.1}$$

where $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ with $\mathbf{E}$ singular and the input vector $\boldsymbol{u} \in \mathbb{R}^m$. We intend to decouple (4.1.1) into differential and algebraic parts, using projector and matrix chains. The construction of these projector and matrix chains is based on the definition of the tractability index below.

Definition 4.1.1 (Tractability index [42]) *If we assume that (4.1.1) is solvable, i.e., the matrix pair* $(\mathbf{E}, \mathbf{A})$ *is regular. We define a matrix and projector chain by setting* $\mathbf{E}_0 := \mathbf{E}$ *and* $\mathbf{A}_0 := \mathbf{A}$*, then*

$$\mathbf{E}_{j+1} := \mathbf{E}_j - \mathbf{A}_j\mathbf{Q}_j, \quad \mathbf{A}_{j+1} := \mathbf{A}_j\mathbf{P}_j, \text{ for } j \geq 0, \tag{4.1.2}$$

where $\mathbf{Q}_j$ *are projectors onto* $\operatorname{Ker}\mathbf{E}_j$ *and* $\mathbf{P}_j = \mathrm{I} - \mathbf{Q}_j$*. There exists an index* μ *such that* $\mathbf{E}_\mu$ *is nonsingular and all* $\mathbf{E}_j$ *are singular for all* $0 \leq j < \mu - 1$*. This type of index is called the* tractability index *and we say that the system (4.1.1) has tractability index-*μ*.*

Next, we use the matrix and projector chain defined in (4.1.2) to decouple (4.1.1) as follows. For the initial step, we set: $\mathbf{E}_0 := \mathbf{E}$, $\mathbf{A}_0 := \mathbf{A}$. Then (4.1.1) can be written as

$$\mathbf{E}_0\dot{\boldsymbol{x}} = \mathbf{A}_0\boldsymbol{x} + \mathbf{B}\boldsymbol{u}. \tag{4.1.3}$$

We then choose projector $\mathbf{Q}_0$ such that it projects onto the nullspace of $\mathbf{E}_0$, i.e., $\operatorname{Im}\mathbf{Q}_0 = \operatorname{Ker}\mathbf{E}_0$ and its complementary projector $\mathbf{P}_0 := \mathrm{I} - \mathbf{Q}_0$.
At step 1, we have to define matrices, $\mathbf{E}_1 := \mathbf{E}_0 - \mathbf{A}_0\mathbf{Q}_0, \quad \mathbf{A}_1 := \mathbf{A}_0\mathbf{P}_0$, which satisfy the identities

$$\mathbf{E}_1\mathbf{P}_0 = \mathbf{E}_0, \quad \mathbf{A}_1 - \mathbf{E}_1\mathbf{Q}_0 = \mathbf{A}_0. \tag{4.1.4}$$

Substituting the identities (4.1.4) into (4.1.3), we obtain:

$$\mathbf{E}_1\Big[\mathbf{P}_0\dot{\boldsymbol{x}} + \mathbf{Q}_0\boldsymbol{x}\Big] = \mathbf{A}_1\boldsymbol{x} + \mathbf{B}\boldsymbol{u}. \tag{4.1.5}$$

If we assume $\mathbf{E}_1$ to be nonsingular, then (4.1.5) can be written as

$$\mathbf{P}_0\dot{\boldsymbol{x}} + \mathbf{Q}_0\boldsymbol{x} = \mathbf{E}_1^{-1}\Big[\mathbf{A}_1\boldsymbol{x} + \mathbf{B}\boldsymbol{u}\Big]. \tag{4.1.6}$$

Since $\mathbf{E}_1$ nonsingular, then we say that the DAE (4.1.1) is of tractability index-1 or index-1 DAE, since the subscript of $\mathbf{E}_1$ is $\mu = 1$. We can observe that by left multiplying (4.1.6) by $\mathbf{P}_0$ and $\mathbf{Q}_0$, we obtain the differential and algebraic parts, respectively of the DAE (4.1.1). Thus, the decoupled equivalent system of (4.1.1) with its output equation can be written as:

$$\dot{\boldsymbol{x}}_P = \mathbf{P}_0\mathbf{E}_1^{-1}\mathbf{A}_0\boldsymbol{x}_P + \mathbf{P}_0\mathbf{E}_1^{-1}\mathbf{B}\boldsymbol{u}, \tag{4.1.7a}$$
$$\boldsymbol{x}_Q = \mathbf{Q}_0\mathbf{E}_1^{-1}\mathbf{A}_0\boldsymbol{x}_P + \mathbf{Q}_0\mathbf{E}_1^{-1}\mathbf{B}\boldsymbol{u}, \tag{4.1.7b}$$
$$\boldsymbol{y} = \mathbf{C}^{\mathrm{T}}\boldsymbol{x}_P + \mathbf{C}^{\mathrm{T}}\boldsymbol{x}_Q, \tag{4.1.7c}$$

where $\boldsymbol{x}_P := \mathbf{P}_0\boldsymbol{x}$ and $\boldsymbol{x}_Q := \mathbf{Q}_0\boldsymbol{x}$. Thus (4.1.7a) and (4.1.7b) are the differential and algebraic parts of system (4.1.1) and (4.1.7c) is the decomposed output equation.
If $\mathbf{E}_1$ is singular, we need to repeat the process iteratively as follows: Assume matrices $\mathbf{E}_j, \mathbf{A}_j \in \mathbb{R}^{n\times n}$ and the projectors $\mathbf{Q}_j \in \mathbb{R}^{n\times n}$ onto $\operatorname{Ker}\mathbf{E}_j$, and $\mathbf{P}_j = \mathrm{I} - \mathbf{Q}_j,\ j > 0$ satisfy: $\mathbf{E}_j\mathbf{Q}_j = 0, \quad \mathbf{Q}_j^2 = \mathbf{Q}_j, \quad \mathbf{Q}_j + \mathbf{P}_j = \mathrm{I}$. We also assume that the following form of system (4.1.1) holds, then

$$\mathbf{E}_j\Big[\mathbf{P}_{j-1}\cdots\mathbf{P}_0\dot{\boldsymbol{x}} + \mathbf{Q}_0\boldsymbol{x} + \cdots + \mathbf{Q}_{j-1}\boldsymbol{x}\Big] = \mathbf{A}_j\boldsymbol{x} + \mathbf{B}\boldsymbol{u}. \tag{4.1.8}$$

Equation (4.1.8) coincides with (4.1.5) for $j = 1$. Then, we define the matrices

$$\mathbf{E}_{j+1} := \mathbf{E}_j - \mathbf{A}_j\mathbf{Q}_j, \quad \mathbf{A}_{j+1} := \mathbf{A}_j\mathbf{P}_j,$$

which satisfy the identities

$$\mathbf{E}_{j+1}\mathbf{P}_j = \mathbf{E}_j, \quad \mathbf{A}_{j+1} - \mathbf{E}_{j+1}\mathbf{Q}_j = \mathbf{A}_j.$$

Using the above identities in (4.1.8), we obtain [42]

$$\mathbf{E}_{j+1}\Big[\mathbf{P}_j \cdots \mathbf{P}_0\dot{\boldsymbol{x}} + \mathbf{Q}_0\boldsymbol{x} + \cdots + \mathbf{Q}_j\boldsymbol{x}\Big] = \mathbf{A}_{j+1}\boldsymbol{x} + \mathbf{B}\boldsymbol{u}. \tag{4.1.9}$$

This procedure can be continued indefinitely, but after a finite number of iterations, we end up with a non-singular matrix $\mathbf{E}_\mu$. Then, we have $\mathbf{E}_{\mu+j} = \mathbf{E}_\mu$, $\mathbf{A}_{\mu+j} = \mathbf{A}_\mu \; \forall j \geq 0$. The index-$\mu$ is the so called the tractability index of the DAE (4.1.1) or simply the index of the DAE. For $j = \mu - 1$, the form (4.1.9) becomes:

$$\mathbf{E}_\mu\Big[\mathbf{P}_{\mu-1} \cdots \mathbf{P}_0\dot{\boldsymbol{x}} + \mathbf{Q}_0\boldsymbol{x} + \cdots + \mathbf{Q}_{\mu-1}\boldsymbol{x}\Big] = \mathbf{A}_\mu\boldsymbol{x} + \mathbf{B}\boldsymbol{u}. \tag{4.1.10}$$

Since $\mathbf{E}_\mu$ is nonsingular, we have,

$$\mathbf{P}_{\mu-1} \cdots \mathbf{P}_0\dot{\boldsymbol{x}} + \mathbf{Q}_0\boldsymbol{x} + \cdots + \mathbf{Q}_{\mu-1}\boldsymbol{x} = \mathbf{E}_\mu^{-1}\Big[\mathbf{A}_\mu\boldsymbol{x} + \mathbf{B}\boldsymbol{u}\Big]. \tag{4.1.11}$$

Equation (4.1.11) is the generalization of (4.1.6). Also, the projectors form a generalized decomposition of the identity,

$$\mathrm{I} = \mathbf{P}_{\mu-1} \cdots \mathbf{P}_0 + \mathbf{Q}_0 + \cdots + \mathbf{Q}_{\mu-1}. \tag{4.1.12}$$

It can be proved that the projectors product $\mathbf{P}_{\mu-1} \cdots \mathbf{P}_0$ in this decomposition are not projectors, i.e. $(\mathbf{P}_{\mu-1} \cdots \mathbf{P}_0)^2 \neq \mathbf{P}_{\mu-1} \cdots \mathbf{P}_0$, $\forall \mu > 1$, if we only use Definition (4.1.2) to construct projector chain. Moreover, to decompose higher index DAEs ($\mu > 1$) into differential and algebraic parts, we need to use other decompositions of the identity matrix. Thus, we need to choose somewhat special projectors $\mathbf{Q}_j$ within (4.1.2), in order to obtain an appropriate tool for decoupling the DAE (4.1.1). This is done by introducing an additional constraint [42]

$$\mathbf{Q}_j\mathbf{Q}_i = 0, \quad j > i \tag{4.1.13}$$

in the projector and matrix chain construction. We use this condition (4.1.13) to obtain the absorption properties below:

$$\mathbf{P}_j\mathbf{Q}_i = \mathbf{Q}_i, \quad \mathbf{Q}_j\mathbf{P}_i = \mathbf{Q}_j, \quad \forall\, j > i,$$

which in turn imply, $\mathbf{P}_j\mathbf{P}_{j-1}\cdots\mathbf{P}_0 = \mathrm{I} - \sum_{i=0}^{j}\mathbf{Q}_i$, $\forall j > 0$. We use the absorption properties to come up with other decompositions of the identity matrix. Hence to decouple (4.1.11), we need to first choose special projectors $\mathbf{Q}_j$, $j > 0$ that satisfy (4.1.13). We note that these special projectors that satisfy (4.1.13) exist in practice and their construction is well discussed in [44]. According to [42], to decouple (4.1.1), we need to decompose the identity matrix into two ways:

$$\mathrm{I} = \mathbf{P}_0 + \mathbf{Q}_0 = \Pi_{\mu-1} + \Pi_{\mu-2}\mathbf{Q}_{\mu-1} + \cdots + \Pi_0\mathbf{Q}_1 + \mathbf{Q}_0, \tag{4.1.14}$$

$$\mathrm{I} = \mathbf{P}_{\mu-1} + \mathbf{Q}_{\mu-1} = \Pi_0^* + \mathbf{Q}_0\Pi_1^* + \cdots + \mathbf{Q}_{\mu-2}\Pi_{\mu-1}^* + \mathbf{Q}_{\mu-1}, \tag{4.1.15}$$

where $\Pi_j := \mathbf{P}_0\mathbf{P}_1\cdots\mathbf{P}_j, \quad \Pi_j^* = \mathbf{P}_j\mathbf{P}_{j+1}\cdots\mathbf{P}_{\mu-1}, \quad j = 0, 1, \ldots, \mu-1$. If (4.1.13) holds then both decompositions of identity matrix are made up of mutually orthogonal projectors. We can now use these two decompositions to decompose higher index DAE into differential and algebraic equations. This done as follows: Decomposition (4.1.14) is used to define the differential and algebraic components:

$$\boldsymbol{x}_P := \Pi_{\mu-1}\boldsymbol{x}, \quad \boldsymbol{x}_{Q,0} := \mathbf{Q}_0\boldsymbol{x}, \quad \boldsymbol{x}_{Q,i} := \Pi_{i-1}\mathbf{Q}_i\boldsymbol{x}, \quad i = 1, \ldots \mu-1. \tag{4.1.16}$$

The second decomposition (4.1.15) is used to derive the differential and algebraic parts. In general, using the decompositions (4.1.14) and (4.1.15), provided (4.1.13) is valid, we can decompose DAEs with arbitrary index into differential and algebraic parts. Without loss of generality, if the DAE (4.1.1) is of tractability index-μ, then its decoupled system is given by

$$\dot{\boldsymbol{x}}_P = \Pi_0^*\mathbf{E}_\mu^{-1}(\mathbf{A}_\mu\boldsymbol{x}_P + \mathbf{B}\boldsymbol{u}), \quad \boldsymbol{x}_P(0) = \Pi_{\mu-1}\boldsymbol{x}(0), \tag{4.1.17a}$$

$$\boldsymbol{x}_{Q,\mu-1} = \Pi_{\mu-2}\mathbf{Q}_{\mu-1}\mathbf{E}_\mu^{-1}(\mathbf{A}_\mu\boldsymbol{x}_P + \mathbf{B}\boldsymbol{u}), \tag{4.1.17b}$$

$$\boldsymbol{x}_{Q,i} = \Pi_{i-1}\mathbf{Q}_i\Pi_{i+1}^*\mathbf{E}_\mu^{-1}(\mathbf{A}_\mu\boldsymbol{x}_P + \mathbf{B}\boldsymbol{u}) + \sum_{j=i+1}^{\mu-1}\Pi_{i-1}\mathbf{Q}_{i,j}\dot{\boldsymbol{x}}_{Q,j}, \quad i = \mu-2, \ldots 0, \tag{4.1.17c}$$

where, $\mathbf{Q}_{i,j} = \begin{cases} \mathbf{Q}_i\mathbf{Q}_{i+1}, & j = i+1, \\ \mathbf{Q}_i\mathbf{P}_{i+1}\dots\mathbf{P}_{j-1}\mathbf{Q}_j, & j > i+1 \end{cases}$

Then, solution (4.1.1) can be obtained using the formula $\boldsymbol{x} = \boldsymbol{x}_P + \sum_{i=0}^{\mu-1}\boldsymbol{x}_{Q,i}$. Equations (4.1.17a)-(4.1.17c) can be solved in the following way: first, the differential part $\boldsymbol{x}_P$ is computed from the purely differential equation (4.1.17a); then the algebraic parts are computed, starting from the last one, $\boldsymbol{x}_{Q,\mu-1}$, given by (4.1.17b), and substituting the computed values in the last but one equation for $\boldsymbol{x}_{Q,\mu-2}$, given by (4.1.17c) for $i = \mu-2$, and so on, up to the first equation for $\boldsymbol{x}_{Q,0}$. We observe that at each substitution, an additional time derivative appears. In the next example, we illustrate the decoupling of index-1 DAE using März decoupling procedure.

Example 4.1.1 Consider a semi-explicit index-1 DAE with the following system matrices:

$$\mathbf{E} = \begin{pmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{pmatrix}. \tag{4.1.18}$$

We assume $\mathbf{E}_{11} \in \mathbb{R}^{n_1,n_1}$ and $\mathbf{A}_{21}\mathbf{E}_{11}^{-1}\mathbf{E}_{12} - \mathbf{A}_{22} \in \mathbb{R}^{n_2\times n_2}$ are nonsingular blocks due to index-1 property and $n = n_1 + n_2$ is the dimension of the DAE. Let $\mathbf{E}_0 = \mathbf{E}$ and $\mathbf{A}_0 = \mathbf{A}$. We can then choose projectors $\mathbf{Q}_0$ and $\mathbf{P}_0$ given by

$$\mathbf{Q}_0 = \begin{pmatrix} \mathbf{0} & -\mathbf{E}_{11}^{-1}\mathbf{E}_{12} \\ \mathbf{0} & \mathrm{I} \end{pmatrix} \quad \text{and} \quad \mathbf{P}_0 = \begin{pmatrix} \mathrm{I} & \mathbf{E}_{11}^{-1}\mathbf{E}_{12} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{4.1.19}$$

such that $\operatorname{Im}\mathbf{Q}_0 = \operatorname{Ker}\mathbf{E}_0$ and $\mathbf{P}_0 = \mathrm{I} - \mathbf{Q}_0$. Next, we compute $\mathbf{E}_1 = \mathbf{E}_0 - \mathbf{A}_0\mathbf{Q}_0$ given by:

$$\mathbf{E}_1 = \begin{pmatrix} \mathbf{E}_{11} & (\mathrm{I} + \mathbf{A}_{11}\mathbf{E}_{11}^{-1})\mathbf{E}_{12} - \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{21}\mathbf{E}_{11}^{-1}\mathbf{E}_{12} - \mathbf{A}_{22} \end{pmatrix}. \tag{4.1.20}$$

Since $\mathbf{E}_1$ is nonsingular, thus (4.1.18) is indeed an index-1 DAE. Substituting Equations (4.1.18)-(4.1.20) into (4.1.7), we obtain the decoupled system of (4.1.18) using März decoupling procedure. This leads to a decoupled system of dimension $2n$. Hence this decoupling procedure does not preserve the dimension of the DAE.

In [42, Sec. 2], it is shown that the projectors $\mathbf{P}_j$ may be chosen such that they are canonical, i.e., the related decoupling becomes complete. This implies that (4.1.17) can be decoupled completely using the canonical projectors. In [42], März also compared these projectors with spectral projector decoupling and she was able to show that $\mathbf{P}_0\mathbf{P}_1 \cdots \mathbf{P}_{\mu-1}$ represents in fact the projector onto the subspace corresponding to the finite eigenvalues of the matrix pencil $(\mathbf{E}, \mathbf{A})$ along its infinite eigenspace.

Remark 4.1.1 *The tractability index-μ is independent of the choice of the projectors $\mathbf{Q}_j$ and it coincides with the differentiation and Kronecker index for the case of linear constant coefficient DAEs. The tractability index concept is numerically feasible compared to other index concepts because it does not involve computing derivatives arrays.*

However, the drawback of the projector and matrix approach used to be the computationally expensive construction of projectors $\mathbf{Q}_j$ onto the nullspace of $\mathbf{E}_j$ for large-scale sparse matrices. The standard way to compute these projectors is to use SVD or alike decompositions to find the nullspace of the singular matrix $\mathbf{E}_j$, which can be very expensive for very large-size matrices. Fortunately, they have been successful development in efficient construction of such projectors, see [66] for more details. This is briefly discussed below. We discuss a fast way to construct projector $\mathbf{Q}_j$ onto the nullspace of $\mathbf{E}_j$ of large sparse matrix. This approach uses the sparse LU decomposition- based routine presented in [66], called LUQ. This routine decomposes a singular sparse matrix $\mathbf{E}_j$, into $\mathbf{E}_j^{\mathrm{T}} = \mathbf{L}_j \begin{pmatrix} \mathbf{U}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{R}_j$, where $\mathbf{L}_j, \mathbf{R}_j \in \mathbb{R}^{n \times n}$ are nonsingular matrices, $\mathbf{U}_j \in \mathbb{R}^{r \times r}$ is a nonsingular upper triangular matrix, r is the rank $\mathbf{E}_j$. Using this routine as a starting step and using the fact that the nullspace of $\mathbf{E}_j$ can be computed via its left nullspace of $\mathbf{E}_j^{\mathrm{T}}$, we can compute projectors $\mathbf{Q}_j$ onto nullspace of $\mathbf{E}_j$ in an optimal way. Hence, this algorithm can be used to compute projectors $\mathbf{Q}_j$ onto the nullspace of $\mathbf{E}_j$ for large sparse matrices and it is numerically tested on large-scale sparse matrices, see [66]. We have discussed how to decouple constant coefficients linear DAEs using the März decoupling procedure. This procedure can be implemented numerically and leads to good solutions. However the März decomposition procedure has two main limitations.

(i) It can easily proved that the DAE (4.1.1) of index-μ with dimension n leads to a decoupled system (4.1.17) of total dimension $n(\mu + 1)$. Thus the März decoupling procedure does not preserve the dimension of the DAEs, see [44].
(ii) This decoupling procedure does not also preserve the spectrum of the matrix pencil.

Hence it is impractical to apply model order reduction on the decoupled system (4.1.17).

Chapter 5

Decoupling of DAEs using special bases

In Chapter 4, we have already discussed that the März decoupling procedure which uses projector and matrix chain to decouple the DAE into one differential and μ algebraic parts. However, this decoupling procedure leads to a much larger decoupled system of total dimension $n(\mu + 1)$. This limits us from using März decoupling procedure in its original form to apply model order reduction. The reason of the increase in the dimension of the decoupled system is due to the use of projectors whose column rank is always less than their respective dimension. This introduces some redundancy into the decoupled system. In this Chapter, we present away of avoiding this redundancy. The main idea is to decouple the DAE using the linearly independent columns of the projector matrices intend of using the projectors matrices themselves.

5.1 Modification of März decoupling procedure

In this Section, we propose a procedure to modify the März decoupling procedure in order to preserve the mathematical properties of the DAE. The main idea of this procedure is to use bases of the projectors instead of the full projectors. Our main tool is the Rank-Nullity theorem below.

Theorem 5.1.1 (Rank-Nullity Theorem [47]) *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be vector spaces over a field* $\mathbb{F}$*, and let* $T : \mathbf{V} \to \mathbf{W}$ *be a linear transformation. Assuming the dimension of* $\mathbf{V}$ *is finite, then* $\dim(\mathbf{V}) = \dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T))$.

The proof can be found in [47].

5.2 Index-1 DAEs

In Section 4.1, we observed that when decoupling index-1 DAEs using März decoupling procedure leads to a decoupled system of dimension $2n$. In this Section, we propose a procedure which can lead to a decoupled system which preserves the dimension of index-1 DAEs. This can be done as follows: Recall from Section 4.1, we can decouple index-1 DAEs using März decoupling procedure leading to

$$\dot{\boldsymbol{x}}_P = \mathbf{P}_0\mathbf{E}_1^{-1}\mathbf{A}_0\boldsymbol{x}_P + \mathbf{P}_0\mathbf{E}_1^{-1}\mathbf{B}\boldsymbol{u}, \tag{5.2.1a}$$

$$\boldsymbol{x}_Q = \mathbf{Q}_0\mathbf{E}_1^{-1}\mathbf{A}_0\boldsymbol{x}_P + \mathbf{Q}_0\mathbf{E}_1^{-1}\mathbf{B}\boldsymbol{u}, \tag{5.2.1b}$$

$$\boldsymbol{y} = \mathbf{C}^{\mathrm{T}}\boldsymbol{x}_P + \mathbf{C}^{\mathrm{T}}\boldsymbol{x}_Q, \tag{5.2.1c}$$

where $\boldsymbol{x}_P := \mathbf{P}_0\boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{x}_Q := \mathbf{Q}_0\boldsymbol{x} \in \mathbb{R}^n$ are the differential and algebraic components, respectively. The solution of the DAE is obtained using the formula, $\boldsymbol{x} = \boldsymbol{x}_P + \boldsymbol{x}_Q \in \mathbb{R}^n$, leading to a decomposed output equation (5.2.1c). We can observe that decoupled system (5.2.1) is of dimension $2n$. This is because the projectors $\mathbf{Q}_0, \mathbf{P}_0 \in \mathbb{R}^{n\times n}$ introduces some redundancy in the decoupled system as a result, we obtained $2n$ linearly dependent equations. We can remove this redundancy as follows: Using the rank-nullity Theorem 5.1.1. Let $n_p = \mathrm{rank}(\mathbf{E}_0)$, $n_q = n - n_p$, and let us consider a basis matrix $(\boldsymbol{q}, \boldsymbol{p}) = \{\boldsymbol{q}_1, \cdots, \boldsymbol{q}_{n_q}, \boldsymbol{p}_1, \cdots, \boldsymbol{p}_{n_p}\} \in \mathbb{R}^n$ made of n_q independent columns of projection matrix $\mathbf{Q}_0$ and n_p independent columns of the complementary projection matrix $\mathbf{P}_0$, such that,

$$\mathbf{Q}_0\boldsymbol{q} = \boldsymbol{q}, \quad \mathbf{Q}_0\boldsymbol{p} = 0, \quad \mathbf{P}_0\boldsymbol{q} = 0, \quad \mathbf{P}_0\boldsymbol{p} = \boldsymbol{p}, \tag{5.2.2}$$

holds. Then, we can expand $\boldsymbol{x}$ with respect to the new basis, obtaining $\boldsymbol{x} = \boldsymbol{q}\xi_q + \boldsymbol{p}\xi_p, \quad \xi_q \in \mathbb{R}^{n_q}, \quad \xi_p \in \mathbb{R}^{n_p}$, which implies that $\boldsymbol{x}_P = \boldsymbol{p}\xi_p \quad \text{and} \quad \boldsymbol{x}_Q = \boldsymbol{q}\xi_q$. Since $(\boldsymbol{q}, \boldsymbol{p})$ is a basis matrix, it is invertible, and let $\left(\boldsymbol{q}^{*\mathrm{T}} \ \boldsymbol{p}^{*\mathrm{T}}\right)^{\mathrm{T}}$ be its inverse, where $\boldsymbol{q}^{*\mathrm{T}} \in \mathbb{R}^{n_q \times n}$ and $\boldsymbol{p}^{*\mathrm{T}} \in \mathbb{R}^{n_p \times n}$. Then, we have

$$\boldsymbol{q}^{*\mathrm{T}}\boldsymbol{q} = \mathrm{I}, \quad \boldsymbol{q}^{*\mathrm{T}}\boldsymbol{p} = 0, \quad \boldsymbol{p}^{*\mathrm{T}}\boldsymbol{q} = 0, \quad \boldsymbol{p}^{*\mathrm{T}}\boldsymbol{p} = \mathrm{I}, \tag{5.2.3}$$

which gives $\boldsymbol{p}^{*\mathrm{T}}\boldsymbol{x} = \xi_p \quad \text{and} \quad \boldsymbol{q}^{*\mathrm{T}}\boldsymbol{x} = \xi_q$. In terms of ξ_p and ξ_q, system (5.2.1) simplifies to

$$\begin{aligned} \xi_p' &= \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u}, \\ \xi_q &= \mathbf{A}_q\xi_p + \mathbf{B}_q\boldsymbol{u}, \\ \mathbf{y} &= \mathbf{C}_p^{\mathrm{T}}\xi_p + \mathbf{C}_q^{\mathrm{T}}\xi_q, \end{aligned} \tag{5.2.4}$$

where
$\mathbf{A}_p = \boldsymbol{p}^{*\mathrm{T}}\mathbf{E}_1^{-1}\mathbf{A}_0\boldsymbol{p} \in \mathbb{R}^{n_p \times n_p}$, $\mathbf{B}_p = \boldsymbol{p}^{*\mathrm{T}}\mathbf{E}_1^{-1}\mathbf{B} \in \mathbb{R}^{n_p \times m}$, $\mathbf{A}_q = \boldsymbol{q}^{*\mathrm{T}}\mathbf{E}_1^{-1}\mathbf{A}_0\boldsymbol{p} \in \mathbb{R}^{n_q \times n_p}$, $\mathbf{B}_q = \boldsymbol{q}^{*\mathrm{T}}\mathbf{E}_1^{-1}\mathbf{B} \in \mathbb{R}^{n_q \times m}$ and $\mathbf{C}_q = \boldsymbol{q}^{\mathrm{T}}\mathbf{C} \in \mathbb{R}^{n_q \times \ell}$, $\mathbf{C}_p = \boldsymbol{p}^{\mathrm{T}}\mathbf{C} \in \mathbb{R}^{n_p \times \ell}$. We can observe that the total dimension of the decoupled system is $n = n_p + n_q$, which is equal to the dimension of the decoupled system. This system also preserves the spectrum of the matrix pair $(\mathbf{E}, \mathbf{A})$ since it can easily be proved that $\sigma_f(\mathbf{E}, \mathbf{A}) = \sigma(\mathbf{A}_p)$. Thus this procedure preserve the dimension of the DAE in contrast with the März decoupling procedure. The example below illustrates how one can decouple index-1 DAE using the modified März decoupling procedure for index-1 DAEs.

Example 5.2.1 In this example, we use system matrices (4.1.18) from Example 4.1.1, for comparison. Thus, we use the same procedure (4.1.18)–(4.1.20) to construct matrix and projector chain: $\mathbf{E}_0, \mathbf{E}_1$ and $\mathbf{Q}_0, \mathbf{P}_0$. Here, we only need to construct the bases of projectors $\mathbf{P}_0$ and $\mathbf{Q}_0$, and their respective inverses given by

$$\boldsymbol{p}_0 = \begin{pmatrix} \mathrm{I} \\ \mathbf{0} \end{pmatrix}, \ \boldsymbol{q}_0 = \begin{pmatrix} -\mathbf{E}_{11}^{-1}\mathbf{E}_{12} \\ \mathrm{I} \end{pmatrix} \quad \text{and} \quad \boldsymbol{p}^{*\mathrm{T}} = \begin{pmatrix} \mathrm{I} & \mathbf{E}_{11}^{-1}\mathbf{E}_{12} \end{pmatrix}, \ \boldsymbol{q}^{*\mathrm{T}} = \begin{pmatrix} \mathbf{0} & \mathrm{I} \end{pmatrix}. \tag{5.2.5}$$

Finally substituting (4.1.18)-(4.1.20) and (5.2.5) into (5.2.4), we obtain the modified decoupled system of (4.1.18) with coefficient matrices given by

$$\mathbf{A}_p = \mathbf{E}_{11}^{-1}\Big[\mathbf{A}_{11} - \Big(\mathbf{A}_{11}\mathbf{E}_{11}^{-1}\mathbf{E}_{12} - \mathbf{A}_{12}\Big)\Big(\mathbf{A}_{21}\mathbf{E}_{11}^{-1}\mathbf{E}_{12} - \mathbf{A}_{22}\Big)^{-1}\mathbf{A}_{21}\Big] \in \mathbb{R}^{n_1\times n_1},$$
$$\mathbf{B}_p = \mathbf{E}_{11}^{-1}\Big[\mathbf{B}_{1} - \Big(\mathbf{A}_{11}\mathbf{E}_{11}^{-1}\mathbf{E}_{12} - \mathbf{A}_{12}\Big)\Big(\mathbf{A}_{21}\mathbf{E}_{11}^{-1}\mathbf{E}_{12} - \mathbf{A}_{22}\Big)^{-1}\mathbf{B}_{2}\Big] \in \mathbb{R}^{n_1\times m},$$
$$\mathbf{A}_q = \Big(\mathbf{A}_{21}\mathbf{E}_{11}^{-1}\mathbf{E}_{12} - \mathbf{A}_{22}\Big)^{-1}\mathbf{A}_{21} \in \mathbb{R}^{n_2\times n_1}, \quad \mathbf{B}_q = \Big(\mathbf{A}_{21}\mathbf{E}_{11}^{-1}\mathbf{E}_{12} - \mathbf{A}_{22}\Big)^{-1}\mathbf{B}_{2} \in \mathbb{R}^{n_2\times m},$$
$$\mathbf{C}_p = \mathbf{C}_1 \in \mathbb{R}^{n_1\times \ell}, \quad \mathbf{C}_q = \mathbf{C}_2 - \mathbf{E}_{12}^{\mathrm{T}}\mathbf{E}_{11}^{-T}\mathbf{C}_1 \in \mathbb{R}^{n_2\times \ell}.$$

We can observe that the DAE (4.1.18) is decoupled into n_1 differential equations and n_2 algebraic equations, whose total dimension is $n = n_1 + n_2$. Hence the dimension of the DAE system is preserved. If we compare Example 5.2.1 and Example 4.1.1, we observe that the Example 5.2.1 preserves the dimension and the stability of the DAE while Example 4.1.1 does not.

5.3 Index-2 DAEs

In this Section, we assume (4.1.1) is a DAE of tractability index-2. Thus, we need to assume that projectors $\mathbf{Q}_0$ and $\mathbf{Q}_1$ satisfy the condition (4.1.13), that is, $\mathbf{Q}_1\mathbf{Q}_0 = 0$. We can, then decouple system (4.1.1) using the März decoupling procedure. Substituting $\mu = 2$ into (4.1.17), we obtain a decoupled system for index-2 DAEs given by

$$\dot{\boldsymbol{x}}_P = \mathbf{P}_0\mathbf{P}_1\mathbf{E}_2^{-1}\mathbf{A}_2\boldsymbol{x}_P + \mathbf{P}_0\mathbf{P}_1\mathbf{E}_2^{-1}\mathbf{B}\boldsymbol{u}, \tag{5.3.1a}$$
$$\boldsymbol{x}_{Q,1} = \mathbf{Q}_1\mathbf{E}_2^{-1}\mathbf{A}_2\boldsymbol{x}_P + \mathbf{Q}_1\mathbf{E}_2^{-1}\mathbf{B}\boldsymbol{u}, \tag{5.3.1b}$$
$$\boldsymbol{x}_{Q,0} = \mathbf{Q}_0\mathbf{P}_1\mathbf{E}_2^{-1}\mathbf{A}_2\boldsymbol{x}_P + \mathbf{Q}_0\mathbf{P}_1\mathbf{E}_2^{-1}\mathbf{B}\boldsymbol{u} + \mathbf{Q}_0\mathbf{Q}_1\dot{\boldsymbol{x}}_{Q,1}, \tag{5.3.1c}$$
$$\boldsymbol{y} = \mathbf{C}^{\mathrm{T}}\boldsymbol{x}_P + \mathbf{C}^{\mathrm{T}}\boldsymbol{x}_{Q,0} + \mathbf{C}^{\mathrm{T}}\boldsymbol{x}_{Q,1}. \tag{5.3.1d}$$

We observe that $\boldsymbol{x}_P, \boldsymbol{x}_{Q,1}, \boldsymbol{x}_{Q,0} \in \mathbb{R}^n$. We can observe that this time the decoupled system (5.3.1) is of dimension $3n$. Next, we need to modify system (5.3.1) as follows. We first construct basis column matrices from the projectors. The starting point is the same as that of index-1 DAEs. Let $k_0 = \dim(\operatorname{Ker}\mathbf{E})$, this implies that $n_0 = n - k_0$, and let us consider a basis matrix $(\boldsymbol{p}, \boldsymbol{q}) = \{\boldsymbol{p}_1, \cdots, \boldsymbol{p}_{n_0}, \boldsymbol{q}_1, \cdots, \boldsymbol{q}_{k_0}\} \in \mathbb{R}^n$ made of k_0 independent columns of projection matrix $\mathbf{Q}_0$ and n_0 independent columns of the complementary projection matrix $\mathbf{P}_0$, such that,

$$\mathbf{Q}_0\boldsymbol{q} = \boldsymbol{q}, \quad \mathbf{Q}_0\boldsymbol{p} = 0, \quad \mathbf{P}_0\boldsymbol{q} = 0, \quad \mathbf{P}_0\boldsymbol{p} = \boldsymbol{p}, \tag{5.3.2}$$

holds. Then the inverse of the basis matrix is $(\boldsymbol{p}, \boldsymbol{q})^{-1} = \left(\boldsymbol{p}^{*\mathrm{T}} \ \boldsymbol{q}^{*\mathrm{T}}\right)^{\mathrm{T}}$, where $\boldsymbol{q}^{*\mathrm{T}} \in \mathbb{R}^{k_0 \times n}$ and $\boldsymbol{p}^{*\mathrm{T}} \in \mathbb{R}^{n_0 \times n}$. Then, we have

$$\boldsymbol{q}^{*\mathrm{T}}\boldsymbol{q} = \mathrm{I}_{k_0}, \quad \boldsymbol{q}^{*\mathrm{T}}\boldsymbol{p} = \mathbf{0}, \quad \boldsymbol{p}^{*\mathrm{T}}\boldsymbol{q} = \mathbf{0}, \quad \boldsymbol{p}^{*\mathrm{T}}\boldsymbol{p} = \mathrm{I}_{n_0}. \tag{5.3.3}$$

Next, we use the basis matrix $(\boldsymbol{p}, \boldsymbol{q})$ and its inverse $\left(\boldsymbol{p}^{*\mathrm{T}} \ \boldsymbol{q}^{*\mathrm{T}}\right)^{\mathrm{T}}$ as the starting basis for the construction of new bases for index-2 DAEs. From (4.1.14), setting $\mu = 2$, we can decompose the identity matrix as follows,

$$\mathrm{I} = \mathbf{P}_0\mathbf{P}_1 + \mathbf{P}_0\mathbf{Q}_1 + \mathbf{Q}_0, \tag{5.3.4}$$

in order to obtain the differential and algebraic components for index-2 DAEs. We need to construct the basis of projector products $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{Q}_1$. We note that for the case of index-1 DAEs, we can always have a differential part but this is not always the case for higher index systems depending on the spectrum of the matrix pencil $(\mathbf{E}, \mathbf{A})$. We note that in this sense linear systems can ne viewed as index-1 DAEs without a differential part. If the spectrum of the matrix pencil has no finite spectrum, i.e., $\sigma(\mathbf{E}, \mathbf{A}) = \sigma_\infty(\mathbf{E}, \mathbf{A})$ or $\det(\lambda\mathbf{E} - \mathbf{A}) = c \in \mathbb{C} \setminus \{0\}$, $\forall\lambda \in \mathbb{C}$, then decoupled system has no differential part otherwise it has a differential part. This implies that $\mathbf{P}_0\mathbf{P}_1 = 0$ or $\mathbf{P}_0\mathbf{P}_1 \neq 0$ depending on the spectrum of the index-2 DAEs. If $\mathbf{P}_0\mathbf{P}_1 = 0$ then we have no differential part otherwise we have a differential part. We consider both cases in the Sections below.

5.3.1 Index-2 DAEs with a differential part

Here, we assume that the matrix pencil of (4.1.1) has at least one finite eigenvalue, that is, $\mathbf{P}_0\mathbf{P}_1 \neq 0$. We, then construct the bases for $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{Q}_1$ as follows:
If we right multiply (5.3.4) by the basis column matrix $\boldsymbol{p}$ of complementary projection matrix $\mathbf{P}_0$ and simplifying, we obtain,

$$\boldsymbol{p} = \mathbf{P}_0\mathbf{P}_1\boldsymbol{p} + \mathbf{P}_0\mathbf{Q}_1\boldsymbol{p}, \tag{5.3.5}$$

since $\mathbf{Q}_0\boldsymbol{p} = 0$. Then, if we left multiply (5.3.5) by $\boldsymbol{p}^{*\mathrm{T}}$, we obtain:

$$\boldsymbol{p}^{*\mathrm{T}}\boldsymbol{p} = \boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_0\mathbf{P}_1\boldsymbol{p} + \boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_0\mathbf{Q}_1\boldsymbol{p}. \tag{5.3.6}$$

We already know that $\boldsymbol{p}^{*\mathrm{T}}\boldsymbol{p} = \mathrm{I}_{n_0}$ and if we let $\mathbf{Z}_{\boldsymbol{p}_0} = \boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_0\mathbf{P}_1\boldsymbol{p} = \boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_1\boldsymbol{p}$ and $\mathbf{Z}_{\boldsymbol{q}_0} = \boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_0\mathbf{Q}_1\boldsymbol{p} = \boldsymbol{p}^{*\mathrm{T}}\mathbf{Q}_1\boldsymbol{p}$, we have,

$$\mathrm{I}_{n_0} = \mathbf{Z}_{\boldsymbol{p}_0} + \mathbf{Z}_{\boldsymbol{q}_0}. \tag{5.3.7}$$

Then, we can come up with the Theorem below [2].

Theorem 5.3.1 *Let $\mathbf{Z}_{\boldsymbol{p}_0} = \boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_1\boldsymbol{p}$ and $\mathbf{Z}_{\boldsymbol{q}_0} = \boldsymbol{p}^{*\mathrm{T}}\mathbf{Q}_1\boldsymbol{p}$, then $\mathbf{Z}_{\boldsymbol{p}_0}, \mathbf{Z}_{\boldsymbol{q}_0} \in \mathbb{R}^{n_0 \times n_0}$ are projectors in $\mathbb{R}^{n_0}$ provided the constraint condition $\mathbf{Q}_1\mathbf{Q}_0 = 0$ holds. Moreover they are orthogonal complimentary to each other, i.e., $\mathrm{I}_{n_0} = \mathbf{Z}_{\boldsymbol{p}_0} + \mathbf{Z}_{\boldsymbol{q}_0}$.*

Proof 5.3.1 *We need to show that $\mathbf{Z}_{\boldsymbol{p}_0}^2 = \mathbf{Z}_{\boldsymbol{p}_0}$ and $\mathbf{Z}_{\boldsymbol{q}_0}^2 = \mathbf{Z}_{\boldsymbol{q}_0}$ as follows. In this prove we assume that projector $\mathbf{Q}_0$ and $\mathbf{Q}_1$ are chosen such that $\mathbf{Q}_1\mathbf{Q}_0 = 0$ holds. Then,*

$$\begin{aligned}
\mathbf{Z}_{\boldsymbol{p}_0}^2 &= (\boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_0\mathbf{P}_1\boldsymbol{p})^2 = \boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_0\mathbf{P}_1\boldsymbol{p}\boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_0\mathbf{P}_1\boldsymbol{p},\\
&= \boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_0\mathbf{P}_1\mathbf{P}_0\mathbf{P}_1\boldsymbol{p},\\
&= \boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_0(\mathbf{P}_1 - \mathbf{P}_1\mathbf{Q}_0)\mathbf{P}_1\boldsymbol{p},\\
&= \boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_0(\mathbf{P}_1 - \mathbf{Q}_0)\mathbf{P}_1\boldsymbol{p}, \textit{ Since } \mathbf{P}_1\mathbf{Q}_0 = \mathbf{Q}_0, \textit{ iff } \quad \mathbf{Q}_1\mathbf{Q}_0 = 0.\\
&= \boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_0\mathbf{P}_1\boldsymbol{p} = \mathbf{Z}_{\boldsymbol{p}_0}.
\end{aligned}$$

Also

$$\begin{aligned}
\mathbf{Z}_{\boldsymbol{q}_0}^2 &= (\boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_0\mathbf{Q}_1\boldsymbol{p})^2 = \boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_0\mathbf{Q}_1\boldsymbol{p}\boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_0\mathbf{Q}_1\boldsymbol{p},\\
&= \boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_0\mathbf{Q}_1\mathbf{P}_0\mathbf{Q}_1\boldsymbol{p},\\
&= \boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_0\mathbf{Q}_1^2\boldsymbol{p}, \textit{ Since } \mathbf{Q}_1\mathbf{P}_0 = \mathbf{Q}_1, \textit{ iff } \quad \mathbf{Q}_1\mathbf{Q}_0 = 0.\\
&= \boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_0\mathbf{Q}_1\boldsymbol{p} = \mathbf{Z}_{\boldsymbol{q}_0}.
\end{aligned}$$

Hence proved as required that $\mathbf{Z}_{\boldsymbol{p}_0}$ and $\mathbf{Z}_{\boldsymbol{q}_0}$ are projectors and are orthogonal compliment to each other.

We need now to construct the bases of the projectors $\mathbf{Z}_{\boldsymbol{p}_0}$ and $\mathbf{Z}_{\boldsymbol{q}_0}$. Let $k_1 = \dim(\operatorname{Im}\mathbf{Z}_{\boldsymbol{q}_0})$, this implies $n_1 = n_0 - k_1$, and let us consider a basis matrix $(\mathbf{z}_{\boldsymbol{p}_0}, \mathbf{z}_{\boldsymbol{q}_0}) \in \mathbb{R}^{n_0}$ made of n_1 independent columns of projection matrix $\mathbf{Z}_{\boldsymbol{p}_0}$ and k_1 independent columns of the complementary projection matrix $\mathbf{Z}_{\boldsymbol{q}_0}$, such that,

$$\mathbf{Z}_{\boldsymbol{p}_0}\mathbf{z}_{\boldsymbol{p}_0} = \mathbf{z}_{\boldsymbol{p}_0}, \quad \mathbf{Z}_{\boldsymbol{p}_0}\mathbf{z}_{\boldsymbol{q}_0} = 0, \quad \mathbf{Z}_{\boldsymbol{q}_0}\mathbf{z}_{\boldsymbol{p}_0} = 0, \quad \mathbf{Z}_{\boldsymbol{q}_0}\mathbf{z}_{\boldsymbol{q}_0} = \mathbf{z}_{\boldsymbol{q}_0}, \tag{5.3.8}$$

holds. Since $(\mathbf{z}_{\boldsymbol{p}_0}, \mathbf{z}_{\boldsymbol{q}_0})$ is a basis matrix, it is nonsingular, and let $\left(\mathbf{z}^{*\mathrm{T}}_{\boldsymbol{p}_0}\ \mathbf{z}^{*\mathrm{T}}_{\boldsymbol{q}_0}\right)^{\mathrm{T}}$ be its inverse, where $\mathbf{z}^{*\mathrm{T}}_{\boldsymbol{p}_0} \in \mathbb{R}^{n_1 \times n_0}$ and $\mathbf{z}^{*\mathrm{T}}_{\boldsymbol{q}_0} \in \mathbb{R}^{k_1 \times n_0}$. Then, we have

$$\mathbf{z}^{*\mathrm{T}}_{\boldsymbol{p}_0}\mathbf{z}_{\boldsymbol{p}_0} = \mathrm{I}_{n_1}, \quad \mathbf{z}^{*\mathrm{T}}_{\boldsymbol{p}_0}\mathbf{z}_{\boldsymbol{q}_0} = 0, \quad \mathbf{z}^{*\mathrm{T}}_{\boldsymbol{q}_0}\mathbf{z}_{\boldsymbol{p}_0} = 0, \quad \mathbf{z}^{*\mathrm{T}}_{\boldsymbol{q}_0}\mathbf{z}_{\boldsymbol{q}_0} = \mathrm{I}_{k_1}. \tag{5.3.9}$$

Hence the basis of projectors products $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{Q}_1$ are given by $\boldsymbol{p}\mathbf{z}_{\boldsymbol{p}_0} \in \mathbb{R}^{n \times n_1}$ and $\boldsymbol{p}\mathbf{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{n \times k_1}$, respectively, where $n_0 = k_1 + n_1$ such that,

$$\mathbf{P}_0\mathbf{P}_1\boldsymbol{p}\mathbf{z}_{\boldsymbol{p}_0} = \boldsymbol{p}\mathbf{z}_{\boldsymbol{p}_0}, \quad \mathbf{P}_0\mathbf{P}_1\mathbf{z}_{\boldsymbol{q}_0} = 0, \quad \mathbf{P}_0\mathbf{Q}_1\boldsymbol{p}\mathbf{z}_{\boldsymbol{p}_0} = 0, \quad \mathbf{P}_0\mathbf{Q}_1\mathbf{z}_{\boldsymbol{q}_0} = \mathbf{z}_{\boldsymbol{q}_0}. \tag{5.3.10}$$

Then, we can now expand $\boldsymbol{x}$ with respect to the new bases, obtaining

$$\boldsymbol{x} = \boldsymbol{p}\mathbf{z}_{\boldsymbol{p}_0}\xi_p + \boldsymbol{p}\mathbf{z}_{\boldsymbol{q}_0}\xi_{q,1} + \boldsymbol{q}\xi_{q,0}, \quad \xi_p \in \mathbb{R}^{n_1}, \quad \xi_{q,1} \in \mathbb{R}^{k_1}, \quad \xi_{q,0} \in \mathbb{R}^{k_0}, \tag{5.3.11}$$

which implies that ,

$$\boldsymbol{x}_P = \boldsymbol{p}\mathbf{z}_{\boldsymbol{p}_0}\xi_p, \quad \boldsymbol{x}_{Q,1} = \boldsymbol{p}\mathbf{z}_{\boldsymbol{q}_0}\xi_{q,1}, \quad \boldsymbol{x}_{Q,0} = \boldsymbol{q}\xi_{q,0}. \tag{5.3.12}$$

The inverses of $\boldsymbol{p}\mathbf{z}_{\boldsymbol{p}_0} \in \mathbb{R}^{n \times n_1}$ and $\boldsymbol{p}\mathbf{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{n \times k_1}$ are given by $\mathbf{z}^{*\mathrm{T}}_{\boldsymbol{p}_0}\boldsymbol{p}^{*\mathrm{T}} \in \mathbb{R}^{n_1 \times n}$ and $\mathbf{z}^{*\mathrm{T}}_{\boldsymbol{q}_0}\boldsymbol{p}^{*\mathrm{T}} \in \mathbb{R}^{k_1 \times n}$ such that

$$\mathbf{z}^{*\mathrm{T}}_{\boldsymbol{p}_0}\boldsymbol{p}^{*\mathrm{T}}\boldsymbol{p}\mathbf{z}_{\boldsymbol{p}_0} = \mathrm{I}_{n_1}, \quad \mathbf{z}^{*\mathrm{T}}_{\boldsymbol{p}_0}\boldsymbol{p}^{*\mathrm{T}}\boldsymbol{p}\mathbf{z}_{\boldsymbol{q}_0} = 0, \quad \mathbf{z}^{*\mathrm{T}}_{\boldsymbol{q}_0}\boldsymbol{p}^{*\mathrm{T}}\boldsymbol{p}\mathbf{z}_{\boldsymbol{p}_0} = 0, \quad \mathbf{z}^{*\mathrm{T}}_{\boldsymbol{q}_0}\boldsymbol{p}^{*\mathrm{T}}\boldsymbol{p}\mathbf{z}_{\boldsymbol{q}_0} = \mathrm{I}_{k_1}. \tag{5.3.13}$$

Thus,

$$\mathbf{z}^{*\mathrm{T}}_{\boldsymbol{p}_0}\boldsymbol{p}^{*\mathrm{T}}\boldsymbol{x} = \xi_p, \quad \mathbf{z}^{*\mathrm{T}}_{\boldsymbol{q}_0}\boldsymbol{p}^{*\mathrm{T}}\boldsymbol{x} = \xi_{q,1}, \quad \boldsymbol{q}^{*\mathrm{T}}\boldsymbol{x} = \xi_{q,0}. \tag{5.3.14}$$

Substituting (5.3.12) into system (5.3.1) and simplifying we obtain,

$$\xi_p' = \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u}, \tag{5.3.15a}$$

$$\xi_{q,1} = \mathbf{A}_{q,1}\xi_p + \mathbf{B}_{q,1}\boldsymbol{u}, \tag{5.3.15b}$$

$$\xi_{q,0} = \mathbf{A}_{q,0}\xi_p + \mathbf{B}_{q,0}\boldsymbol{u} + \mathbf{A}_{\boldsymbol{q}_{0,1}}\xi_{q,1}', \tag{5.3.15c}$$

$$\mathbf{y} = \mathbf{C}_p^{\mathrm{T}}\,\xi_p + \mathbf{C}_{q,1}^{\mathrm{T}}\,\xi_{q,1} + \mathbf{C}_{q,0}^{\mathrm{T}}\,\xi_{q,0}, \tag{5.3.15d}$$

where

$\mathbf{A}_p = \mathbf{z}^{*\mathrm{T}}_{\boldsymbol{p}_0}\boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_1\mathbf{E}_2^{-1}\mathbf{A}_2\boldsymbol{p}\mathbf{z}_{\boldsymbol{p}_0} \in \mathbb{R}^{n_p \times n_p}$, $\mathbf{B}_p = \mathbf{z}^{*\mathrm{T}}_{\boldsymbol{p}_0}\boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_1\mathbf{E}_2^{-1}\mathbf{B} \in \mathbb{R}^{n_p \times m}$, $\mathbf{A}_{q,1} = \mathbf{z}^{*\mathrm{T}}_{\boldsymbol{q}_0}\boldsymbol{p}^{*\mathrm{T}}\mathbf{Q}_1\mathbf{E}_2^{-1}\mathbf{A}_2\boldsymbol{p}\mathbf{z}_{\boldsymbol{p}_0} \in \mathbb{R}^{k_1 \times n_p}$,

$\mathbf{B}_{q,1} = \mathbf{z}_{\boldsymbol{q}_0}^{*\mathrm{T}} \boldsymbol{p}^{*\mathrm{T}} \mathbf{Q}_1 \mathbf{E}_2^{-1} \mathbf{B} \in \mathbb{R}^{k_1 \times m}$, $\mathbf{A}_{q,0} = \boldsymbol{q}^{*\mathrm{T}} \mathbf{P}_1 \mathbf{E}_2^{-1} \mathbf{A}_2 \boldsymbol{p} \mathbf{z}_{\boldsymbol{p}_0} \in \mathbb{R}^{k_0 \times n_p}$, $\mathbf{B}_{q,0} = \boldsymbol{q}^{*\mathrm{T}} \mathbf{P}_1 \mathbf{E}_2^{-1} \mathbf{B} \in \mathbb{R}^{k_0 \times m}$, $\mathbf{A}_{\boldsymbol{q}_{0,1}} = \boldsymbol{q}^{*\mathrm{T}} \mathbf{Q}_1 \boldsymbol{p} \mathbf{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{k_0 \times k_1}$, $\mathbf{C}_p = \mathbf{z}_{\boldsymbol{p}_0}^{\mathrm{T}} \boldsymbol{p}^{\mathrm{T}} \mathbf{C} \in \mathbb{R}^{n_p \times \ell}$, $\mathbf{C}_{q,1} = \mathbf{z}_{\boldsymbol{q}_0}^{\mathrm{T}} \boldsymbol{p}^{\mathrm{T}} \mathbf{C} \in \mathbb{R}^{k_1 \times \ell}$, $\mathbf{C}_{q,0} = \boldsymbol{q}^{\mathrm{T}} \mathbf{C} \in \mathbb{R}^{k_0}$. If we apply initial condition $\xi_{\boldsymbol{p}_0}(0) = \mathbf{z}_{\boldsymbol{p}_0}^{*\mathrm{T}} \boldsymbol{p}^{*\mathrm{T}} \boldsymbol{x}(0)$, where $\boldsymbol{x}(0)$ is a consistent initial condition, we can solve the differential part (5.3.15a), and then solve algebraic parts (5.3.15b) and (5.3.15c). We can see that the number of differential equations is equal to $n_p = n_1$ and $n_q = k_1 + k_0$ is the total number of algebraic equations and the total system dimension is $n = n_p + n_q$. It can also easily be shown that $\sigma_f(\mathbf{E}, \mathbf{A}) = \sigma(\mathbf{A_p})$ still holds. Thus the number of differential equation is always equal to the total algebraic multiplicity of finite eigenvalues of the matrix pencil $(\mathbf{E}, \mathbf{A})$.

5.3.2 Index-2 DAEs without a differential part

Here, we assume that the matrix pencil of (4.1.1) has no finite eigenvalues. From (5.3.4), this implies that $\mathbf{P}_0 \mathbf{P}_1 = 0$, thus the decomposition of the identity reduces to

$$\mathrm{I} = \mathbf{P}_0 \mathbf{Q}_1 + \mathbf{Q}_0.$$

Then, the decoupled system (5.3.1) reduces to

$$\boldsymbol{x}_{Q,1} = \mathbf{Q}_1 \mathbf{E}_2^{-1} \mathbf{B} \boldsymbol{u}, \tag{5.3.16a}$$

$$\boldsymbol{x}_{Q,0} = \mathbf{Q}_0 \mathbf{P}_1 \mathbf{E}_2^{-1} \mathbf{B} \boldsymbol{u} + \mathbf{Q}_0 \mathbf{Q}_1 \dot{\boldsymbol{x}}_{Q,1}, \tag{5.3.16b}$$

$$\boldsymbol{y} = \mathbf{C}^{\mathrm{T}} \boldsymbol{x}_{Q,1} + \mathbf{C}^{\mathrm{T}} \boldsymbol{x}_{Q,0}. \tag{5.3.16c}$$

We can observe that the decoupled system has only algebraic parts of total dimension $2n$. The modification of this decoupled system is done in the same way as the case of index -1 DAEs since $\mathbf{P}_0 \mathbf{Q}_1 = \mathbf{P}_0$. If, we let $k_0 = \dim(\operatorname{Im} \mathbf{Q}_0)$, this implies $k_1 = n - k_0$. Then, we consider a basis matrix $(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{R}^n$ made of k_0 independent columns of projection matrix $\mathbf{Q}_0$ and k_1 independent columns of the product of projection matrix $\mathbf{P}_0 \mathbf{Q}_1$. Then the inverse of the basis matrix is denoted by $\left(\boldsymbol{p}^{*\mathrm{T}} \; \boldsymbol{q}^{*\mathrm{T}}\right)^{\mathrm{T}}$, where $\boldsymbol{q}^{*\mathrm{T}} \in \mathbb{R}^{k_0 \times n}$ and $\boldsymbol{p}^{*\mathrm{T}} \in \mathbb{R}^{k_1 \times n}$. Thus, (5.3.16) simplifies to

$$\xi_{q,1} = \mathbf{B}_{q,1} \boldsymbol{u}, \tag{5.3.17a}$$

$$\xi_{q,0} = \mathbf{B}_{q,0} \boldsymbol{u} + \mathbf{A}_{\boldsymbol{q}_{0,1}} \xi_{\boldsymbol{q}_0}^{'}, \tag{5.3.17b}$$

$$\boldsymbol{y} = \mathbf{C}_{q,1}^{\mathrm{T}} \xi_{q,1} + \mathbf{C}_{q,0}^{\mathrm{T}} \xi_{q,0}, \tag{5.3.17c}$$

where $\mathbf{B}_{q,1} = \boldsymbol{p}^{*\mathrm{T}}\mathbf{Q}_1\mathbf{E}_2^{-1}\mathbf{B} \in \mathbb{R}^{k_1\times m}$, $\mathbf{B}_{q,0} = \boldsymbol{q}^{*\mathrm{T}}\mathbf{P}_1\mathbf{E}_2^{-1}\mathbf{B} \in \mathbb{R}^{k_0\times m}$, $\mathbf{A}_{q_{0,1}} = \boldsymbol{q}^{*\mathrm{T}}\mathbf{Q}_1\boldsymbol{p} \in \mathbb{R}^{k_0\times k_1}$, $\mathbf{C}_{q,1} = \boldsymbol{p}^{\mathrm{T}}\mathbf{C} \in \mathbb{R}^{k_1\times \ell}$ and $\mathbf{C}_{q,0} = \boldsymbol{q}^{\mathrm{T}}\mathbf{C} \in \mathbb{R}^{k_0\times \ell}$. We can observe that this time we do not need to apply any initial condition rather the input function has to be smooth enough. In order to solve (5.3.17), we first solve algebraic part (5.3.17a) and then solve (5.3.17b). The total number of algebraic equations is equal to the dimension n of the system, thus the dimension of the DAE is preserved. We have seen that index-2 DAEs can be decoupled in two ways depending on the eigenvalues of the matrix pencil $(\mathbf{E}, \mathbf{A})$. Thus special has to be taken when implementing this procedure. In Example 5.3.1 and 5.3.2, we illustrate the modified decoupling of index-2 DAEs with and without differential part, respectively.

Example 5.3.1 This example originates from [60]. In this example we consider a system composed of two rotating masses as shown in Figure 5.1. The two rotating parts

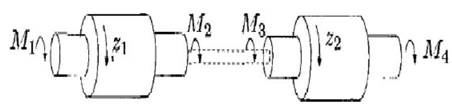

Figure 5.1: Two interconnected rotating masses.

are described by the torques M_1, M_2, M_3 and M_4 and the angular velocities z_1 and z_2. The system of equations describing this system is a DAE of dimension 4 with system matrices

$$\mathbf{E} = \begin{pmatrix} J_1 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} M_1 \\ M_4 \end{pmatrix}, \tag{5.3.18}$$

where $\boldsymbol{x} = \begin{pmatrix} z_1 & z_2 & M_2 & M_3 \end{pmatrix}^{\mathrm{T}}$ and let $J_1, J_2 \geq 0$. We are interested only with the velocities thus $\mathbf{C} = \mathbf{B}$. Since the matrix pencil $(\mathbf{E}, \mathbf{A})$ is regular , that is $\det(\lambda\mathbf{E}-\mathbf{A}) = (J_1+J_2)\lambda \neq 0$, thus the DAE is solvable. Next, we checked the tractability index of the DAE and we found out that it is of tractability index-2. We then chose projectors

$$\mathbf{Q}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{Q}_1 = \begin{pmatrix} \frac{J_2}{J_1+J_2} & -\frac{J_2}{J_1+J_2} & 0 & 0 \\ -\frac{J_1}{J_1+J_2} & \frac{J_1}{J_1+J_2} & 0 & 0 \\ \frac{J_1J_2}{J_1+J_2} & -\frac{J_1J_2}{J_1+J_2} & 0 & 0 \\ -\frac{J_1J_2}{J_1+J_2} & \frac{J_1J_2}{J_1+J_2} & 0 & 0 \end{pmatrix},$$

such that $\mathbf{Q}_1\mathbf{Q}_0 = 0$ holds true and the corresponding complementary projectors can be obtained as $\mathbf{P}_i = \mathrm{I} - \mathbf{Q}_i$, $i = 0, 1$. The last values of the matrix chains are given by,

$$\mathbf{E}_2 = \begin{pmatrix} J_1 & 0 & -1 & 0 \\ 0 & J_2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This system is indeed an index-2 DAE since $\mathbf{E}_2$ is nonsingular. We can easily check that $\mathbf{P}_0\mathbf{P}_1 \neq 0$, thus the decoupled system of the DAE takes the form (5.3.15). We can now use the procedure derived in Section 5.3.1 to decouple the DAE as follows: We need to first constructed the new basis vector $(\boldsymbol{p}, \boldsymbol{q})$ and their corresponding inverses $\left(\boldsymbol{p}^{*\mathrm{T}} \; \boldsymbol{q}^{*\mathrm{T}}\right)^{\mathrm{T}}$ given by

$$\boldsymbol{p} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{p}^{*\mathrm{T}} = \boldsymbol{p}^{\mathrm{T}}, \quad \boldsymbol{q}^{*\mathrm{T}} = \boldsymbol{q}^{\mathrm{T}},$$

for the projector $\mathbf{Q}_0$ and its complementary $\mathbf{P}_0$, respectively. Using Theorem 5.3.1, we construct another pair of projector matrices $\mathbf{Z}_{\boldsymbol{p}_0}$ and $\mathbf{Z}_{\boldsymbol{q}_0}$ given by

$$\mathbf{Z}_{\boldsymbol{p}_0} = \boldsymbol{p}^{*\mathrm{T}}\mathbf{P}_1\boldsymbol{p} = \frac{1}{J_1 + J_2}\begin{pmatrix} J_1 & J_2 \\ J_1 & J_2 \end{pmatrix}, \quad \mathbf{Z}_{\boldsymbol{q}_0} = \boldsymbol{p}^{*\mathrm{T}}\mathbf{Q}_1\boldsymbol{p} = \frac{1}{J_1 + J_2}\begin{pmatrix} J_2 & -J_2 \\ -J_1 & J_1 \end{pmatrix}$$

and their respective bases and inverses are given by

$$\mathbf{z}_{\boldsymbol{p}_0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{z}_{\boldsymbol{q}_0} = \begin{pmatrix} J_2 \\ -J_1 \end{pmatrix}, \quad \text{and} \quad \mathbf{z}^{*\mathrm{T}}_{\boldsymbol{p}_0} = \frac{1}{J_1 + J_2}\begin{pmatrix} J_1 & J_2 \end{pmatrix}, \quad \mathbf{z}^{*\mathrm{T}}_{\boldsymbol{q}_0} = \frac{1}{J_1 + J_2}\begin{pmatrix} 1 & -1 \end{pmatrix}. \tag{5.3.19}$$

Thus, substituting equation (5.3.18) – (5.3.19) into the modified decoupled system (5.3.15), we obtain decoupled system with system matrices $\mathbf{A}_p = 0$, $\mathbf{B}_p = \frac{1}{J_1+J_2}\begin{pmatrix} 1 & 1 \end{pmatrix}$, $\mathbf{A}_{q,1} = 0$, $\mathbf{B}_{q,1} = \begin{pmatrix} 0 & 0 \end{pmatrix}$, $\mathbf{A}_{q,0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{B}_{q,0} = \frac{1}{J_1+J_2}\begin{pmatrix} -J_2 & J_1 \\ J_2 & -J_1 \end{pmatrix}$, $\mathbf{A}_{q_{0,1}} = J_1 J_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\mathbf{C}_p = \frac{1}{J_1+J_2}\begin{pmatrix} J_1 & J_2 \end{pmatrix}$, $\mathbf{C}_{q,1} = \frac{1}{J_1+J_2}\begin{pmatrix} 1 & -1 \end{pmatrix}$, $\mathbf{C}_{q,0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. We observe that $n_p = 1, k_1 = 1$ and $k_0 = 2$, thus the decoupled system has only 1 differential equation and 3 algebraic equations. This is leads to a decoupled system with total dimension 4 which is equal to the dimension of the DAE (5.3.18). On simplifying, this decoupled system reduces to an ODE given by

$$\begin{aligned} \xi_p' &= \frac{1}{J_1 + J_2} \begin{pmatrix} 1 & 1 \end{pmatrix} \boldsymbol{u} \\ \boldsymbol{y} &= \frac{1}{J_1 + J_2} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} \xi_p. \end{aligned} \tag{5.3.20}$$

Applying the initial value $\xi_p(0)$ on the differential part, we can obtain the desired solution of the DAE (5.3.18).

Example 5.3.2 Consider a simple RL network in Figure. 5.2. Using the modified nodal analysis on this network leads to DAE with system matrices ,

$$\mathbf{E} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -G & G & 0 \\ G & -G & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} e_1 \\ e_2 \\ \iota_L \end{pmatrix}, \quad \boldsymbol{u} = \iota.$$

We can choose the control output matrix as $\mathbf{C} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^{\mathrm{T}}$. The matrix pencil is regular

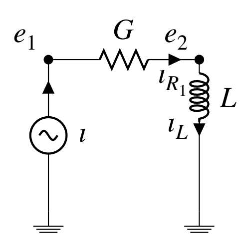

Figure 5.2: Simple RL network.

since $\det(\lambda\mathbf{E} - \mathbf{A}) = G > 0$, this system is solvable and its matrix pencil $(\mathbf{E}, \mathbf{A})$ has only infinite eigenvalues. Thus its decoupled system has no differential part. We can choose special projectors, $\mathbf{Q}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathbf{Q}_1 = \begin{pmatrix} 0 & 0 & L \\ 0 & 0 & L \\ 0 & 0 & 1 \end{pmatrix}$, such that $\mathbf{Q}_1\mathbf{Q}_0 = 0$ holds. Then, we have, $\mathbf{E}_2 = \begin{pmatrix} G & -G & 0 \\ -G & G & 1 \\ 0 & -1 & L \end{pmatrix}$. Since the $\mathbf{E}_2$ is nonsingular. Thus, this is an index-2 DAE. We can easily check that $\mathbf{P}_0\mathbf{P}_1 = 0$, thus the decoupled system of the DAE takes the form (5.3.17). We can now use the procedure derived in Section 5.3.2 to decouple the DAE as follows. The linearly independent columns and their respective inverses of projector $\mathbf{Q}_0$

and its complimentary $\mathbf{P}_0$ are given by

$$\boldsymbol{p} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{p}^{*\mathrm{T}} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{q}^{*\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus, substituting the above system matrices and bases into (5.3.17), and simplifying we obtain the decoupled system with system matrices given by

$$\mathbf{B}_{q,1} = 1, \; \mathbf{B}_{q,0} = \begin{pmatrix} G^{-1} \\ 0 \end{pmatrix}, \mathbf{A}_{q_{0,1}} = \begin{pmatrix} L \\ L \end{pmatrix}, \; \mathbf{C}_{q,1} = 1, \; \mathbf{C}_{q,0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This decoupled system leads to an output solution which coincides with the exact solution given by $\boldsymbol{y} = (G^{-1} + 1)\boldsymbol{u} + 2L\boldsymbol{u}'$.

5.4 Index-μ DAEs

In this Section, we generalize the procedure of modifying the März decoupling procedure. Assume (4.1.1) is of tractability index-μ, then using the März decoupling procedure leads to a decoupled system of the form (4.1.17). Then, its compact form is given by:

$$\boldsymbol{x}_P' = \mathbf{A}_P \boldsymbol{x}_P + \mathbf{B}_P \boldsymbol{u}, \quad \boldsymbol{x}_P(0) = \Pi_{\mu-1} x(0), \tag{5.4.1a}$$

$$\boldsymbol{x}_{Q,\mu-1} = \mathbf{A}_{Q,\mu-1} \boldsymbol{x}_P + \mathbf{B}_{Q,\mu-1} \boldsymbol{u}, \tag{5.4.1b}$$

$$\boldsymbol{x}_{Q,i} = \mathbf{A}_{Q,i} \boldsymbol{x}_P + \mathbf{B}_{Q,i} \boldsymbol{u} + \sum_{j=i+1}^{\mu-1} \mathbf{A}_{\mathbf{Q}_{i,j}} \boldsymbol{x}_{Q,j}', \; i = \mu - 2, \dots 0, \tag{5.4.1c}$$

$$\boldsymbol{y} = \mathbf{C}^{\mathrm{T}} \boldsymbol{x}_P + \mathbf{C}^{\mathrm{T}} \sum_{i=0}^{\mu-1} \boldsymbol{x}_{Q,i}. \tag{5.4.1d}$$

where

$$\mathbf{A}_P := \Pi_0^* \mathbf{E}_\mu^{-1} \mathbf{A}_\mu, \quad \mathbf{B}_P := \Pi_0^* \mathbf{E}_\mu^{-1} \mathbf{B}, \quad \mathbf{A}_{Q,\mu-1} := \Pi_{\mu-2} \mathbf{Q}_{\mu-1} \mathbf{E}_\mu^{-1} \mathbf{A}_\mu,$$
$$\mathbf{B}_{Q,\mu-1} := \Pi_{\mu-2} \mathbf{Q}_{\mu-1} \mathbf{E}_\mu^{-1} \mathbf{B}, \quad \mathbf{A}_{Q,0} := \Pi_{i-1} \mathbf{Q}_i \Pi_{i+1}^* \mathbf{E}_\mu^{-1} \mathbf{A}_\mu,$$
$$\mathbf{B}_{Q,0} := \Pi_{i-1} \mathbf{Q}_i \Pi_{i+1}^* \mathbf{E}_\mu^{-1} \mathbf{B}, \quad \mathbf{A}_{\mathbf{Q}_{i,j}} := \Pi_{i-1} \mathbf{Q}_{i,j}, \quad \mathbf{Q}_{i,j} = \begin{cases} \mathbf{Q}_i \mathbf{Q}_{i+1}, & j = i+1, \\ \mathbf{Q}_i \mathbf{P}_{i+1} \dots \mathbf{P}_{j-1} \mathbf{Q}_j, & j > i+1. \end{cases}$$

We have already discussed that this decomposition increases the dimension of the system dimension to $(\mu + 1)n$ and it does not preserve the stability of the DAE. Thus, we need to generalize the modification of the März decoupling procedure for higher index DAEs. This is done by generalizing the modification procedures derived in Section 5.2 and 5.3 for index-1 and -2 DAEs, respectively. This is done as follows. We modify the system (5.4.1) by constructing basis column matrices for the projectors and the projector products in the decomposition (4.1.14). From (4.1.14), we have:

$$\mathbf{I}_n = \mathbf{Q}_0 + \sum_{i=1}^{\mu-1} \Pi_{i-1}\mathbf{Q}_i + \Pi_{\mu-1}. \tag{5.4.2}$$

In Section 5.3, we mentioned that higher index DAEs have a possibility of having a purely algebraic system depending on the nature of the spectrum of the matrix pencil $(\mathbf{E}, \mathbf{A})$. This implies the projector product $\Pi_{\mu-1}$ can vanish to zero depending on the matrix pencil of the DAE (4.1.1). Thus in this section, we also consider two cases of compact decomposition of the DAE (4.1.1) depending on the spectrum of the matrix pencil $(\mathbf{E}, \mathbf{A})$. In both cases the starting point is the same as that of index-1 as presented in Section 5.2.

5.4.1 Index-μ DAEs with a differential part

Here, we assume that the spectrum of the matrix pencil of (4.1.1) has at least one finite eigenvalue, this implies that $\Pi_{\mu-1} \neq 0$. Let $k_0 = \dim(\operatorname{Ker}\mathbf{E}_0)$, $n_0 = n - k_0$, and let us consider an orthonormal basis matrix $(\boldsymbol{p}_0, \boldsymbol{q}_0) = (\boldsymbol{p}_{0,1}, \ldots, \boldsymbol{p}_{0,n_0}, \boldsymbol{q}_{0,1}, \ldots, \boldsymbol{q}_{0,k_0}) \in \mathbb{R}^n$ which contains k_0 independent vectors $\boldsymbol{q}_{0,i}$ which span $\operatorname{Ker}\mathbf{E}_0$. Since $(\boldsymbol{p}_0, \boldsymbol{q}_0)$ is a basis matrix, it is invertible, and let $(\boldsymbol{p}_0^{*\mathrm{T}}, \boldsymbol{q}_0^{*\mathrm{T}})^{\mathrm{T}}$ be its inverse, with $\boldsymbol{q}_0^* \in \mathbb{R}^{n\times k_0}$ and $\boldsymbol{p}_0^* \in \mathbb{R}^{n\times n_0}$. Then, we have

$$\boldsymbol{q}_0^{*\mathrm{T}}\boldsymbol{q}_0 = \mathbf{I}_{k_0},\ \boldsymbol{q}_0^{*\mathrm{T}}\boldsymbol{p}_0 = 0,\ \boldsymbol{p}_0^{*\mathrm{T}}\boldsymbol{q}_0 = 0,\ \boldsymbol{p}_0^{*\mathrm{T}}\boldsymbol{p}_0 = \mathbf{I}_{n_0}, \tag{5.4.3}$$

and also

$$\boldsymbol{q}_0\boldsymbol{q}_0^{*\mathrm{T}} + \boldsymbol{p}_0\boldsymbol{p}_0^{*\mathrm{T}} = \mathbf{I}_{n_0}. \tag{5.4.4}$$

The previous relations imply that we can represent the projectors $\mathbf{Q}_0$ and $\mathbf{P}_0$ as

$$\mathbf{Q}_0 = \boldsymbol{q}_0\boldsymbol{q}_0^{*\mathrm{T}}, \quad \mathbf{P}_0 = \boldsymbol{p}_0\boldsymbol{p}_0^{*\mathrm{T}}. \tag{5.4.5}$$

We note that, by construction we have

$$\mathbf{Q}_0 \boldsymbol{q}_0 = \boldsymbol{q}_0, \quad \mathbf{Q}_0 \boldsymbol{p}_0 = 0, \quad \mathbf{P}_0 \boldsymbol{q}_0 = 0, \quad \mathbf{P}_0 \boldsymbol{p}_0 = \boldsymbol{p}_0. \tag{5.4.6}$$

Then we take the following steps:
Step: 0 if $\mu > 1$:
By construction, (5.4.2) can be written as,

$$\mathbf{I}_{n_0} = \boldsymbol{p}_0^{*\mathrm{T}} \Pi_0 \mathbf{Q}_1 \boldsymbol{p}_0 + \boldsymbol{p}_0^{*\mathrm{T}} \sum_{i=2}^{\mu-1} \Pi_{i-1} \mathbf{Q}_i \boldsymbol{p}_0 + \boldsymbol{p}_0^{*\mathrm{T}} \Pi_{\mu-1} \boldsymbol{p}_0$$

Then,

$$\mathbf{I}_{n_0} = \mathbf{Z}_{\boldsymbol{q}_0} + \mathbf{Z}_{\boldsymbol{p}_0}, \tag{5.4.7}$$

with $\mathbf{Z}_{\boldsymbol{p}_0} := \boldsymbol{p}_0^{*\mathrm{T}} \sum_{\substack{i=2 \\ \mu>2}}^{\mu-1} \Pi_{i-1} \mathbf{Q}_i \boldsymbol{p}_0 + \boldsymbol{p}_0^{*\mathrm{T}} \Pi_{\mu-1} \boldsymbol{p}_0, \quad \mathbf{Z}_{\boldsymbol{q}_0} := \boldsymbol{p}_0^{*\mathrm{T}} \Pi_0 \mathbf{Q}_1 \boldsymbol{p}_0$. Using the general form of Theorem 5.3.1, $\mathbf{Z}_{\boldsymbol{p}_0}$ and $\mathbf{Z}_{\boldsymbol{q}_0}$ are mutually orthogonal projectors, acting in $\mathbb{R}^{n_0}$. Let $k_1 = \dim(\operatorname{Im} \mathbf{Z}_{\boldsymbol{q}_0})$, and $n_1 = n_0 - k_1$, and let us consider a basis matrix $(z_{\boldsymbol{p}_0}, z_{\boldsymbol{q}_0}) \in \mathbb{R}^{n_0}$ made of n_1 independent columns of projection matrix $\mathbf{Z}_{\boldsymbol{p}_0}$ and k_1 independent columns of the complementary projection matrix $\mathbf{Z}_{\boldsymbol{q}_0}$. We denote by $(z_{\boldsymbol{p}_0}^{*\mathrm{T}}, z_{\boldsymbol{q}_0}^{*\mathrm{T}})^{\mathrm{T}}$ the inverse of $(z_{\boldsymbol{p}_0}, z_{\boldsymbol{q}_0})$, such that

$$z_{\boldsymbol{p}_0}^{*\mathrm{T}} z_{\boldsymbol{p}_0} = \mathbf{I}_{n_1},\ z_{\boldsymbol{p}_0}^{*\mathrm{T}} z_{\boldsymbol{q}_0} = 0,\ z_{\boldsymbol{q}_0}^{*\mathrm{T}} z_{\boldsymbol{p}_0} = 0,\ z_{\boldsymbol{q}_0}^{*\mathrm{T}} z_{\boldsymbol{q}_0} = \mathbf{I}_{k_1},\ z_{\boldsymbol{p}_0} z_{\boldsymbol{p}_0}^{*\mathrm{T}} + z_{\boldsymbol{q}_0} z_{\boldsymbol{q}_0}^{*\mathrm{T}} = \mathbf{I}_{n_0}. \tag{5.4.8}$$

Then, we can represent $\mathbf{Z}_{\boldsymbol{p}_0}$ and $\mathbf{Z}_{\boldsymbol{q}_0}$ as $\mathbf{Z}_{\boldsymbol{p}_0} = z_{\boldsymbol{p}_0} z_{\boldsymbol{p}_0}^{*\mathrm{T}}$ and $\mathbf{Z}_{\boldsymbol{q}_0} = z_{\boldsymbol{q}_0} z_{\boldsymbol{q}_0}^{*\mathrm{T}}$, respectively. Then, we have

$$\mathbf{Z}_{\boldsymbol{p}_0} z_{\boldsymbol{p}_0} = z_{\boldsymbol{p}_0}, \quad \mathbf{Z}_{\boldsymbol{p}_0} z_{\boldsymbol{q}_0} = 0, \quad \mathbf{Z}_{\boldsymbol{q}_0} z_{\boldsymbol{p}_0} = 0, \quad \mathbf{Z}_{\boldsymbol{q}_0} z_{\boldsymbol{q}_0} = z_{\boldsymbol{q}_0}. \tag{5.4.9}$$

We can see that if $\mu = 2$ then the bases of the projector products $\{\mathbf{Q}_0, \Pi_0 \mathbf{Q}_1, \Pi_1\}$ in (5.4.2) are $\{\boldsymbol{q}_0, \boldsymbol{p}_0 z_{\boldsymbol{q}_0}, \boldsymbol{p}_0 z_{\boldsymbol{p}_0}\}$, respectively.

Step: 1 if $\mu > 2$:
Using the identities (5.4.8) and (5.4.9) on (5.4.7) leads to

$$\mathbf{I}_{n_1} = \mathbf{Z}_{\boldsymbol{q}_1} + \mathbf{Z}_{\boldsymbol{p}_1}, \tag{5.4.10}$$

with $\mathbf{Z}_{\boldsymbol{p}_1} := z_{\boldsymbol{p}_0}^{*\mathrm{T}} \boldsymbol{p}_0^{*\mathrm{T}} \sum_{\substack{i=3 \\ \mu>3}}^{\mu-1} \Pi_{i-1} \mathbf{Q}_i \boldsymbol{p}_0 z_{\boldsymbol{p}_0} + z_{\boldsymbol{p}_0}^{*\mathrm{T}} \boldsymbol{p}_0^{*\mathrm{T}} \Pi_{\mu-1} \boldsymbol{p}_0 z_{\boldsymbol{p}_0}$, $\quad \mathbf{Z}_{\boldsymbol{q}_1} := z_{\boldsymbol{p}_0}^{*\mathrm{T}} \boldsymbol{p}_0^{*\mathrm{T}} \Pi_1 \mathbf{Q}_2 \boldsymbol{p}_0 z_{\boldsymbol{p}_0}$. We can also see that the projectors are mutually orthogonal projectors, acting in $\mathbb{R}^{n_1}$.
Let $k_2 = \dim(\operatorname{Im} \mathbf{Z}_{\boldsymbol{q}_1})$, and $n_2 = n_1 - k_2$, and let us consider a basis matrix $(z_{\boldsymbol{p}_1}, z_{\boldsymbol{q}_1}) \in \mathbb{R}^{n_1}$ made of n_2 independent columns of projection matrix $\mathbf{Z}_{\boldsymbol{p}_0}$ and k_2 independent columns of the complementary projection matrix $\mathbf{Z}_{\boldsymbol{q}_1}$. We denote by $(z_{\boldsymbol{p}_1}^*, z_{\boldsymbol{q}_1}^*)^{\mathrm{T}}$ the inverse of $(z_{\boldsymbol{p}_1}, z_{\boldsymbol{q}_1})$, such that

$$z_{\boldsymbol{p}_1}^{*\mathrm{T}} z_{\boldsymbol{p}_1} = \mathbf{I}_{n_2}, \quad z_{\boldsymbol{p}_1}^{*\mathrm{T}} z_{\boldsymbol{q}_1} = 0, \quad z_{\boldsymbol{q}_1}^{*\mathrm{T}} z_{\boldsymbol{p}_1} = 0, \quad z_{\boldsymbol{q}_1}^{*\mathrm{T}} z_{\boldsymbol{q}_1} = \mathbf{I}_{k_2}, \quad z_{\boldsymbol{p}_1} z_{\boldsymbol{p}_1}^{*\mathrm{T}} + z_{\boldsymbol{q}_1} z_{\boldsymbol{q}_1}^{*\mathrm{T}} = \mathbf{I}_{n_1}.$$

Then, we can represent $\mathbf{Z}_{\boldsymbol{p}_1}$ and $\mathbf{Z}_{\boldsymbol{q}_1}$ as $\mathbf{Z}_{\boldsymbol{p}_1} = z_{\boldsymbol{p}_1} z_{\boldsymbol{p}_1}^{*\mathrm{T}}$ and $\mathbf{Z}_{\boldsymbol{q}_1} = z_{\boldsymbol{q}_1} z_{\boldsymbol{q}_1}^{*\mathrm{T}}$, respectively. Then, we have

$$\mathbf{Z}_{\boldsymbol{p}_1} z_{\boldsymbol{p}_1} = z_{\boldsymbol{p}_1}, \quad \mathbf{Z}_{\boldsymbol{p}_1} z_{\boldsymbol{q}_1} = 0, \quad \mathbf{Z}_{\boldsymbol{q}_1} z_{\boldsymbol{p}_1} = 0, \quad \mathbf{Z}_{\boldsymbol{q}_1} z_{\boldsymbol{q}_1} = z_{\boldsymbol{q}_1}. \tag{5.4.11}$$

We can also see that if $\mu = 3$ then the bases of the projector products $\{\mathbf{Q}_0, \Pi_0 \mathbf{Q}_1, \Pi_1 \mathbf{Q}_2, \Pi_2\}$ are $\{\boldsymbol{q}_0, \boldsymbol{p}_0 z_{\boldsymbol{q}_0}, \boldsymbol{p}_0 z_{\boldsymbol{p}_0} z_{\boldsymbol{q}_1}, \boldsymbol{p}_0 z_{\boldsymbol{p}_0} z_{\boldsymbol{p}_1}\}$ respectively.

Step: j if $\mu > j + 1$:
It's interesting to see that this process is an iterative process and the *jth* iteration leads to an identity matrix given by,

$$\mathbf{I}_{n_j} = \mathbf{Z}_{\boldsymbol{q}_j} + \mathbf{Z}_{\boldsymbol{p}_j}, \quad j = 1, \ldots, \mu - 2, \quad \mu > 2, \tag{5.4.12}$$

with

$$\mathbf{Z}_{\boldsymbol{p}_j} := z_{\boldsymbol{p}_{j-1}}^{*\mathrm{T}} \cdots z_{\boldsymbol{p}_0}^{*\mathrm{T}} \boldsymbol{p}_0^{*\mathrm{T}} \sum_{\substack{i=j+2 \\ j<\mu-2}}^{\mu-1} \Pi_{i-1} \mathbf{Q}_i \boldsymbol{p}_0 z_{\boldsymbol{p}_0} \cdots z_{\boldsymbol{p}_{j-1}} + z_{\boldsymbol{p}_{j-1}}^{*\mathrm{T}} \cdots z_{\boldsymbol{p}_0}^{*\mathrm{T}} \boldsymbol{p}_0^{*\mathrm{T}} \Pi_{\mu-1} \boldsymbol{p}_0 z_{\boldsymbol{p}_0} \cdots z_{\boldsymbol{p}_{j-1}},$$

$$\mathbf{Z}_{\boldsymbol{q}_j} := z_{\boldsymbol{p}_{j-1}}^{*\mathrm{T}} \cdots z_{\boldsymbol{p}_0}^{*\mathrm{T}} \boldsymbol{p}_0^{*\mathrm{T}} \Pi_j \mathbf{Q}_{j+1} \boldsymbol{p}_0 z_{\boldsymbol{p}_0} \cdots z_{\boldsymbol{p}_{j-1}}.$$

These projectors are also mutually orthogonal projectors, acting in $\mathbb{R}^{n_j}$.
Let $k_{j+1} = \dim(\operatorname{Im} \mathbf{Z}_{\boldsymbol{q}_j})$, and $n_{j+1} = n_j - k_{j+1}$, and let us consider a basis matrix $(z_{\boldsymbol{p}_j}, z_{\boldsymbol{q}_j}) \in \mathbb{R}^{n_j}$ made of n_{j+1} independent columns of projection matrix $\mathbf{Z}_{\boldsymbol{p}_j}$ and k_{j+1} independent columns of the complementary projection matrix $\mathbf{Z}_{\boldsymbol{q}_j}$. We denote by $(z^{*\mathrm{T}}_{\boldsymbol{p}_j}, z^{*\mathrm{T}}_{\boldsymbol{q}_j})^{\mathrm{T}}$ the inverse of $(z_{\boldsymbol{p}_j}, z_{\boldsymbol{q}_j})$, such that

$$z^{*\mathrm{T}}_{\boldsymbol{p}_j} z_{\boldsymbol{p}_j} = \mathbf{I}_{n_{j+1}}, \quad z^{*\mathrm{T}}_{\boldsymbol{p}_j} z_{\boldsymbol{q}_j} = 0, \quad z^{*\mathrm{T}}_{\boldsymbol{q}_j} z_{\boldsymbol{p}_j} = 0, \quad z^{*\mathrm{T}}_{\boldsymbol{q}_j} z_{\boldsymbol{q}_j} = \mathbf{I}_{k_{j+1}}, \quad z_{\boldsymbol{p}_j} z^{*\mathrm{T}}_{\boldsymbol{p}_j} + z_{\boldsymbol{q}_j} z^{*\mathrm{T}}_{\boldsymbol{q}_j} = \mathbf{I}_{n_j}.$$

Then, we can represent $\mathbf{Z}_{\boldsymbol{p}_j}, \mathbf{Z}_{\boldsymbol{q}_j}$ as $\mathbf{Z}_{\boldsymbol{p}_j} = z_{\boldsymbol{p}_j} z^{*\mathrm{T}}_{\boldsymbol{p}_j}, \quad \mathbf{Z}_{\boldsymbol{q}_j} = z_{\boldsymbol{q}_j} z^{*\mathrm{T}}_{\boldsymbol{q}_j}$, respectively. We, then s have

$$\mathbf{Z}_{\boldsymbol{p}_j} z_{\boldsymbol{p}_j} = z_{\boldsymbol{p}_j}, \quad \mathbf{Z}_{\boldsymbol{p}_j} z_{\boldsymbol{q}_j} = 0, \quad \mathbf{Z}_{\boldsymbol{q}_j} z_{\boldsymbol{p}_j} = 0, \quad \mathbf{Z}_{\boldsymbol{q}_j} z_{\boldsymbol{q}_j} = z_{\boldsymbol{q}_j}. \tag{5.4.13}$$

Hence the bases of the projector products $\{\mathbf{Q}_0, \Pi_0\mathbf{Q}_1, \ldots, \Pi_{i-1}\mathbf{Q}_i, \ldots, \Pi_{\mu-1}\}$ in (5.4.2) are given by $\{\boldsymbol{q}_0, \boldsymbol{p}_0 z_{\boldsymbol{q}_0}, \ldots, \boldsymbol{p}_0 z_{\boldsymbol{p}_0} \cdots z_{\boldsymbol{p}_{i-2}} z_{\boldsymbol{q}_{i-1}}, \ldots, \boldsymbol{p}_0 z_{\boldsymbol{p}_0} \cdots z_{\boldsymbol{p}_{\mu-2}}\}$, $i = 2, \ldots, \mu-1$, respectively. Thus we can now expand $\boldsymbol{x}$ with respect to these bases, obtaining,

$$\boldsymbol{x} = \boldsymbol{p}_0 z_{\boldsymbol{p}_0} \cdots z_{\boldsymbol{p}_{\mu-2}} \xi_p + \boldsymbol{q}_0 \xi_{q,0} + \boldsymbol{p}_0 z_{\boldsymbol{q}_0} \xi_{q,1} + \sum_{i=2}^{\mu-1} \boldsymbol{p}_0 z_{\boldsymbol{p}_0} \cdots z_{\boldsymbol{p}_{i-2}} z_{\boldsymbol{q}_{i-1}} \xi_{q,i}, \tag{5.4.14}$$

where $\xi_p \in \mathbb{R}^{n_{\mu-1}}$, $\xi_{q,i} \in \mathbb{R}^{k_i}$, $\xi_{q,0} \in \mathbb{R}^{k_i}$,$i = 0, \ldots, \mu-1$ and with inversion expressions

$$\begin{aligned} &\xi_p = z^{*\mathrm{T}}_{\boldsymbol{p}_{\mu-2}} \cdots z^{*\mathrm{T}}_{\boldsymbol{p}_0} \boldsymbol{p}^{*\mathrm{T}}_0 \boldsymbol{x}_P, \quad \xi_{q,0} = \boldsymbol{q}^{*\mathrm{T}}_0 \boldsymbol{x}_{Q,0}, \quad \xi_{q,1} = z^{*\mathrm{T}}_{\boldsymbol{q}_0} \boldsymbol{p}^{*\mathrm{T}}_0 \boldsymbol{x}_{Q,1}, \\ &\xi_{q,i} = z^{*\mathrm{T}}_{\boldsymbol{q}_{i-1}} z^{*\mathrm{T}}_{\boldsymbol{p}_{i-2}} \cdots z^{*\mathrm{T}}_{\boldsymbol{p}_0} \boldsymbol{p}^{*\mathrm{T}}_0 \boldsymbol{x}_{Q,i}, \quad i = 2, \ldots, \mu-1. \end{aligned} \tag{5.4.15}$$

Substituting the variables in (5.4.14) and (5.4.15) into (5.4.1) leads to modified decoupled system given by

$$\xi_p' = \mathbf{A}_p \xi_p + \mathbf{B}_p \boldsymbol{u}, \tag{5.4.16a}$$

$$\xi_{q,\mu-1} = \mathbf{A}_{q,\mu-1} \xi_p + \mathbf{B}_{q,\mu-1} \boldsymbol{u}, \tag{5.4.16b}$$

$$\xi_{q,i} = \mathbf{A}_{q,i} \xi_p + \mathbf{B}_{q,i} \boldsymbol{u} + \sum_{j=i+1}^{\mu-1} \mathbf{A}_{\boldsymbol{q}_{i,j}} \xi_{q,j}', \quad i = \mu-2, \ldots 2, \tag{5.4.16c}$$

$$\xi_{q,1} = \mathbf{A}_{q,1} \xi_p + \mathbf{B}_{q,1} \boldsymbol{u} + \sum_{j=2}^{\mu-1} \mathbf{A}_{\boldsymbol{q}_{1,j}} \xi_{q,j}', \tag{5.4.16d}$$

$$\xi_{q,0} = \mathbf{A}_{q,0} \xi_p + \mathbf{B}_{q,0} \boldsymbol{u} + \sum_{j=1}^{\mu-1} \mathbf{A}_{\boldsymbol{q}_{0,j}} \xi_{q,j}', \tag{5.4.16e}$$

$$\boldsymbol{y} = \mathbf{C}_p^{\mathrm{T}}\xi_p + \sum_{i=0}^{\mu-1}\mathbf{C}_{q,i}^{\mathrm{T}}\xi_{q,i}, \tag{5.4.16f}$$

where

$$\begin{aligned}
&\mathbf{A}_p := \boldsymbol{z}_{\boldsymbol{p}_{\mu-2}}^{*\mathrm{T}}\cdots\boldsymbol{z}_{\boldsymbol{p}_0}^{*\mathrm{T}}\boldsymbol{p}_0^{*\mathrm{T}}\mathbf{A}_P\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0}\cdots\boldsymbol{z}_{\boldsymbol{p}_{\mu-2}} \in \mathbb{R}^{n_p\times n_p}, \quad \mathbf{B}_p := \boldsymbol{z}_{\boldsymbol{p}_{\mu-2}}^{*\mathrm{T}}\cdots\boldsymbol{z}_{\boldsymbol{p}_0}^{*\mathrm{T}}\boldsymbol{p}_0^{*\mathrm{T}}\mathbf{B}_P \in \mathbb{R}^{n_p\times m},\\
&\mathbf{A}_{q,\mu-1} := \boldsymbol{z}_{\boldsymbol{q}_{\mu-2}}^{*\mathrm{T}}\boldsymbol{z}_{\boldsymbol{p}_{\mu-3}}^{*\mathrm{T}}\cdots\boldsymbol{z}_{\boldsymbol{p}_0}^{*\mathrm{T}}\boldsymbol{p}_0^{*\mathrm{T}}\mathbf{A}_{Q,\mu-1}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0}\cdots\boldsymbol{z}_{\boldsymbol{p}_{\mu-2}} \in \mathbb{R}^{k_{\mu-1}\times n_p},\\
&\mathbf{B}_{q,\mu-1} := \boldsymbol{z}_{\boldsymbol{q}_{\mu-2}}^{*\mathrm{T}}\boldsymbol{z}_{\boldsymbol{p}_{\mu-3}}^{*\mathrm{T}}\cdots\boldsymbol{z}_{\boldsymbol{p}_0}^{*\mathrm{T}}\boldsymbol{p}_0^{*\mathrm{T}}\mathbf{B}_{Q,\mu-1} \in \mathbb{R}^{k_{\mu-1}\times m},\\
&\mathbf{A}_{q,i} := \boldsymbol{z}_{\boldsymbol{q}_{i-1}}^{*\mathrm{T}}\boldsymbol{z}_{\boldsymbol{p}_{i-2}}^{*\mathrm{T}}\cdots\boldsymbol{z}_{\boldsymbol{p}_0}^{*\mathrm{T}}\boldsymbol{p}_0^{*\mathrm{T}}\mathbf{A}_{Q,i}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0}\cdots\boldsymbol{z}_{\boldsymbol{p}_{\mu-2}} \in \mathbb{R}^{k_i\times n_p},\\
&\mathbf{B}_{q,i} := \boldsymbol{z}_{\boldsymbol{q}_{i-1}}^{*\mathrm{T}}\boldsymbol{z}_{\boldsymbol{p}_{i-2}}^{*\mathrm{T}}\cdots\boldsymbol{z}_{\boldsymbol{p}_0}^{*\mathrm{T}}\boldsymbol{p}_0^{*\mathrm{T}}\mathbf{B}_{Q,i} \in \mathbb{R}^{k_i\times m},\\
&\mathbf{A}_{\boldsymbol{q}_{i,j}} := \boldsymbol{z}_{\boldsymbol{q}_{i-1}}^{*\mathrm{T}}\boldsymbol{z}_{\boldsymbol{p}_{i-2}}^{*\mathrm{T}}\cdots\boldsymbol{z}_{\boldsymbol{q}_0}^{*\mathrm{T}}\boldsymbol{p}_0^{*\mathrm{T}}\mathbf{A}_{\mathbf{Q}_{i,j}}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0}\cdots\boldsymbol{z}_{\boldsymbol{p}_{j-2}}\boldsymbol{z}_{\boldsymbol{q}_{j-1}} \in \mathbb{R}^{k_i\times k_j}\\
&\mathbf{A}_{q,1} := \boldsymbol{z}_{\boldsymbol{q}_0}^{*\mathrm{T}}\boldsymbol{p}_0^{*\mathrm{T}}\mathbf{A}_{Q,1}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0}\cdots\boldsymbol{z}_{\boldsymbol{p}_{\mu-2}} \in \mathbb{R}^{k_1\times n_p}, \quad \mathbf{B}_{q,1} := \boldsymbol{z}_{\boldsymbol{q}_0}^{*\mathrm{T}}\boldsymbol{p}_0^{*\mathrm{T}}\mathbf{B}_{Q,1} \in \mathbb{R}^{k_1\times m},\\
&\mathbf{A}_{\boldsymbol{q}_{1,j}} := \boldsymbol{z}_{\boldsymbol{p}_0}^{*\mathrm{T}}\boldsymbol{p}_0^{*\mathrm{T}}\mathbf{A}_{\mathbf{Q}_{1,j}}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0}\cdots\boldsymbol{z}_{\boldsymbol{p}_{j-2}}\boldsymbol{z}_{\boldsymbol{q}_{j-1}} \in \mathbb{R}^{k_1\times k_j},\\
&\mathbf{A}_{q,0} := \boldsymbol{q}_0^{*\mathrm{T}}\mathbf{A}_{Q,0}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0}\cdots\boldsymbol{z}_{\boldsymbol{p}_{\mu-2}} \in \mathbb{R}^{k_0\times n_p}, \quad \mathbf{B}_{q,0} := \boldsymbol{q}_0^{*\mathrm{T}}\mathbf{B}_{Q,0} \in \mathbb{R}^{k_0\times m},\\
&\mathbf{A}_{\boldsymbol{q}_{0,j}} := \begin{cases}\boldsymbol{q}_0^{*\mathrm{T}}\mathbf{A}_{\mathbf{Q}_{0,j}}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{q}_0}, & \text{If } j=1,\\ \boldsymbol{q}_0^{*\mathrm{T}}\mathbf{A}_{\mathbf{Q}_{0,j}}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0}\cdots\boldsymbol{z}_{\boldsymbol{p}_{j-2}}\boldsymbol{z}_{\boldsymbol{q}_{j-1}}, & \text{Otherwise.}\end{cases}\\
&\mathbf{C}_p^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0}\cdots\boldsymbol{z}_{\boldsymbol{p}_{\mu-2}} \in \mathbb{R}^{\ell\times n_p}, \quad \mathbf{C}_{q,0}^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\boldsymbol{q}_0 \in \mathbb{R}^{\ell\times k_0}, \quad \mathbf{C}_{q,1}^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{\ell\times k_1},\\
&\mathbf{C}_{q,i}^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0}\cdots\boldsymbol{z}_{\boldsymbol{p}_{i-2}}\boldsymbol{z}_{\boldsymbol{q}_{i-1}} \in \mathbb{R}^{\ell\times k_i} \quad n_p = n_{\mu-1}.
\end{aligned}$$

We can observe that, (5.4.16) can be written in a compact form given by

$$\begin{aligned}
\xi_p' &= \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u},\\
-\mathcal{L}\xi_q' &= \mathbf{A}_q\xi_p - \xi_q + \mathbf{B}_q\boldsymbol{u},\\
\boldsymbol{y} &= \mathbf{C}_p^{\mathrm{T}}\,\xi_p + \mathbf{C}_q^{\mathrm{T}}\,\xi_q,
\end{aligned} \tag{5.4.17}$$

where $\xi_p \in \mathbb{R}^{n_p}$, $\mathbf{A}_p \in \mathbb{R}^{n_p,n_p}, \mathbf{B}_p \in \mathbb{R}^{n_p,m}$, $\xi_q = (\xi_{q,\mu-1},\dots,\xi_{q,0})^{\mathrm{T}} \in \mathbb{R}^{n_q}$, $\mathbf{A}_q = (\mathbf{A}_{q,\mu-1},\dots,\mathbf{A}_{q,0})^{\mathrm{T}} \in \mathbb{R}^{n_q\times n_p}$, $\mathbf{B}_q = (\mathbf{B}_{q,\mu-1},\dots,\mathbf{B}_{q,0})^{\mathrm{T}} \in \mathbb{R}^{n_q\times m}$, $\mathbf{C}_q = (\mathbf{C}_{q,\mu-1}^{\mathrm{T}},\dots,\mathbf{C}_{q,0}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{n_q\times\ell}$ and $\mathcal{L} \in \mathbb{R}^{n_q\times n_q}$ is a strictly lower triangular nilpotent matrix of index-μ with entries $\mathbf{A}_{\boldsymbol{q}_{i,j}}$ as defined in the decoupled system (5.4.16). n_p and $n_q = \sum_{i=0}^{\mu-1}k_i$ is the number of differential and algebraic equations, respectively and $n = n_p + n_q$ is the dimension of the DAE. Thus, the decoupled system (5.4.17) preserves the dimension of the DAE (4.1.1). It can be proved that the decoupled system (5.4.17) can be written in the form:

$$\xi_p' = \mathbf{A}_p \xi_p + \mathbf{B}_p \boldsymbol{u} \tag{5.4.18a}$$

$$\xi_q = \sum_{i=0}^{\mu-1} \mathcal{L}^i \mathbf{A}_q \mathbf{A}_p^i \xi_p + \sum_{i=1}^{\mu-1} \sum_{k=0}^{i-1} \mathcal{L}^i \mathbf{A}_q \mathbf{A}_p^k \mathbf{B}_p \boldsymbol{u}^{(i-k-1)} + \sum_{i=0}^{\mu-1} \mathcal{L}^i \mathbf{B}_q \boldsymbol{u}^{(i)}, \tag{5.4.18b}$$

$$\boldsymbol{y} = \mathbf{C}_p^{\mathrm{T}} \xi_p + \mathbf{C}_q^{\mathrm{T}} \xi_q, \tag{5.4.18c}$$

where $\boldsymbol{u}^{(i)} \in \mathbb{R}^m$ is the *ith* derivative of the input data. Equation (5.4.18a) and (5.4.18b) are the differential and algebraic part of DAE (4.1.1), respectively. Next, we analyze the initial value of the DAE (4.1.1) as follows: Using system (5.4.18), we have: $\xi(0) := \begin{pmatrix} \xi_p(0) \\ \xi_q(0) \end{pmatrix}$, where

$$\xi_q(0) = \sum_{i=0}^{\mu-1} \mathcal{L}^i \mathbf{A}_q \mathbf{A}_p^i \xi_p(0) + \sum_{i=1}^{\mu-1} \sum_{k=0}^{i-1} \mathcal{L}^i \mathbf{A}_q \mathbf{A}_p^k \mathbf{B}_p \boldsymbol{u}^{(i-k-1)}(0) + \sum_{i=0}^{\mu-1} \mathcal{L}^i \mathbf{B}_q \boldsymbol{u}^{(i)}(0). \tag{5.4.19}$$

We observe that $\xi_p(0)$ can be chosen arbitrary while $\xi_q(0)$ has to be chosen such that the hidden constraint (5.4.19) is satisfied. Thus the initial value $\boldsymbol{x}(0)$ of DAE (4.1.1) has to be consistent initial value and the input data has to be at least $\mu - 1$ times differentiable. In this approach we take care of this since, If we apply initial condition $\xi_p(0) = \boldsymbol{z}_{\boldsymbol{p}_{\mu-2}}^{*\mathrm{T}} \cdots \boldsymbol{z}_{\boldsymbol{p}_0}^{*\mathrm{T}} \boldsymbol{p}_0^{*\mathrm{T}} \boldsymbol{x}(0)$, where $\boldsymbol{x}(0)$ is a consistent initial condition, we can solve the system (5.4.18) hierarchically by numerically integration of differential part (5.4.18a) and then compute the algebraic solutions using (5.4.18b). Then the desired output solutions are obtained using (5.4.18c). It can be proved that $\sigma_f(\mathbf{E}, \mathbf{A}) = \sigma(\mathbf{A}_p)$, thus system (5.4.18) preserves stability of the DAEs. The number of differential equation is always equal to the total algebraic multiplicity of finite eigenvalues of the matrix pencil $(\mathbf{E}, \mathbf{A})$.

5.4.2 Index-μ DAEs without a differential part

Here, we assume that the spectrum of the matrix pencil of (4.1.1) has no finite eigenvalues, this implies $\Pi_{\mu-1} = 0$. Thus, (5.4.2), reduces to

$$\mathbf{I}_n = \mathbf{Q}_0 + \sum_{i=1}^{\mu-1} \Pi_{i-1} \mathbf{Q}_i. \tag{5.4.20}$$

Repeating steps from (5.4.3) to (5.4.9). It is nice to see that if $\mu = 2$, then (5.4.20) simplifies to, $\mathbf{I}_n = \mathbf{Q}_0 + \Pi_0\mathbf{Q}_1 = \mathbf{Q}_0 + \mathbf{P}_0$. Thus if $\mu = 2$ then the bases of the projector products $\{\mathbf{Q}_0, \Pi_0\mathbf{Q}_1\}$ in (5.4.20) are $\{\boldsymbol{q}_0, \boldsymbol{p}_0\}$, respectively otherwise we follow the steps below:

Step: 0 if $\mu > 2$:
Then by construction, (5.4.20) can be written as,

$$\mathbf{I}_{n_0} = \boldsymbol{p}_0^{*\mathrm{T}}\Pi_0\mathbf{Q}_1\boldsymbol{p}_0 + \boldsymbol{p}_0^{*\mathrm{T}}\sum_{i=2}^{\mu-1}\Pi_{i-1}\mathbf{Q}_i\boldsymbol{p}_0.$$

Then,

$$\mathbf{I}_{n_0} = \mathbf{Z}_{\boldsymbol{q}_0} + \mathbf{Z}_{\boldsymbol{p}_0}, \tag{5.4.21}$$

with $\mathbf{Z}_{\boldsymbol{p}_0} := \boldsymbol{p}_0^{*\mathrm{T}}\sum_{\substack{i=2\\ \mu>2}}^{\mu-1}\Pi_{i-1}\mathbf{Q}_i\boldsymbol{p}_0, \quad \mathbf{Z}_{\boldsymbol{q}_0} := \boldsymbol{p}_0^{*\mathrm{T}}\Pi_0\mathbf{Q}_1\boldsymbol{p}_0$. Using the general form of Theorem 5.3.1, $\mathbf{Z}_{\boldsymbol{p}_0}$ and $\mathbf{Z}_{\boldsymbol{q}_0}$ are mutually orthogonal projectors, acting in $\mathbb{R}^{n_0}$.
Let $k_1 = \dim(\mathrm{Im}\,\mathbf{Z}_{\boldsymbol{q}_0})$, and $n_1 = n_0 - k_1$, and let us consider a basis matrix $(z_{\boldsymbol{p}_0}, z_{\boldsymbol{q}_0}) \in \mathbb{R}^{n_0}$ made of n_1 independent columns of projection matrix $\mathbf{Z}_{\boldsymbol{p}_0}$ and k_1 independent columns of the complementary projection matrix $\mathbf{Z}_{\boldsymbol{q}_0}$. We denote by $(z_{\boldsymbol{p}_0}^{*\mathrm{T}}, z_{\boldsymbol{q}_0}^{*\mathrm{T}})^{\mathrm{T}}$ the inverse of $(z_{\boldsymbol{p}_0}, z_{\boldsymbol{q}_0})$, such that

$$z_{\boldsymbol{p}_0}^{*\mathrm{T}}z_{\boldsymbol{p}_0} = \mathbf{I}_{n_1},\ z_{\boldsymbol{p}_0}^{*\mathrm{T}}z_{\boldsymbol{q}_0} = 0,\ z_{\boldsymbol{q}_0}^{*\mathrm{T}}z_{\boldsymbol{p}_0} = 0,\ z_{\boldsymbol{q}_0}^{*\mathrm{T}}z_{\boldsymbol{q}_0} = \mathbf{I}_{k_1},\ z_{\boldsymbol{p}_0}z_{\boldsymbol{p}_0}^{*\mathrm{T}} + z_{\boldsymbol{q}_0}z_{\boldsymbol{q}_0}^{*\mathrm{T}} = \mathbf{I}_{n_0} \tag{5.4.22}$$

Then, we can represent $\mathbf{Z}_{\boldsymbol{p}_0}$ and $\mathbf{Z}_{\boldsymbol{q}_0}$ as

$$\mathbf{Z}_{\boldsymbol{p}_0} = z_{\boldsymbol{p}_0}z_{\boldsymbol{p}_0}^{*\mathrm{T}}, \quad \mathbf{Z}_{\boldsymbol{q}_0} = z_{\boldsymbol{q}_0}z_{\boldsymbol{q}_0}^{*\mathrm{T}},$$

and we have

$$\mathbf{Z}_{\boldsymbol{p}_0}z_{\boldsymbol{p}_0} = z_{\boldsymbol{p}_0}, \quad \mathbf{Z}_{\boldsymbol{p}_0}z_{\boldsymbol{q}_0} = 0, \quad \mathbf{Z}_{\boldsymbol{q}_0}z_{\boldsymbol{p}_0} = 0, \quad \mathbf{Z}_{\boldsymbol{q}_0}z_{\boldsymbol{q}_0} = z_{\boldsymbol{q}_0}. \tag{5.4.23}$$

Thus, if $\mu = 3$ then the bases of the projector products $\{\mathbf{Q}_0, \Pi_0\mathbf{Q}_1, \Pi_1\mathbf{Q}_2\}$ in (5.4.20) are $\{\boldsymbol{q}_0, \boldsymbol{p}_0 z_{\boldsymbol{q}_0}, \boldsymbol{p}_0 z_{\boldsymbol{p}_0}\}$, respectively.
Step: 1 if $\mu > 3$:

Using the identities (5.4.22) and (5.4.23) on (5.4.21) leads to

$$\mathbf{I}_{n_1} = \mathbf{Z}_{\boldsymbol{q}_1} + \mathbf{Z}_{\boldsymbol{p}_1}, \tag{5.4.24}$$

with $\mathbf{Z}_{\boldsymbol{p}_1} := z^{*\mathrm{T}}_{\boldsymbol{p}_0}\boldsymbol{p}_0^{*\mathrm{T}}\sum_{\substack{i=3\\ \mu>3}}^{\mu-1}\Pi_{i-1}\mathbf{Q}_i\boldsymbol{p}_0 z_{\boldsymbol{p}_0}, \quad \mathbf{Z}_{\boldsymbol{q}_1} := z^{*\mathrm{T}}_{\boldsymbol{p}_0}\boldsymbol{p}_0^{*\mathrm{T}}\Pi_1\mathbf{Q}_2\boldsymbol{p}_0 z_{\boldsymbol{p}_0}$. We can also see that the projectors are mutually orthogonal projectors, acting in $\mathbb{R}^{n_1}$.

Let $k_2 = \dim(\operatorname{Im}\mathbf{Z}_{\boldsymbol{q}_1})$, and $n_2 = n_1 - k_2$, and let us consider a basis matrix $(z_{\boldsymbol{p}_1}, z_{\boldsymbol{q}_1}) \in \mathbb{R}^{n_1}$ made of n_2 independent columns of projection matrix $\mathbf{Z}_{\boldsymbol{p}_0}$ and k_2 independent columns of the complementary projection matrix $\mathbf{Z}_{\boldsymbol{q}_1}$. We denote by $(z^{*\mathrm{T}}_{\boldsymbol{p}_1}, z^{*\mathrm{T}}_{\boldsymbol{q}_1})^{\mathrm{T}}$ the inverse of $(z_{\boldsymbol{p}_1}, z_{\boldsymbol{q}_1})$, such that

$$z^{*\mathrm{T}}_{\boldsymbol{p}_1} z_{\boldsymbol{p}_1} = \mathbf{I}_{n_2},\ z^{*\mathrm{T}}_{\boldsymbol{p}_1} z_{\boldsymbol{q}_1} = 0,\ z^{*\mathrm{T}}_{\boldsymbol{q}_1} z_{\boldsymbol{p}_1} = 0,\ z^{*\mathrm{T}}_{\boldsymbol{q}_1} z_{\boldsymbol{q}_1} = \mathbf{I}_{k_2},\ z_{\boldsymbol{p}_1} z^{*\mathrm{T}}_{\boldsymbol{p}_1} + z_{\boldsymbol{q}_1} z^{*\mathrm{T}}_{\boldsymbol{q}_1} = \mathbf{I}_{n_1}.$$

Then, we can represent $\mathbf{Z}_{\boldsymbol{p}_1}$ and $\mathbf{Z}_{\boldsymbol{q}_1}$ as $\mathbf{Z}_{\boldsymbol{p}_1} = z_{\boldsymbol{p}_1} z^{*\mathrm{T}}_{\boldsymbol{p}_1}, \quad \mathbf{Z}_{\boldsymbol{q}_1} = z_{\boldsymbol{q}_1} z^{*\mathrm{T}}_{\boldsymbol{q}_1}$, and we have

$$\mathbf{Z}_{\boldsymbol{p}_1} z_{\boldsymbol{p}_1} = z_{\boldsymbol{p}_1}, \quad \mathbf{Z}_{\boldsymbol{p}_1} z_{\boldsymbol{q}_1} = 0, \quad \mathbf{Z}_{\boldsymbol{q}_1} z_{\boldsymbol{p}_1} = 0, \quad \mathbf{Z}_{\boldsymbol{q}_1} z_{\boldsymbol{q}_1} = z_{\boldsymbol{q}_1}. \tag{5.4.25}$$

Thus, if $\mu = 4$ then the bases of the projector products $\{\mathbf{Q}_0, \Pi_0\mathbf{Q}_1, \Pi_1\mathbf{Q}_2, \Pi_2\mathbf{Q}_3\}$ in (5.4.20) are $\{\boldsymbol{q}_0, \boldsymbol{p}_0 z_{\boldsymbol{q}_0}, \boldsymbol{p}_0 z_{\boldsymbol{p}_0} z_{\boldsymbol{q}_1}, \boldsymbol{p}_0 z_{\boldsymbol{p}_0} z_{\boldsymbol{p}_1}\}$, respectively. This a recursive process which can easily be generalized.

Step: j if $\mu > j+2$:

The *jth* iteration leads to an identity matrix given by,

$$\mathbf{I}_{n_j} = \mathbf{Z}_{\boldsymbol{q}_j} + \mathbf{Z}_{\boldsymbol{p}_j}, \quad j = 1, \ldots, \mu-2, \quad \mu > 2, \tag{5.4.26}$$

with $\mathbf{Z}_{\boldsymbol{p}_j} := z^{*\mathrm{T}}_{\boldsymbol{p}_{j-1}} \cdots z^{*\mathrm{T}}_{\boldsymbol{p}_0}\boldsymbol{p}_0^{*\mathrm{T}} \sum_{\substack{i=j+2\\ j<\mu-2}}^{\mu-1} \Pi_{i-1}\mathbf{Q}_i\boldsymbol{p}_0 z_{\boldsymbol{p}_0} \cdots z_{\boldsymbol{p}_{j-1}}$,

$\mathbf{Z}_{\boldsymbol{q}_j} := z^{*\mathrm{T}}_{\boldsymbol{p}_{j-1}} \cdots z^{*\mathrm{T}}_{\boldsymbol{p}_0}\boldsymbol{p}_0^{*\mathrm{T}}\Pi_j\mathbf{Q}_{j+1}\boldsymbol{p}_0 z_{\boldsymbol{p}_0} \cdots z_{\boldsymbol{p}_{j-1}}$. These projectors are also mutually orthogonal projectors, acting in $\mathbb{R}^{n_j}$. Let $k_{j+1} = \dim(\operatorname{Im}\mathbf{Z}_{\boldsymbol{q}_j})$, and $n_{j+1} = n_j - k_{j+1}$, and let us consider a basis matrix $(z_{\boldsymbol{p}_j}, z_{\boldsymbol{q}_j}) \in \mathbb{R}^{n_j}$ made of n_{j+1} independent columns of projection matrix $\mathbf{Z}_{\boldsymbol{p}_j}$ and k_{j+1} independent columns of the complementary projection matrix $\mathbf{Z}_{\boldsymbol{q}_j}$.

We denote by $(\boldsymbol{z}_{\boldsymbol{p}_j}^{*\mathrm{T}}, \boldsymbol{z}_{\boldsymbol{q}_j}^{*\mathrm{T}})^{\mathrm{T}}$ the inverse of $(\boldsymbol{z}_{\boldsymbol{p}_j}, \boldsymbol{z}_{\boldsymbol{q}_j})$, such that

$$\begin{aligned}
&\boldsymbol{z}_{\boldsymbol{p}_j}^{*\mathrm{T}}\boldsymbol{z}_{\boldsymbol{p}_j} = \mathbf{I}_{n_{j+1}}, \quad \boldsymbol{z}_{\boldsymbol{p}_j}^{*\mathrm{T}}\boldsymbol{z}_{\boldsymbol{q}_j} = 0, \quad \boldsymbol{z}_{\boldsymbol{q}_j}^{*\mathrm{T}}\boldsymbol{z}_{\boldsymbol{p}_j} = 0, \quad \boldsymbol{z}_{\boldsymbol{q}_j}^{*\mathrm{T}}\boldsymbol{z}_{\boldsymbol{q}_j} = \mathbf{I}_{k_{j+1}},\\
&\boldsymbol{z}_{\boldsymbol{p}_j}\boldsymbol{z}_{\boldsymbol{p}_j}^{*\mathrm{T}} + \boldsymbol{z}_{\boldsymbol{q}_j}\boldsymbol{z}_{\boldsymbol{q}_j}^{*\mathrm{T}} = \mathbf{I}_{n_j}.
\end{aligned} \tag{5.4.27}$$

Then, we can represent $\mathbf{Z}_{\boldsymbol{p}_j}, \mathbf{Z}_{\boldsymbol{q}_j}$ as $\mathbf{Z}_{\boldsymbol{p}_j} = \boldsymbol{z}_{\boldsymbol{p}_j}\boldsymbol{z}_{\boldsymbol{p}_j}^{*\mathrm{T}}, \quad \mathbf{Z}_{\boldsymbol{q}_j} = \boldsymbol{z}_{\boldsymbol{q}_j}\boldsymbol{z}_{\boldsymbol{q}_j}^{*\mathrm{T}}$, and we have

$$\mathbf{Z}_{\boldsymbol{p}_j}\boldsymbol{z}_{\boldsymbol{p}_j} = \boldsymbol{z}_{\boldsymbol{p}_j}, \quad \mathbf{Z}_{\boldsymbol{p}_j}\boldsymbol{z}_{\boldsymbol{q}_j} = 0, \quad \mathbf{Z}_{\boldsymbol{q}_j}\boldsymbol{z}_{\boldsymbol{p}_j} = 0, \quad \mathbf{Z}_{\boldsymbol{q}_j}\boldsymbol{z}_{\boldsymbol{q}_j} = \boldsymbol{z}_{\boldsymbol{q}_j}. \tag{5.4.28}$$

Hence the bases of the projector products $\{\mathbf{Q}_0, \Pi_0\mathbf{Q}_1, \Pi_1\mathbf{Q}_2, \ldots, \Pi_{i-1}\mathbf{Q}_i, \ldots, \Pi_{\mu-2}\mathbf{Q}_{\mu-1}\}$ in (5.4.20) are $\{\boldsymbol{q}_0, \boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{q}_0}, \boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{q}_1}, \ldots, \boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0}\cdots\boldsymbol{z}_{\boldsymbol{p}_{i-3}}\boldsymbol{z}_{\boldsymbol{q}_{i-2}}, \ldots, \boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0}\cdots\boldsymbol{z}_{\boldsymbol{p}_{\mu-2}}\}$, $i = 3, \ldots, \mu-1$, respectively. Thus, we can now expand $\boldsymbol{x}$ with respect to these bases, obtaining,

$$\boldsymbol{x} = \boldsymbol{q}_0\xi_{q,0} + \boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{q}_0}\xi_{q,1} + \sum_{i=2}^{\mu-2}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0}\cdots\boldsymbol{z}_{\boldsymbol{p}_{i-2}}\boldsymbol{z}_{\boldsymbol{q}_{i-1}}\xi_{q,i} + \boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0}\cdots\boldsymbol{z}_{\boldsymbol{p}_{\mu-3}}\xi_{q,\mu-1}, \tag{5.4.29}$$

where $\xi_{\mu-1} \in \mathbb{R}^{n_{\mu-2}}$, $\xi_{q,i} \in \mathbb{R}^{k_i}$, $\xi_{q,0} \in \mathbb{R}^{k_i}$,$i = 0, \ldots, \mu-2$ and with inversion expressions

$$\begin{aligned}
&\xi_{\mu-1} = \boldsymbol{z}_{\boldsymbol{p}_{\mu-3}}^{*\mathrm{T}}\cdots\boldsymbol{z}_{\boldsymbol{p}_0}^{*\mathrm{T}}\boldsymbol{p}_0^{*\mathrm{T}}\boldsymbol{x}_{Q,\mu-1}, \quad \xi_{q,0} = \boldsymbol{q}_0^{*\mathrm{T}}\boldsymbol{x}_{Q,0}, \quad \xi_{q,1} = \boldsymbol{z}_{\boldsymbol{q}_0}^{*\mathrm{T}}\boldsymbol{p}_0^{*\mathrm{T}}\boldsymbol{x}_{Q,1},\\
&\xi_{q,i} = \boldsymbol{z}_{\boldsymbol{q}_{i-1}}^{*\mathrm{T}}\boldsymbol{z}_{\boldsymbol{p}_{i-2}}^{*\mathrm{T}}\cdots\boldsymbol{z}_{\boldsymbol{p}_0}^{*\mathrm{T}}\boldsymbol{p}_0^{*\mathrm{T}}\boldsymbol{x}_{Q,i}, \quad i = 2, \ldots, \mu-2.
\end{aligned} \tag{5.4.30}$$

If we substitute the variables in (5.4.29) and (5.4.30) into (5.4.1) leads to modified decoupled system given by

$$\xi_{q,\mu-1} = \mathbf{B}_{q,\mu-1}\boldsymbol{u}, \tag{5.4.31a}$$

$$\xi_{q,i} = \mathbf{B}_{q,i}\boldsymbol{u} + \sum_{j=i+1}^{\mu-1}\mathbf{A}_{\boldsymbol{q}_{i,j}}\xi'_{q,j}, \quad i = \mu-2, \ldots 2, \tag{5.4.31b}$$

$$\xi_{q,1} = \mathbf{B}_{q,1}\boldsymbol{u} + \sum_{j=2}^{\mu-1}\mathbf{A}_{\boldsymbol{q}_{1,j}}\xi'_{q,j}, \tag{5.4.31c}$$

$$\xi_{q,0} = \mathbf{B}_{q,0}\boldsymbol{u} + \sum_{j=1}^{\mu-1}\mathbf{A}_{\boldsymbol{q}_{0,j}}\xi'_{q,j}, \tag{5.4.31d}$$

$$\boldsymbol{y} = \sum_{i=0}^{\mu-1}\mathbf{C}_{q,i}^{\mathrm{T}}\xi_{q,i}, \tag{5.4.31e}$$

where

$$\mathbf{B}_{q,\mu-1} := \boldsymbol{z}^{*\mathrm{T}}_{\boldsymbol{p}_{\mu-3}} \cdots \boldsymbol{z}^{*\mathrm{T}}_{\boldsymbol{p}_0} \boldsymbol{p}_0^{*\mathrm{T}} \mathbf{B}_{Q,\mu-1} \in \mathbb{R}^{k_{\mu-1}\times m}, \quad \mathbf{B}_{q,i} := \boldsymbol{z}^{*\mathrm{T}}_{\boldsymbol{q}_{i-1}} \boldsymbol{z}^{*\mathrm{T}}_{\boldsymbol{p}_{i-2}} \cdots \boldsymbol{z}^{*\mathrm{T}}_{\boldsymbol{p}_0} \boldsymbol{p}_0^{*\mathrm{T}} \mathbf{B}_{Q,i} \in \mathbb{R}^{k_i\times m},$$
$$\mathbf{B}_{q,0} := \boldsymbol{q}_0^{*\mathrm{T}} \mathbf{B}_{Q,0} \in \mathbb{R}^{k_0,m}, \quad \mathbf{B}_{q,1} := \boldsymbol{z}^{*\mathrm{T}}_{\boldsymbol{q}_0} \boldsymbol{p}_0^{*\mathrm{T}} \mathbf{B}_{Q,1} \in \mathbb{R}^{k_1\times m},$$
$$\mathbf{A}_{\boldsymbol{q}_{i,j}} := \begin{cases} \boldsymbol{z}^{*\mathrm{T}}_{\boldsymbol{q}_{i-1}} \boldsymbol{z}^{*\mathrm{T}}_{\boldsymbol{p}_{i-2}} \cdots \boldsymbol{z}^{*\mathrm{T}}_{\boldsymbol{p}_0} \boldsymbol{p}_0^{*\mathrm{T}} \mathbf{A}_{\mathbf{Q}_{i,j}} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{p}_0} \cdots \boldsymbol{z}_{\boldsymbol{p}_{\mu-3}}, & \text{If} \quad j = \mu - 1, \\ \boldsymbol{z}^{*\mathrm{T}}_{\boldsymbol{q}_{i-1}} \boldsymbol{z}^{*\mathrm{T}}_{\boldsymbol{p}_{i-2}} \cdots \boldsymbol{z}^{*\mathrm{T}}_{\boldsymbol{p}_0} \boldsymbol{p}_0^{*\mathrm{T}} \mathbf{A}_{\mathbf{Q}_{i,j}} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{p}_0} \cdots \boldsymbol{z}_{\boldsymbol{p}_{j-2}} \boldsymbol{z}_{\boldsymbol{q}_{j-1}}, & \text{Otherwise,} \end{cases}$$
$$\mathbf{A}_{\boldsymbol{q}_{1,j}} := \begin{cases} \boldsymbol{z}^{*\mathrm{T}}_{\boldsymbol{q}_0} \boldsymbol{p}_0^{*\mathrm{T}} \mathbf{A}_{\mathbf{Q}_{1,j}} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{p}_0} \cdots \boldsymbol{z}_{\boldsymbol{p}_{\mu-3}}, & \text{If,} \quad j = \mu - 1, \\ \boldsymbol{z}^{*\mathrm{T}}_{\boldsymbol{q}_0} \boldsymbol{p}_0^{*\mathrm{T}} \mathbf{A}_{\mathbf{Q}_{1,j}} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{p}_0} \cdots \boldsymbol{z}_{\boldsymbol{p}_{j-2}} \boldsymbol{z}_{\boldsymbol{q}_{j-1}}, & \text{Otherwise.} \end{cases}$$
$$\mathbf{A}_{\boldsymbol{q}_{0,j}} := \begin{cases} \boldsymbol{q}_0^{*\mathrm{T}} \mathbf{A}_{\mathbf{Q}_{0,j}} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{q}_0}, & \text{If,} \quad j = 1, \\ \boldsymbol{q}_0^{*\mathrm{T}} \mathbf{A}_{\mathbf{Q}_{0,j}} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{p}_0} \cdots \boldsymbol{z}_{\boldsymbol{p}_{j-2}} \boldsymbol{z}_{\boldsymbol{q}_{j-1}}, & \text{If,} \quad 2 \le j \le \mu - 2, \\ \boldsymbol{q}_0^{*\mathrm{T}} \mathbf{A}_{\mathbf{Q}_{0,j}} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{p}_0} \cdots \boldsymbol{z}_{\boldsymbol{p}_{\mu-3}}, & \text{If,} \quad j = \mu - 1. \end{cases}$$
$$\mathbf{C}^{\mathrm{T}}_{q,0} = \mathbf{C}^{\mathrm{T}} \boldsymbol{q}_0 \in \mathbb{R}^{\ell\times k_0}, \ \mathbf{C}^{\mathrm{T}}_{q,1} = \mathbf{C}^{\mathrm{T}} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{\ell\times k_1}, \ \mathbf{C}^{\mathrm{T}}_{q,i} = \mathbf{C}^{\mathrm{T}} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{p}_0} \cdots \boldsymbol{z}_{\boldsymbol{p}_{i-2}} \boldsymbol{z}_{\boldsymbol{q}_{i-1}} \in \mathbb{R}^{\ell\times k_i}.$$

We can observe that equations (5.4.31) can be written as

$$-\mathcal{L}\xi_q' = -\xi_q + \mathbf{B}_q \boldsymbol{u} \tag{5.4.32a}$$
$$\boldsymbol{y} = \mathbf{C}_q^{\mathrm{T}} \xi_q, \tag{5.4.32b}$$

where $\xi_q = (\xi_{q,\mu-1}, \ldots, \xi_{q,0})^{\mathrm{T}} \in \mathbb{R}^n$, $\mathbf{B}_q = (\mathbf{B}_{q,\mu-1}, \ldots, \mathbf{B}_{q,0})^{\mathrm{T}} \in \mathbb{R}^{n\times m}$, $\mathbf{C}_q = (\mathbf{C}^{\mathrm{T}}_{q,\mu-1}, \ldots, \mathbf{C}^{\mathrm{T}}_{q,0})^{\mathrm{T}} \in \mathbb{R}^{n\times\ell}$, $\mathcal{L} \in \mathbb{R}^{n\times n}$ is a strictly lower triangular nilpotent matrix of index μ. It can also be proved that the decoupled system (5.4.32) can be written in the form:

$$\boldsymbol{y} = \mathbf{C}_q^{\mathrm{T}} \sum_{i=0}^{\mu-1} \mathcal{L}^i \mathbf{B}_q \boldsymbol{u}^{(i)}, \tag{5.4.33}$$

where $\boldsymbol{u}^{(i)} \in \mathbb{R}^m$ is the *ith* derivative of the input data. We observe that, we have only algebraic equations and their solutions can be computed exactly. We can also observe that $n = \sum_{i=0}^{\mu-1} k_i$ is the total number of algebraic equations which is also equal to the dimension of the DAE. Thus the decoupled system (5.4.32) preserves the dimension of the DAE. For comparison with the DAE (2.3.1), we can rewrite either system (5.4.18) or (5.4.32) in the descriptor form given by

$$\tilde{\mathbf{E}}\xi' = \tilde{\mathbf{A}}\xi + \tilde{\mathbf{B}}\boldsymbol{u}, \tag{5.4.34a}$$
$$\boldsymbol{y} = \tilde{\mathbf{C}}^{\mathrm{T}}\xi, \tag{5.4.34b}$$

where: if the spectrum of the matrix pencil $(\mathbf{E}, \mathbf{A})$ has at least one finite eigenvalue, then $\tilde{\mathbf{E}} = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathcal{L} \end{pmatrix} \in \mathbb{R}^{n\times n}$, $\tilde{\mathbf{A}} = \begin{pmatrix} \mathbf{A}_p & 0 \\ \mathbf{A}_q & -\mathrm{I} \end{pmatrix} \in \mathbb{R}^{n\times n}$, $\tilde{\mathbf{B}} = \begin{pmatrix} \mathbf{B}_p \\ \mathbf{B}_q \end{pmatrix} \in \mathbb{R}^{n\times m}$, $\tilde{\mathbf{C}} = \begin{pmatrix} \mathbf{C}_p \\ \mathbf{C}_q \end{pmatrix} \in \mathbb{R}^{n\times \ell}$ and if the spectrum of the matrix pencil $(\mathbf{E}, \mathbf{A})$ has no finite eigenvalue, then $\tilde{\mathbf{E}} = -\mathcal{L} \in \mathbb{R}^{n\times n}$, $\tilde{\mathbf{A}} = -\mathrm{I} \in \mathbb{R}^{n\times n}$, $\tilde{\mathbf{B}} = \mathbf{B}_q \in \mathbb{R}^{n\times m}$, $\tilde{\mathbf{C}} = \mathbf{C}_q \in \mathbb{R}^{n\times \ell}$. We can observe that this form reveals the interconnection structure of the DAE (2.3.1). Moreover it can be proved that systems (2.3.1) and (5.4.34) are equivalent. This implies that also their respective matrix pencils $(\mathbf{E}, \mathbf{A})$ and $(\tilde{\mathbf{E}}, \tilde{\mathbf{A}})$ are equivalent. If we consider DAEs whose matrix pencil $(\mathbf{E}, \mathbf{A})$ has at least one finite eigenvalue, we can show that they have same spectrum , since we can easily show that $\det(\lambda\tilde{\mathbf{E}} - \tilde{\mathbf{A}}) = \det(\lambda \mathrm{I} - \mathbf{A}_p)$, since $\det(\mathrm{I} - \lambda\mathcal{L}) = (1)^{n_q}$. This identity shows that the finite eigenvalues of the matrix pencil $(\mathbf{E}, \mathbf{A})$ coincide with the (possibly complex) eigenvalues of the matrix $\mathbf{A}_p$ of the differential part, which are exactly n_p, counting their multiplicity, i.e., $\sigma(\mathbf{A}_p) = \sigma_f(\mathbf{E}, \mathbf{A})$. Thus, the differential part of the decoupled system inherits the stability properties of DAEs.

5.4.3 Decoupling of index-3 DAEs

In Section 5.2 and 5.3, we have discussed the decoupling of index-1 and-2 DAEs, respectively. These decoupled system can be written in the descriptor form (5.4.34) and it is easy to check that for the case of index-1 and -2 DAEs, nilpotent matrices are given by $\mathcal{L} = \mathbf{0}$ and $\mathcal{L} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{\boldsymbol{q}_{0,1}} & \mathbf{0} \end{pmatrix}$, respectively. In this Section, we discuss how to decouple index-3 DAEs using the generalized procedure in the previous Section. Thus we need to assume that the DAE (4.1.1) is an index-3 DAE, i.e., $\mu = 3$. We also assume that the projectors are constructed such that (4.1.13) holds true. Thus index-3 DAEs can be decoupled as follows: If system (4.1.1) has a matrix pencil with at least one finite eigenvalue then decoupled system take the form (5.4.16). Thus substituting $\mu = 3$ into (5.4.16) and simplifying, we obtain a modified decoupled system for index-3 DAEs given by:

$$\xi_p' = \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u}, \tag{5.4.35a}$$

$$\xi_{q,2} = \mathbf{A}_{q,2}\xi_p + \mathbf{B}_{q,2}\boldsymbol{u}, \tag{5.4.35b}$$

$$\xi_{q,1} = \mathbf{A}_{q,1}\xi_p + \mathbf{B}_{q,1}\boldsymbol{u} + \mathbf{A}_{\boldsymbol{q}_{1,2}}\xi_{q,2}', \tag{5.4.35c}$$

$$\xi_{q,0} = \mathbf{A}_{q,0}\xi_p + \mathbf{B}_{q,0}\boldsymbol{u} + \mathbf{A}_{\boldsymbol{q}_{0,1}}\xi_{q,1}' + \mathbf{A}_{\boldsymbol{q}_{0,2}}\xi_{q,2}', \tag{5.4.35d}$$

$$\boldsymbol{y} = \mathbf{C}_p^{\mathrm{T}}\xi_p + \mathbf{C}_{q,2}^{\mathrm{T}}\xi_{q,2} + \mathbf{C}_{q,1}^{\mathrm{T}}\xi_{q,1} + \mathbf{C}_{q,0}^{\mathrm{T}}\xi_{q,0} \tag{5.4.35e}$$

where

$$
\begin{aligned}
&\mathbf{A}_p := \boldsymbol{z}_{\boldsymbol{p}_1}^{*\mathrm{T}} \boldsymbol{z}_{\boldsymbol{p}_0}^{*\mathrm{T}} \boldsymbol{p}_0^{*\mathrm{T}} \mathbf{A}_P \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{p}_1} \in \mathbb{R}^{n_p \times n_p}, && \mathbf{B}_p := \boldsymbol{z}_{\boldsymbol{p}_1}^{*\mathrm{T}} \boldsymbol{z}_{\boldsymbol{p}_0}^{*\mathrm{T}} \boldsymbol{p}_0^{*\mathrm{T}} \mathbf{B}_P \in \mathbb{R}^{n_p \times m},\\
&\mathbf{A}_{q,2} := \boldsymbol{z}_{\boldsymbol{q}_1}^{*\mathrm{T}} \boldsymbol{z}_{\boldsymbol{p}_0}^{*\mathrm{T}} \boldsymbol{p}_0^{*\mathrm{T}} \mathbf{A}_{Q,2} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{p}_1} \in \mathbb{R}^{k_2 \times n_p}, && \mathbf{B}_{q,2} := \boldsymbol{z}_{\boldsymbol{q}_1}^{*\mathrm{T}} \boldsymbol{z}_{\boldsymbol{p}_0}^{*\mathrm{T}} \boldsymbol{p}_0^{*\mathrm{T}} \mathbf{B}_{Q,2} \in \mathbb{R}^{k_2 \times m},\\
&\mathbf{A}_{q,1} := \boldsymbol{z}_{\boldsymbol{q}_0}^{*\mathrm{T}} \boldsymbol{p}_0^{*\mathrm{T}} \mathbf{A}_{Q,1} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{p}_1} \in \mathbb{R}^{k_1 \times n_p}, && \mathbf{B}_{q,1} := \boldsymbol{z}_{\boldsymbol{q}_0}^{*\mathrm{T}} \boldsymbol{p}_0^{*\mathrm{T}} \mathbf{B}_{Q,1} \in \mathbb{R}^{k_1 \times m},\\
&\mathbf{A}_{q,0} := \boldsymbol{q}_0^{*\mathrm{T}} \mathbf{A}_{Q,0} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{p}_1} \in \mathbb{R}^{k_0 \times n_p}, && \mathbf{B}_{q,0} := \boldsymbol{q}_0^{*\mathrm{T}} \mathbf{B}_{Q,0} \in \mathbb{R}^{k_0 \times m},\\
&\mathbf{A}_{\boldsymbol{q}_{1,2}} := \boldsymbol{z}_{\boldsymbol{q}_0}^{*\mathrm{T}} \boldsymbol{p}_0^{*\mathrm{T}} \mathbf{A}_{\mathbf{Q}_{1,2}} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{q}_1} \in \mathbb{R}^{k_1 \times k_2}, && \mathbf{A}_{\boldsymbol{q}_{0,1}} := \boldsymbol{q}_0^{*\mathrm{T}} \mathbf{A}_{\mathbf{Q}_{0,1}} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{k_0 \times k_1},\\
&\mathbf{A}_{\boldsymbol{q}_{0,2}} := \boldsymbol{q}_0^{*\mathrm{T}} \mathbf{A}_{\mathbf{Q}_{0,2}} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{q}_1} \in \mathbb{R}^{k_0 \times k_2}, && \mathbf{C}_p^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{p}_1} \in \mathbb{R}^{n_p \times \ell},\\
&\mathbf{C}_{q,2}^{\mathrm{T}} := \mathbf{C}^{\mathrm{T}} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{q}_1} \in \mathbb{R}^{k_2 \times \ell}, \mathbf{C}_{q,1}^{\mathrm{T}} := \mathbf{C}^{\mathrm{T}} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{k_1 \times \ell}, && \mathbf{C}_{q,0}^{\mathrm{T}} := \mathbf{C}^{\mathrm{T}} \boldsymbol{q}_0 \in \mathbb{R}^{k_0 \times \ell},
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{A}_P := \mathbf{P}_0 \mathbf{P}_1 \mathbf{P}_2 \mathbf{E}_3^{-1} \mathbf{A}_3 \in \mathbb{R}^{n \times n}, && \mathbf{B}_P := \mathbf{P}_0 \mathbf{P}_1 \mathbf{P}_2 \mathbf{E}_3^{-1} \mathbf{B} \in \mathbb{R}^{n \times m},\\
&\mathbf{A}_{Q,2} := \mathbf{P}_0 \mathbf{P}_1 \mathbf{Q}_2 \mathbf{E}_3^{-1} \mathbf{A}_3 \in \mathbb{R}^{n \times n}, && \mathbf{B}_{Q,2} := \mathbf{P}_0 \mathbf{P}_1 \mathbf{Q}_2 \mathbf{E}_3^{-1} \mathbf{B} \in \mathbb{R}^{n \times m},\\
&\mathbf{A}_{Q,1} := \mathbf{P}_0 \mathbf{Q}_1 \mathbf{P}_2 \mathbf{E}_3^{-1} \mathbf{A}_3 \in \mathbb{R}^{n \times n}, && \mathbf{B}_{Q,1} := \mathbf{P}_0 \mathbf{Q}_1 \mathbf{P}_2 \mathbf{E}_3^{-1} \mathbf{B} \in \mathbb{R}^{n \times m},\\
&\mathbf{A}_{\mathbf{Q}_{1,2}} := \mathbf{P}_0 \mathbf{Q}_1 \mathbf{Q}_2 \in \mathbb{R}^{n \times n}, && \mathbf{A}_{Q,0} := \mathbf{Q}_0 \mathbf{P}_1 \mathbf{P}_2 \mathbf{E}_3^{-1} \mathbf{A}_3 \in \mathbb{R}^{n \times n},\\
&\mathbf{B}_{Q,0} := \mathbf{Q}_0 \mathbf{P}_1 \mathbf{P}_2 \mathbf{E}_3^{-1} \mathbf{B} \in \mathbb{R}^{n \times m}, && \mathbf{A}_{\mathbf{Q}_{0,1}} := \mathbf{Q}_0 \mathbf{Q}_1 \in \mathbb{R}^{n \times n},\\
&\mathbf{A}_{\mathbf{Q}_{0,2}} := \mathbf{Q}_0 \mathbf{P}_1 \mathbf{Q}_2 \in \mathbb{R}^{n \times n}.
\end{aligned}
$$

After re-arranging and simplifying this decoupled system can be simplified to the form (5.4.18) given by:

$$
\begin{aligned}
\xi_p' &= \mathbf{A}_p \xi_p + \mathbf{B}_p \boldsymbol{u},\\
\xi_q &= \sum_{i=0}^{2} \mathcal{L}^i \mathbf{A}_q \mathbf{A}_p^i \xi_p + \sum_{i=1}^{2} \sum_{k=0}^{i-1} \mathcal{L}^i \mathbf{A}_q \mathbf{A}_p^k \mathbf{B}_p \boldsymbol{u}^{(i-k-1)} + \sum_{i=0}^{2} \mathcal{L}^i \mathbf{B}_q \boldsymbol{u}^{(i)},\\
\boldsymbol{y} &= \mathbf{C}_p^{\mathrm{T}} \xi_p + \mathbf{C}_q^{\mathrm{T}} \xi_q,
\end{aligned}
\tag{5.4.36}
$$

where $\xi_q = (\xi_{q,2}, \xi_{q,1}, \xi_{q,0})^{\mathrm{T}} \in \mathbb{R}^{n_q}$, $\mathbf{A}_q = (\mathbf{A}_{q,2}, \mathbf{A}_{q,1}, \mathbf{A}_{q,0})^{\mathrm{T}} \in \mathbb{R}^{n_q \times n_p}$, $\mathbf{B}_q = (\mathbf{B}_{q,2}, \mathbf{B}_{q,1}, \mathbf{B}_{q,0})^{\mathrm{T}} \in \mathbb{R}^{n_q \times m}$, $\mathbf{C}_q = (\mathbf{C}_{q,2}, \mathbf{C}_{q,1}, \mathbf{C}_{q,0})^{\mathrm{T}} \in \mathbb{R}^{n_q \times \ell}$, $n_q = \sum_{i=0}^{2} k_i$ and $\mathcal{L} \in \mathbb{R}^{n_q \times n_q}$ is a strictly lower triangular nilpotent matrix of index-3 given by

$$
\mathcal{L} := \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{\boldsymbol{q}_{1,2}} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{\boldsymbol{q}_{0,2}} & \mathbf{A}_{\boldsymbol{q}_{0,1}} & \mathbf{0} \end{pmatrix}. \tag{5.4.37}
$$

If system (4.1.1) has a matrix pencil with only infinite spectrum then its März decoupled system can be modified into the form (5.4.31). Thus substituting $\mu = 3$ into (5.4.31) and simplifying, we obtain a modified decoupled system of index-3 DAEs given by:

$$\xi_{q,2} = \mathbf{B}_{q,2}\boldsymbol{u}, \tag{5.4.38a}$$
$$\xi_{q,1} = \mathbf{B}_{q,1}\boldsymbol{u} + \mathbf{A}_{\boldsymbol{q}_{1,2}}\xi'_{q,2}, \tag{5.4.38b}$$
$$\xi_{q,0} = \mathbf{B}_{q,0}\boldsymbol{u} + \mathbf{A}_{\boldsymbol{q}_{0,1}}\xi'_{q,1} + \mathbf{A}_{\boldsymbol{q}_{0,2}}\xi'_{q,2}, \tag{5.4.38c}$$
$$\boldsymbol{y} = \mathbf{C}^{\mathrm{T}}_{q,2}\,\xi_{q,2} + \mathbf{C}^{\mathrm{T}}_{q,1}\,\xi_{q,1} + \mathbf{C}^{\mathrm{T}}_{q,0}\,\xi_{q,0}, \tag{5.4.38d}$$

where

$$\mathbf{B}_{q,2} := \boldsymbol{z}^{*\mathrm{T}}_{\boldsymbol{p}_0}\boldsymbol{p}^{*\mathrm{T}}_0\mathbf{B}_{Q,2} \in \mathbb{R}^{k_2\times m},\ \mathbf{B}_{q,1} := \boldsymbol{z}^{*\mathrm{T}}_{\boldsymbol{q}_0}\boldsymbol{p}^{*\mathrm{T}}_0\mathbf{B}_{Q,1} \in \mathbb{R}^{k_1\times m},\ \mathbf{B}_{q,0} := \boldsymbol{q}^{*\mathrm{T}}_0\mathbf{B}_{Q,0} \in \mathbb{R}^{k_0\times m},$$
$$\mathbf{A}_{\boldsymbol{q}_{1,2}} := \boldsymbol{z}^{*\mathrm{T}}_{\boldsymbol{q}_0}\boldsymbol{p}^{*\mathrm{T}}_0\mathbf{A}_{\mathbf{Q}_{1,2}}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0} \in \mathbb{R}^{k_1\times k_2},\ \mathbf{A}_{\boldsymbol{q}_{0,1}} := \boldsymbol{q}^{*\mathrm{T}}_0\mathbf{A}_{\mathbf{Q}_{0,1}}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{k_0\times k_1},\ \mathbf{C}^{\mathrm{T}}_{q,0} = \mathbf{C}^{\mathrm{T}}\boldsymbol{q}_0 \in \mathbb{R}^{k_0\times \ell},$$
$$\mathbf{A}_{\boldsymbol{q}_{0,2}} := \boldsymbol{q}^{*\mathrm{T}}_0\mathbf{A}_{\mathbf{Q}_{0,2}}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0} \in \mathbb{R}^{k_0\times k_2},\ \mathbf{C}^{\mathrm{T}}_{q,2} = \mathbf{C}^{\mathrm{T}}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0} \in \mathbb{R}^{k_2\times \ell},\ \mathbf{C}^{\mathrm{T}}_{q,1} = \mathbf{C}^{\mathrm{T}}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{k_1\times \ell}.$$

Also after re-arranging and simplifying this decoupled system can be simplified into the form (5.4.33) given by:

$$\boldsymbol{y} = \mathbf{C}^{\mathrm{T}}_{q,}\sum_{i=0}^{2}\mathcal{L}^i\mathbf{B}_q\boldsymbol{u}^{(i)}, \tag{5.4.39}$$

where $\xi_q = (\xi_{q,2}, \xi_{q,1}, \xi_{q,0})^{\mathrm{T}} \in \mathbb{R}^n$, $\mathbf{B}_q = (\mathbf{B}_{q,2}, \mathbf{B}_{q,1}, \mathbf{B}_{q,0})^{\mathrm{T}} \in \mathbb{R}^{n\times m}$,
$\mathbf{C}_q = (\mathbf{C}_{q,2}, \mathbf{C}_{q,1}, \mathbf{C}_{q,0})^{\mathrm{T}} \in \mathbb{R}^{n\times m}$, $\mathcal{L} \in \mathbb{R}^{n\times n}$ is a strictly lower triangular nilpotent matrix of index-3 which takes the same form as (5.4.37). Hence index-3 DAEs can be decoupled into the form either (5.4.36) or (5.4.39) depending on the spectrum of the matrix pencil. In the examples below, we illustrate the decoupling of index-3 DAEs.

Example 5.4.1 ([46]) As a simple mechanical example, we consider a car-pendulum system shown in Figure 5.3 that consists of a cart of mass m_1 and a pendulum of length L and of mass m_2. In [46], they linearized nonlinear equations of motion of this multibody system along the equilibrium $\left[0, 0, -L, 0, 0, m_2 g/(2L)\right]$ which yields an index-3 DAE

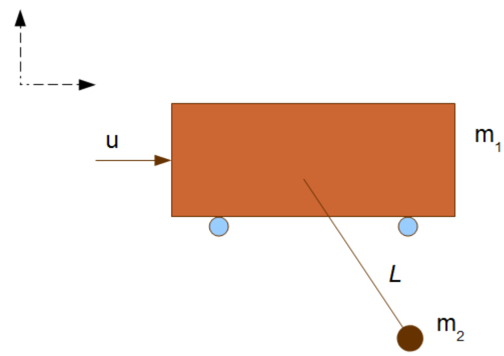

Figure 5.3: A cart-pendulum system

with system matrices :

$$\mathbf{E} = \mathrm{diag}(1,1,1,m_1,m_2,m_2,0), \quad \mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -m_2 g/L & m_2 g/L & 0 & 0 & 0 & 0 & 0 \\ m_2 g/L & -m_2 g/L & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_2 g/L & 0 & 0 & 0 & 2L \\ 0 & 0 & -2L & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{B} = \left[0,0,0,1,0,0,0\right]^{\mathrm{T}}, \quad \mathbf{C} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}. \tag{5.4.40}$$

This system has a regular matrix pencil $(\mathbf{E},\mathbf{A})$ and has the finite eigenvalues $\sigma_f(\mathbf{E},\mathbf{A}) = \{0,0,\pm i\sqrt{(m_1+m_2)g/(m_1 L)}\}$. We constructed special projector chain $\mathbf{Q}_0, \mathbf{Q}_1$ and $\mathbf{Q}_2$ of this system that satisfy (4.1.13) given by,

$$\mathbf{Q}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{m_2}{2L} & 0 & 0 & \frac{m_2}{2L} & 1 \end{pmatrix}, \quad \mathbf{Q}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{Q}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and final matrices on the matrix chain given by,

$$\mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & m_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 0 & \frac{gm_2}{L} & 0 & 0 & m_2 & -2L \\ 0 & 0 & 2L & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{gm_2}{L} & \frac{gm_2}{L} & 0 & 0 & 0 & 0 & 0 \\ \frac{gm_2}{L} & -\frac{gm_2}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $\mathbf{E}_3$ is non-singular, then this system is of tractability index-3 or index-3 DAE. We need to first construct the basis vector $(\boldsymbol{p}, \boldsymbol{q})$ and their corresponding inverses $\begin{pmatrix} \boldsymbol{p}^{*\mathrm{T}} \\ \boldsymbol{q}^{*\mathrm{T}} \end{pmatrix}$ given by

$$\boldsymbol{p} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{m_2}{2L} & 0 & 0 & -\frac{m_2}{2L} \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{p}^{*\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \boldsymbol{q}^{*\mathrm{T}} = \begin{pmatrix} 0 \\ 0 \\ \frac{m_2}{2L} \\ 0 \\ 0 \\ \frac{m_2}{2L} \\ 1 \end{pmatrix}, \tag{5.4.41}$$

for the projector $\mathbf{Q}_0$ and its complementary $\mathbf{P}_0$, respectively. Then, we use the above basis to construct the second basis $(\mathbf{z}_{\boldsymbol{p}_0}, \mathbf{z}_{\boldsymbol{q}_0})$ and their corresponding inverses $\begin{pmatrix} \mathbf{z}_{\boldsymbol{p}_0}^{*\mathrm{T}} \\ \mathbf{z}_{\boldsymbol{q}_0}^{*\mathrm{T}} \end{pmatrix}$ given by,

$$\mathbf{z}_{\boldsymbol{p}_0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \quad \mathbf{z}_{\boldsymbol{q}_0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{z}_{\boldsymbol{p}_0}^{*\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{z}_{\boldsymbol{q}_0}^{*} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{5.4.42}$$

for the projector $\mathbf{Z}_{\boldsymbol{p}_0}$ and its complementary $\mathbf{Z}_{\boldsymbol{q}_0}$, respectively. Thus, we use the above bases to construct the third basis $(\mathbf{z}_{\boldsymbol{p}_1}, \mathbf{z}_{\boldsymbol{q}_1})$ and their corresponding inverses $\begin{pmatrix} \mathbf{z}_{\boldsymbol{p}_1}^{*\mathrm{T}} \\ \mathbf{z}_{\boldsymbol{q}_1}^{*\mathrm{T}} \end{pmatrix}$ given by,

$$\mathbf{z}_{p_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{z}_{q_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{z}^{*\mathrm{T}}_{p_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{z}^{*\mathrm{T}}_{q_1} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \tag{5.4.43}$$

for the projector $\mathbf{Z}_{p_1}$ and its complementary $\mathbf{Z}_{q_1}$, respectively. Substituting (5.4.41)-(5.4.43) into (5.4.35), we obtain the modified decoupled system with matrix coefficients,

$$\mathbf{A}_p = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -m_2 g/(Lm_1) & m_2 g/(Lm_1) & 0 & 0 \\ g/L & -g/L & 0 & 0 \end{pmatrix}, \quad \mathbf{B}_p = \begin{pmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{pmatrix}, \quad \mathbf{A}_{q,2} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}_{q,2} = 0,$$

$$\mathbf{A}_{q,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}_{q,1} = 0, \quad \mathbf{A}_{\boldsymbol{q}_{1,2}} = 0, \quad \mathbf{A}_{q,0} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}_{q,0} = 0, \quad \mathbf{A}_{\boldsymbol{q}_{0,1}} = \frac{m_2}{2L},$$

$$\mathbf{A}_{\boldsymbol{q}_{0,1}} = 0, \quad \mathbf{C}^{\mathrm{T}}_p = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{C}^{\mathrm{T}}_{q,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{C}^{\mathrm{T}}_{q,1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{C}^{\mathrm{T}}_{q,0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

After simplifying the decoupled system reduces to an ODE dynamical system given by,

$$\begin{aligned} \xi_p' &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -m_2 g/(Lm_1) & m_2 g/(Lm_1) & 0 & 0 \\ g/L & -g/L & 0 & 0 \end{pmatrix} \xi_p + \begin{pmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{pmatrix} \boldsymbol{u}, \\ \boldsymbol{y} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xi_p. \end{aligned} \tag{5.4.44}$$

When we computed the eigenvalues of $\mathbf{A}_p$ we found out that they are equal to the finite eigenvalues of the matrix pencil $(\mathbf{E},)$, i.e., $\sigma_f(\mathbf{E}, \mathbf{A}) = \sigma(\mathbf{A}_p)$ as expected. We can observe that DAE dynamical system (5.4.40) of dimension 7 reduces to an ODE system (5.4.44) of dimension 4 even before applying any MOR technique. Hence this approach is not only advantageous in solving DAEs but also in MOR of DAEs.

Example 5.4.2 Consider an index-3 DAE obtained from [63] with system matrices given by

$$\mathbf{E} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 10 \\ 0.1 \\ 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0.04 \\ 30 \\ 1 \end{pmatrix}. \tag{5.4.45}$$

Since the $\det(\lambda\mathbf{E} - \mathbf{A}) = -1$, thus the DAE system (5.4.45) is solvable and its matrix pencil has no finite eigenvalues. Hence we expect its decoupled system to have no differential part. We can then choose the projector chains,

$$\mathbf{Q}_0 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{Q}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{Q}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that satisfy the condition $\mathbf{Q}_j\mathbf{Q}_i = 0,\ j > i,\ i, j = 0, 1, 2$ and the last matrix chain is given by $\mathbf{E}_3 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$. Following the same procedure we discussed earlier, we were able to construct projector bases given by

$$\boldsymbol{p}_0 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{q}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{p}_0^{*\mathrm{T}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{q}_0^{*\mathrm{T}} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix},$$

$$\text{and} \quad \boldsymbol{z}_{\boldsymbol{p}_0} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{z}_{\boldsymbol{q}_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{z}_{\boldsymbol{p}_0}^{*\mathrm{T}} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad \boldsymbol{z}_{\boldsymbol{q}_0}^{*\mathrm{T}} = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

Thus, substituting the above matrices into (5.4.38), we obtain the system matrices given by, $\mathbf{B}_{q,2} = 0$, $\mathbf{B}_{q,1} = -0.1$, $\mathbf{A}_{\boldsymbol{q}_{1,2}} = 1$, $\mathbf{B}_{q,0} = -10.1$, $\mathbf{A}_{\boldsymbol{q}_{0,1}} = 1$, $\mathbf{A}_{\boldsymbol{q}_{0,2}} = 0$, $\mathbf{C}_{q,2} = -29$, $\mathbf{C}_{q,1} = 29.96$, $\mathbf{C}_{q,0} = 0.04$. This leads to a decoupled system given by

$$\begin{aligned} \xi_{q,2} &= 0\boldsymbol{u}, \\ \xi_{q,1} &= -0.1\boldsymbol{u} + \xi'_{q,2}, \\ \xi_{q,0} &= -10.1\boldsymbol{u} + \xi'_{q,1} + 0\xi'_{q,2}, \\ \boldsymbol{y} &= -29\xi_{q,2} + 29.96\xi_{q,1} + 0.04\xi_{q,0}. \end{aligned} \tag{5.4.46}$$

We can observe that it is easy to solve (5.4.46) and its solution coincide with that of (5.4.45) given by $\boldsymbol{y} = -3.4\boldsymbol{u} - 0.004\boldsymbol{u}'$.

In this Chapter, we have discussed how to decouple linear constant DAEs using special bases of projectors and their respective products. We have seen that this approach preserves the dimension and the spectrum of the DAE in contrast with the März decoupling procedure discussed in the previous Chapter. This approach is robust and leads to simple decoupled systems which can be solved using the existing numerical integration techniques for ODEs. This means that one no longer need special numerical integration techniques for DAEs. However, this decoupling procedure and März decoupling procedure use the projected DAE (4.1.11) as the starting projected system in order to decouple the DAE into differential and algebraic parts. We can observe that (4.1.11) involves the inversion of matrix $\mathbf{E}_\mu$ which can be computationally very expensive for large-scale problems. This limits the use of this method on large-scale problems. In the next Chapter, we derive another procedure which do not involve the inversion of matrix $\mathbf{E}_\mu$, i.e., we use (4.1.10) as the starting projected system.

Chapter 6

Decoupling of DAEs without matrix $\mathbf{E}_\mu$ inversion

Consider a DAE of the form

$$\mathbf{E}\dot{\boldsymbol{x}}(t) = \mathbf{A}\boldsymbol{x}(t) + \mathbf{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{6.0.1a}$$

$$\boldsymbol{y}(t) = \mathbf{C}^{\mathrm{T}}\boldsymbol{x}(t), \tag{6.0.1b}$$

where $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{n\times \ell}$, the input vector $\boldsymbol{u}(t) \in \mathbb{R}^m$ and output vector $\boldsymbol{y}(t) \in \mathbb{R}^\ell$ of the system. $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector and $\boldsymbol{x}_0$ is a consistent initial value. In this Section, we use (4.1.10) to decouple the DAE (6.0.1a) instead of (4.1.11). Thus, the projected system of (6.0.1a) is given by:

$$\mathbf{E}_\mu\Big[\mathbf{P}_{\mu-1}\cdots\mathbf{P}_0\dot{\boldsymbol{x}} + \mathbf{Q}_0\boldsymbol{x} + \cdots + \mathbf{Q}_{\mu-1}\boldsymbol{x}\Big] = \mathbf{A}_\mu\boldsymbol{x} + \mathbf{B}\boldsymbol{u}, \tag{6.0.2}$$

where μ is the tractability index the DAE (6.0.1a). This decoupling procedure is different that proposed by März [42] which involves the inversion of matrix $\mathbf{E}_\mu$ which might be computationally expensive and impractical for large scale problems. This decoupling procedure led to explicit differential and algebraic parts. If we use (6.0.2) as the starting projected system, this will lead to an implicit differential part and a linear system. The latter will be computational cheaper to decouple than the former since does not involve matrix inversion.

6.1 Index-1 DAEs

In this Section, we assume that (6.0.1a) is of index-1, i.e., $\mu = 1$. This implies that $\mathbf{E}_1$ must be a nonsingular matrix. Thus substituting $\mu = 1$ into (6.0.2) , we obtain:

$$\mathbf{E}_1\Big[\mathbf{P}_0\dot{\boldsymbol{x}} + \mathbf{Q}_0\boldsymbol{x}\Big] = \mathbf{A}_1\boldsymbol{x} + \mathbf{B}\boldsymbol{u}. \tag{6.1.1}$$

Recall from Section 5.2, the decomposition of $\boldsymbol{x}$ for index-1 DAEs is given by $\boldsymbol{x} = \begin{pmatrix}\boldsymbol{p} & \boldsymbol{q}\end{pmatrix}\begin{pmatrix}\xi_p \\ \xi_q\end{pmatrix}$, where $\boldsymbol{p} \in \mathbb{R}^{n\times n_p}$ and $\boldsymbol{q} \in \mathbb{R}^{n\times n_q}$ are the linearly independent columns of projectors $\mathbf{P}_0$ and $\mathbf{Q}_0$, respectively. $\xi_p \in \mathbb{R}^{n_p}$ and $\xi_q \in \mathbb{R}^{n_q}$ are the projected differential and algebraic variables, respectively and $n = n_p + n_q$ is the dimension of the DAE. Substituting the decomposed $\boldsymbol{x}$ into (6.1.1) and simplifying leads to,

$$\begin{pmatrix}\mathbf{E}_1\boldsymbol{p} & 0\end{pmatrix}\begin{pmatrix}\xi_p \\ \xi_q\end{pmatrix}' = \begin{pmatrix}\mathbf{A}_1\boldsymbol{p} & -\mathbf{E}_1\boldsymbol{q}\end{pmatrix}\begin{pmatrix}\xi_p \\ \xi_q\end{pmatrix} + \mathbf{B}\boldsymbol{u}. \tag{6.1.2}$$

Left multiplying (6.1.2) by $\begin{pmatrix}\hat{\boldsymbol{p}}^{\mathrm{T}} & \hat{\boldsymbol{q}}^{\mathrm{T}}\end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, we obtain

$$\begin{pmatrix}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_1\boldsymbol{p} & 0 \\ \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_1\boldsymbol{p} & 0\end{pmatrix}\begin{pmatrix}\xi_p \\ \xi_q\end{pmatrix}' = \begin{pmatrix}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{A}_1\boldsymbol{p} & -\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_1\boldsymbol{q} \\ \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{A}_1\boldsymbol{p} & -\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_1\boldsymbol{q}\end{pmatrix}\begin{pmatrix}\xi_p \\ \xi_q\end{pmatrix} + \begin{pmatrix}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{B} \\ \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{B}\end{pmatrix}\boldsymbol{u}. \tag{6.1.3}$$

In order to decouple (6.1.2), we need to also construct full column rank matrices $\hat{\boldsymbol{p}}^{\mathrm{T}} \in \mathbb{R}^{n_p\times n}$ and $\hat{\boldsymbol{q}}^{\mathrm{T}} \in \mathbb{R}^{n_q\times n}$ such that $\mathrm{Span}(\hat{\boldsymbol{p}}) = \mathrm{Ker}\,\boldsymbol{q}^{\mathrm{T}}\mathbf{E}_1^{\mathrm{T}}$ and $\mathrm{Span}(\hat{\boldsymbol{q}}) = \mathrm{Ker}\,\boldsymbol{p}^{\mathrm{T}}\mathbf{E}_1^{\mathrm{T}}$, that is, $\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_1\boldsymbol{q} = -\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{A}\boldsymbol{q} = \mathbf{0}$ and $\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_1\boldsymbol{p} = \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}\boldsymbol{p} = \mathbf{0}$. This implies that $\mathrm{Span}(\hat{\boldsymbol{p}}) = \mathrm{Ker}\,\boldsymbol{q}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ and $\mathrm{Span}(\hat{\boldsymbol{q}}) = \mathrm{Ker}\,\mathbf{E}^{\mathrm{T}}$. This is due to the fact that $\mathrm{Ker}\,\mathbf{E}^{\mathrm{T}} \subset \mathrm{Ker}\,\boldsymbol{p}^{\mathrm{T}}\mathbf{E}^{\mathrm{T}} \subset \mathbf{P}_0^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}$. We note that $\hat{\boldsymbol{q}} = \boldsymbol{q}$, if $\mathbf{E}$ is symmetric, i.e., $\mathbf{E} = \mathbf{E}^{\mathrm{T}}$. Thus, (6.1.3), simplifies to

$$\begin{pmatrix} \hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_1\boldsymbol{p} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \xi_p \\ \xi_q \end{pmatrix}' = \begin{pmatrix} \hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{A}_1\boldsymbol{p} & \mathbf{0} \\ \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{A}_1\boldsymbol{p} & -\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_1\boldsymbol{q} \end{pmatrix} \begin{pmatrix} \xi_p \\ \xi_q \end{pmatrix} + \begin{pmatrix} \hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{B} \\ \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{B} \end{pmatrix} \boldsymbol{u}. \tag{6.1.4}$$

Using the fact that $\mathbf{E}_1 = \mathbf{E} - \mathbf{A}\mathbf{Q}_0$ and $\mathbf{A}_1 = \mathbf{A}\mathbf{P}_0$, (6.1.4) simplifies to an implicit decoupled system of (6.0.1) given by

$$\mathbf{E}_p\xi_p' = \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u}, \tag{6.1.5a}$$

$$\mathbf{E}_q\xi_q = \mathbf{A}_q\xi_p + \mathbf{B}_q\boldsymbol{u}, \tag{6.1.5b}$$

$$\boldsymbol{y} = \mathbf{C}_p^{\mathrm{T}}\xi_p + \mathbf{C}_q^{\mathrm{T}}\xi_q, \tag{6.1.5c}$$

where
$\mathbf{E}_p = \hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}\boldsymbol{p}$, $\mathbf{A}_p = \hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{A}\boldsymbol{p} \in \mathbb{R}^{n_p \times n_p}$, $\mathbf{B}_p = \hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{B} \in \mathbb{R}^{n_p \times m}$, $\mathbf{E}_q = -\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{A}\boldsymbol{q} \in \mathbb{R}^{n_q \times n_q}$, $\mathbf{A}_q = \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{A}\boldsymbol{p} \in \mathbb{R}^{n_p \times n_q}$, $\mathbf{B}_q = \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{B} \in \mathbb{R}^{n_q \times m}$ and $\mathbf{C}_p = \boldsymbol{p}^{\mathrm{T}}\mathbf{C} \in \mathbb{R}^{n_p,\ell}$, $\mathbf{C}_q = \boldsymbol{q}^{\mathrm{T}}\mathbf{C} \in \mathbb{R}^{n_q,\ell}$. We note that matrices $\mathbf{E}_p$ and $\mathbf{E}_q$ must be nonsingular. We can observe that (6.1.5) is an implicit version of the decoupled system (5.2.4) and their solutions must coincide. However (6.1.5) is computationally cheaper to derive than (5.2.4). We also note that, it can be proved that $\sigma(\mathbf{E}_p, \mathbf{A}_p) = \sigma_f(\mathbf{E}, \mathbf{A})$. Thus, the implicit system also preserves the dimension and the stability of the DAE.

Example 6.1.1 In this example, we use matrices from Example 5.2.1. Using (5.2.5) and (4.1.18), we can construct $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{q}}$ such that $\boldsymbol{q}^{\mathrm{T}}\mathbf{A}\hat{\boldsymbol{p}} = \mathbf{0}$ and $\boldsymbol{p}^{\mathrm{T}}\mathbf{E}\hat{\boldsymbol{q}} = \mathbf{0}$ given by

$$\hat{\boldsymbol{p}}^{\mathrm{T}} = \begin{pmatrix} \mathrm{I} & -(\mathbf{A}_{12} - \mathbf{A}_{11}\mathbf{Q}_{12})(\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{Q}_{12})^{-1} \end{pmatrix} \quad \text{and} \quad \hat{\boldsymbol{q}}^{\mathrm{T}} = \begin{pmatrix} \mathrm{I} & 0 \end{pmatrix}, \tag{6.1.6}$$

where $\mathbf{Q}_{12} = \mathbf{E}_{11}^{-1}\mathbf{E}_{12}$. Substituting (4.1.18)-(4.1.20), (5.2.5) and (6.1.6) into (6.1.5), we obtain the coefficients of implicit decoupled system given by
$\mathbf{E}_p = \hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}\boldsymbol{p} = \mathbf{E}_{11}$, $\mathbf{A}_p = \hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{A}\boldsymbol{p} = (\mathbf{A}_{11} - (\mathbf{A}_{12} - \mathbf{A}_{11}\mathbf{Q}_{12})(\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{Q}_{12})^{-1}\mathbf{A}_{21})$, $\mathbf{E}_q = -\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{A}\boldsymbol{q} = (\mathbf{A}_{21}\mathbf{Q}_{12} - \mathbf{A}_{22})$, $\mathbf{A}_q = \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{A}\boldsymbol{p} = \mathbf{A}_{21}$, $\mathbf{B}_q = \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{B} = \mathbf{B}_2$, $\mathbf{C}_p = \boldsymbol{p}^{\mathrm{T}}\mathbf{C} = \mathbf{C}_1$ and $\mathbf{C}_q = \boldsymbol{q}^{\mathrm{T}}\mathbf{C} = \mathbf{C}_2 - \mathbf{Q}_{12}^{\mathrm{T}}\mathbf{C}_1$. Hence the implicit decoupled system of (4.1.18) is given by

$$\begin{aligned} \mathbf{E}_{11}\xi_p' = {} & \left[\mathbf{A}_{11} - (\mathbf{A}_{12} - \mathbf{A}_{11}\mathbf{Q}_{12})(\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{Q}_{12})^{-1}\mathbf{A}_{21}\right]\xi_p \\ & + \left[\mathbf{B}_1 - (\mathbf{A}_{12} - \mathbf{A}_{11}\mathbf{Q}_{12})(\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{Q}_{12})^{-1}\mathbf{B}_2\right]\boldsymbol{u}, \end{aligned}$$

$$(\mathbf{A}_{21}\mathbf{Q}_{12} - \mathbf{A}_{22})\xi_q = \mathbf{A}_{21}\xi_p + \mathbf{B}_2\boldsymbol{u},$$
$$\boldsymbol{y} = \mathbf{C}_1^{\mathrm{T}}\xi_p + (\mathbf{C}_2^{\mathrm{T}} - \mathbf{C}_1^{\mathrm{T}}\mathbf{Q}_{12})\xi_q.$$

We can observe that this decoupled system and the explicit decoupled system derived in Example 5.2.1 lead to the same solutions.

6.2 Index-2 DAEs

In this Section, we assume that (6.0.1a) is an index-2 DAE, i.e., $\mu = 2$. Thus the matrix chain $\mathbf{E}_2$ must be nonsingular. Substituting $\mu = 2$ into (6.0.2) , we obtain

$$\mathbf{E}_2\Big[\mathbf{P}_1\mathbf{P}_0\dot{\boldsymbol{x}} + \mathbf{Q}_0\boldsymbol{x} + \mathbf{Q}_1\boldsymbol{x}\Big] = \mathbf{A}_2\mathbf{P}_0\mathbf{P}_1\boldsymbol{x} + \mathbf{B}\boldsymbol{u}. \tag{6.2.1}$$

Projector $\mathbf{Q}_1$ is chosen such that $\operatorname{Im}\mathbf{Q}_1 = \operatorname{Ker}\mathbf{E}_1$, and $\mathbf{P}_1 = \mathrm{I} - \mathbf{Q}_1$ is its complementary projector. However, in order to ensure that the projector products are also projectors, we assume that projectors $\mathbf{Q}_0$ and $\mathbf{Q}_1$ are constructed such that $\mathbf{Q}_1\mathbf{Q}_0 = 0$ holds true. In the previous Chapter, we discussed that for higher index DAEs there is a possibility of obtaining a decoupled system with either a differential part or without a differential part depending on the nature of the spectrum of the matrix pencil. Thus for the case of index-2 DAEs, we shall consider two cases as follows:

6.2.1 Index-2 DAEs with a differential part

Assume that the matrix pencil $(\mathbf{E}, \mathbf{A})$ of (6.0.1a) has at least one finite eigenvalue. Thus using (5.3.11), we can introduce a decomposition

$$\boldsymbol{x} = \begin{pmatrix}\boldsymbol{pz}_{\boldsymbol{p}_0} & \boldsymbol{pz}_{\boldsymbol{q}_0} & \boldsymbol{q}\end{pmatrix}\begin{pmatrix}\xi_p \\ \xi_{q,1} \\ \xi_{q,0}\end{pmatrix} \in \mathbb{R}^{n\times n},$$

where $\xi_p \in \mathbb{R}^{n_p}$, $\xi_{q,1} \in \mathbb{R}^{k_1}$, $\xi_{q,0} \in \mathbb{R}^{k_0}$ and $n = n_p + k_1 + k_0$. Substituting the decomposed $\boldsymbol{x}$ into (6.2.1), we obtain,

$$\begin{pmatrix}\mathbf{E}_2 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} & -\mathbf{E}_2 \mathbf{Q}_0 \mathbf{Q}_1 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{q}_0} & 0\end{pmatrix}\begin{pmatrix}\xi_{\boldsymbol{p}_0} \\ \xi_{\boldsymbol{q}_0} \\ \xi_q\end{pmatrix}' = \begin{pmatrix}\mathbf{A}_2 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} & -\mathbf{E}_2 \mathbf{Q}_1 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{q}_0} & -\mathbf{E}_2 \boldsymbol{q}\end{pmatrix}\begin{pmatrix}\xi_p \\ \xi_{q,1} \\ \xi_{q,0}\end{pmatrix} + \mathbf{B}\boldsymbol{u}. \tag{6.2.2}$$

In order to decouple (6.2.2), we first introduce $\hat{\boldsymbol{p}}^{\mathrm{T}} \in \mathbb{R}^{n_0 \times n}$, $\hat{\boldsymbol{q}}^{\mathrm{T}} \in \mathbb{R}^{k_0 \times n}$, where $n_0 = n_p + k_1$, such that $\hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{E}_2 \boldsymbol{q} = \mathbf{0}$ and $\hat{\boldsymbol{q}}^{\mathrm{T}} \mathbf{E}_2 \boldsymbol{p} = \mathbf{0}$. This implies that $\mathrm{Span}(\hat{\boldsymbol{p}}) = \mathrm{Ker}\, \boldsymbol{q}^{\mathrm{T}} \mathbf{E}_2^{\mathrm{T}}$ and $\mathrm{Span}(\hat{\boldsymbol{q}}) = \mathrm{Ker}\, \boldsymbol{p}^{\mathrm{T}} \mathbf{E}_2^{\mathrm{T}}$. We then construct $\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} \in \mathbb{R}^{k_1, n_0}$, $\hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}} \in \mathbb{R}^{k_0 \times n_0}$, such that $\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{E}_2 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{q}_0} = \mathbf{0}$, $\hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{E}_2 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} = \mathbf{0}$. This implies that $\mathrm{Span}\, \hat{\boldsymbol{z}}_{\boldsymbol{p}_0} = \mathrm{Ker}\, (\hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{E}_2 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{q}_0})^{\mathrm{T}}$ and $\mathrm{Span}\, \hat{\boldsymbol{z}}_{\boldsymbol{q}_0} = \mathrm{Ker}\, (\hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{E}_2 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0})^{\mathrm{T}}$. Multiplying (6.2.2) by $\begin{pmatrix}\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} & \hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} & \hat{\boldsymbol{q}}^{\mathrm{T}}\end{pmatrix}^{\mathrm{T}}$ and simplifying, we obtain

$$\begin{pmatrix}\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{E}_2 \boldsymbol{p} z_{\boldsymbol{p}_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\hat{\boldsymbol{q}}^{\mathrm{T}} \mathbf{E}_2 \mathbf{Q}_1 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{q}_0} & 0\end{pmatrix}\begin{pmatrix}\xi_p \\ \xi_{q,1} \\ \xi_{q,0}\end{pmatrix}' = \begin{pmatrix}\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{A}_2 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} & 0 & 0 \\ \hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{A}_2 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} & -\hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{E}_2 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{q}_0} & 0 \\ \hat{\boldsymbol{q}}^{\mathrm{T}} \mathbf{A}_2 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} & -\hat{\boldsymbol{q}}^{\mathrm{T}} \mathbf{E}_2 \mathbf{Q}_1 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{q}_0} & -\hat{\boldsymbol{q}}^{\mathrm{T}} \mathbf{E}_2 \boldsymbol{q}\end{pmatrix}\begin{pmatrix}\xi_p \\ \xi_{q,1} \\ \xi_{q,0}\end{pmatrix} + \begin{pmatrix}\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{B} \\ \hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{B} \\ \hat{\boldsymbol{q}}^{\mathrm{T}} \mathbf{B}\end{pmatrix}\boldsymbol{u}.$$

From the above system, without loss of generality the implicit decoupled system of (6.0.1) is given by

$$\mathbf{E}_p \xi_p' = \mathbf{A}_p \xi_p + \mathbf{B}_p \boldsymbol{u}, \tag{6.2.3a}$$

$$\mathbf{E}_{q,1} \xi_{q,1} = \mathbf{A}_{q,1} \xi_p + \mathbf{B}_{q,1} \boldsymbol{u}, \tag{6.2.3b}$$

$$\mathbf{E}_{q,0} \xi_{q,0} = \mathbf{A}_{q,0} \xi_p + \mathbf{B}_{q,0} \boldsymbol{u} + \mathbf{A}_{\boldsymbol{q}_{0,1}} \left[\xi_{q,1}' - \xi_{q,1}\right], \tag{6.2.3c}$$

$$\boldsymbol{y} = \mathbf{C}_p^{\mathrm{T}} \xi_p + \mathbf{C}_{q,1}^{\mathrm{T}} \xi_{q,1} + \mathbf{C}_{q,0}^{\mathrm{T}} \xi_{q,0} \tag{6.2.3d}$$

where

$$\begin{aligned}
&\mathbf{E}_p = \hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{E} \boldsymbol{p} z_{\boldsymbol{p}_0} \in \mathbb{R}^{n_p \times n_p}, && \mathbf{A}_p = \hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{A} \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} \in \mathbb{R}^{n_p \times n_p}, && \mathbf{B}_p = \hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{B} \in \mathbb{R}^{n_p \times m}, \\
&\mathbf{E}_{q,1} = \hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{E}_2 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{k_1 \times k_1}, && \mathbf{A}_{q,1} = \hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{A} \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} \in \mathbb{R}^{k_1 \times n_p}, && \mathbf{B}_{q,1} = \hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{B} \in \mathbb{R}^{k_1 \times m}, \\
&\mathbf{E}_{q,0} = -\hat{\boldsymbol{q}}^{\mathrm{T}} \mathbf{A} \boldsymbol{q}_0 \in \mathbb{R}^{k_0 \times k_0}, && \mathbf{A}_{q,0} = \hat{\boldsymbol{q}}^{\mathrm{T}} \mathbf{A} \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} \in \mathbb{R}^{k_0 \times n_p}, && \mathbf{B}_{q,0} = \hat{\boldsymbol{q}}^{\mathrm{T}} \mathbf{B} \in \mathbb{R}^{k_0 \times m}, \\
&\mathbf{A}_{\boldsymbol{q}_{0,1}} = -\hat{\boldsymbol{q}}^{\mathrm{T}} \mathbf{A} \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{k_0 \times k_1}, && \mathbf{C}_p = z_{\boldsymbol{p}_0}^{\mathrm{T}} \boldsymbol{p}^{\mathrm{T}} \mathbf{C} \in \mathbb{R}^{n_p \times \ell}, && \mathbf{C}_{q,1} = z_{\boldsymbol{q}_0}^{\mathrm{T}} \boldsymbol{p}^{\mathrm{T}} \mathbf{C} \in \mathbb{R}^{k_1 \times \ell} \\
&\text{and} \quad \mathbf{C}_{q,0} = \boldsymbol{q}^{\mathrm{T}} \mathbf{C} \in \mathbb{R}^{k_0 \times \ell}.
\end{aligned}$$

We note that matrices $\mathbf{E}_p$, $\mathbf{E}_{q,1}$ and $\mathbf{E}_{q,0}$ must be nonsingular. We can observe that (6.2.3) is an implicit version of (5.3.15) and their solutions must be coincide. This decoupled system also preserves the dimension and the stability of the DAE as its counterpart. Since, we can observe that (6.2.3a), (6.2.3b) and (6.2.3c) are of dimension n_p, k_1 and k_0, respectively and $n = n_p + k_1 + k_0$ is the dimension of the DAE system (6.0.1a). n_p and $k_1 + k_0$ are the dimensions of the differential and algebraic parts, respectively.

6.2.2 Index-2 DAEs without a differential part

Here, we assume that the matrix pencil $(\mathbf{E}, \mathbf{A})$ of (6.0.1a) has no finite eigenvalues, i.e., $\sigma_f(\mathbf{E}, \mathbf{A}) = \emptyset$. This implies that $\mathbf{P}_0\mathbf{P}_1 = 0$, thus (6.2.1) simplifies to,

$$\mathbf{E}_2\Big[\mathbf{P}_1\mathbf{P}_0\dot{\boldsymbol{x}} + \mathbf{Q}_0\boldsymbol{x} + \mathbf{Q}_1\boldsymbol{x}\Big] = \mathbf{B}\boldsymbol{u}. \tag{6.2.4}$$

Recall from Section 5.3.2 that the state-space of index-2 DAE without differential part can decompose $\boldsymbol{x}$ as, $\boldsymbol{x} = \begin{pmatrix}\boldsymbol{p} & \boldsymbol{q}\end{pmatrix}\begin{pmatrix}\xi_{q,1}\\ \xi_{q,0}\end{pmatrix}$, $\xi_{q,1} \in \mathbb{R}^{k_1}$, $\xi_{q,0} \in \mathbb{R}^{k_0}$ and this time $n = k_1 + k_0$ is the dimension of the DAE (6.0.1a). Substituting $\boldsymbol{x}$ into (6.2.4) and simplifying , we obtain:

$$\begin{pmatrix}\mathbf{E}_2\mathbf{P}_1\boldsymbol{p} & 0\end{pmatrix}\begin{pmatrix}\xi_{q,1}\\ \xi_{q,0}\end{pmatrix}' = -\begin{pmatrix}\mathbf{E}_2\mathbf{Q}_1\boldsymbol{p} & \mathbf{E}_2\boldsymbol{q}\end{pmatrix}\begin{pmatrix}\xi_{q,1}\\ \xi_{q,0}\end{pmatrix} + \mathbf{B}\boldsymbol{u}. \tag{6.2.5}$$

In order to decouple (6.2.5), we introduce $\hat{\boldsymbol{p}}^{\mathrm{T}}$, $\hat{\boldsymbol{q}}^{\mathrm{T}}$, such that $\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_2\boldsymbol{q}_0 = 0$, $\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_2\mathbf{Q}_1\boldsymbol{p} = 0$. This implies that Span $\hat{\boldsymbol{p}} = \operatorname{Ker}\boldsymbol{q}^{\mathrm{T}}\mathbf{E}_2^{\mathrm{T}}$ and Span $\hat{\boldsymbol{q}} = \operatorname{Ker}\boldsymbol{p}^{\mathrm{T}}\mathbf{Q}_1^{\mathrm{T}}\mathbf{E}_2^{\mathrm{T}}$. Multiplying (6.2.2) by $\begin{pmatrix}\hat{\boldsymbol{p}}^{\mathrm{T}} & \hat{\boldsymbol{q}}^{\mathrm{T}}\end{pmatrix}^{\mathrm{T}}$, we obtain:

$$\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_2\mathbf{P}_1\boldsymbol{p} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\xi_{q,1}\\ \xi_{q,0}\end{pmatrix}' + \begin{pmatrix}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_2\mathbf{Q}_1\boldsymbol{p} & \mathbf{0}\\ \mathbf{0} & \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_2\boldsymbol{q}\end{pmatrix}\begin{pmatrix}\xi_{q,1}\\ \xi_{q,0}\end{pmatrix} = \begin{pmatrix}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{B}\\ \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{B}\end{pmatrix}\boldsymbol{u}.$$

Thus, from above system, if (6.0.1) is of index-2 and the spectrum of its matrix pencil $(\mathbf{E}, \mathbf{A})$ has only infinite eigenvalues, it can be decoupled into the form

$$\begin{aligned}\mathbf{E}_{q,1}\xi_{q,1} &= \mathbf{B}_{q,1}\boldsymbol{u},\\ \mathbf{E}_{q,0}\xi_{q,0} &= \mathbf{B}_{q,0}\boldsymbol{u} + \mathbf{A}_{\boldsymbol{q}_{0,1}}\xi_{q,1}',\\ \boldsymbol{y} &= \mathbf{C}_{q,1}^{\mathrm{T}}\,\xi_{q,1} + \mathbf{C}_{q,0}^{\mathrm{T}}\,\xi_{q,0},\end{aligned} \tag{6.2.6}$$

where $\mathbf{E}_{q,1} = \hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_2\mathbf{Q}_1\boldsymbol{p} \in \mathbb{R}^{k_1\times k_1}$, $\mathbf{B}_{q,1} = \hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{B} \in \mathbb{R}^{k_1\times m}$, $\mathbf{E}_{q,0} = \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_2\boldsymbol{q} \in \mathbb{R}^{k_0\times k_0}$, $\mathbf{B}_{q,0} = \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{B} \in \mathbb{R}^{k_0\times m}$, $\mathbf{A}_{\boldsymbol{q}_{0,1}} = -\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_2\mathbf{P}_1\boldsymbol{p} \in \mathbb{R}^{k_0\times k_1}$, $\mathbf{C}_{q,1} = \boldsymbol{p}^{\mathrm{T}}\mathbf{C} \in \mathbb{R}^{k_1\times \ell}$ and $\mathbf{C}_{q,0} = \boldsymbol{q}^{\mathrm{T}}\mathbf{C} \in \mathbb{R}^{k_0\times \ell}$. We can observe that $n = k_0 + k_1$ is the dimension of the DAE.

Hence index-2 DAEs can be decoupled in two ways depending on the spectrum of the matrix pencil. This is also illustrated in Example 6.2.1 and 6.2.2.

Example 6.2.1 In this example, we use the same matrices from Example 5.3.1. This DAE has matrix pencil whose spectrum as one finite eigenvalue and it is of index-2. Thus, its decoupled system must have a differential part. We can then use the procedure in Section 6.2.1 to derive its implicit decoupled system. We used the procedure in section 6.2.1 to construct bases $\hat{\boldsymbol{p}}$, $\hat{\boldsymbol{q}}$ and $\hat{z}_{\boldsymbol{p}_0}$, $\hat{z}_{\boldsymbol{q}_0}$ and obtained

$$\hat{\boldsymbol{p}} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\boldsymbol{q}} = \begin{pmatrix} 0 & \frac{1}{J_1} \\ 0 & -\frac{1}{J_2} \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \hat{z}_{\boldsymbol{p}_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{z}_{\boldsymbol{q}_0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{6.2.7}$$

Substituting matrices from Example 5.3.1 and (6.2.7) into (6.2.3) we obtain an implicit decoupled system with system matrices: $\mathbf{E}_p = J_1 + J_2$, $\mathbf{A}_p = 0$, $\mathbf{B}_p = \begin{pmatrix} 1 & 1 \end{pmatrix}$, $\mathbf{E}_{q,1} = -J_1 - J_2$, $\mathbf{A}_{q,1} = 0$, $\mathbf{B}_{q,1} = \begin{pmatrix} 0 & 0 \end{pmatrix}$, $\mathbf{E}_{q,0} = \begin{pmatrix} 1 & 1 \\ -\frac{1}{J_1} & \frac{1}{J_2} \end{pmatrix}$, $\mathbf{A}_{q,0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{B}_{q,0} = \begin{pmatrix} 0 & 0 \\ \frac{1}{J_1} & -\frac{1}{J_2} \end{pmatrix}$, $\mathbf{A}_{\boldsymbol{q}_{0,1}} = \begin{pmatrix} 0 \\ J_1 + J_2 \end{pmatrix}$, $\mathbf{C}_p = \frac{1}{J_1+J_2} \begin{pmatrix} J_1 & J_2 \end{pmatrix}^{\mathrm{T}}$, $\mathbf{C}_{q,1} = \frac{1}{J_1+J_2} \begin{pmatrix} 1 & -1 \end{pmatrix}^{\mathrm{T}}$, $\mathbf{C}_{q,0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}^{\mathrm{T}}$. We can observe that if we simplify this implicit decoupled system, it reduces to an ODE given by

$$\begin{aligned} (J_1 + J_2)\xi_p' &= \begin{pmatrix} 1 & 1 \end{pmatrix} \boldsymbol{u}, \\ \boldsymbol{y} &= \frac{1}{J_1 + J_2} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} \xi_p. \end{aligned} \tag{6.2.8}$$

We can observe that solutions of (5.3.20) and (6.2.8) coincide. We can also observe that (6.2.8) is an implicit version of (5.3.20).

Example 6.2.2 In this example, we use system matrices from Example 5.3.2. This DAE has matrix pencil whose spectrum has no finite eigenvalues and it is of index-2. Thus, its decoupled system has no differential part. Using matrices from Example 5.3.2, we constructed $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{q}}$ using the derived procedure for implicit decoupling of index-2 system without a differential part and obtained : $\hat{\boldsymbol{p}} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\hat{\boldsymbol{q}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$. Substituting the column matrices and those from Example 5.3.2 into (6.2.6), we obtain an implicit decoupled system without a differential part with system matrices given by

$$\mathbf{E}_{q,1} = 1,\ \mathbf{B}_{q,1} = 1,\ \mathbf{E}_{q,0} = \begin{pmatrix} G & -G \\ 0 & -1 \end{pmatrix},\ \mathbf{B}_{q,0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \mathbf{A}_{\boldsymbol{q}_{0,1}} = \begin{pmatrix} 0 \\ -L \end{pmatrix},\ \mathbf{C}_{q,1} = 1,\ \mathbf{C}_{q,0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solving the system lead to an output solution $\boldsymbol{y} = (G^{-1} + 1)\boldsymbol{u} + 2L\boldsymbol{u}'$ which coincides with that in Example 5.3.2.

6.3 Index-3 DAEs

In this Section, we assume that (6.0.1a) is an index-3 DAE, i.e., $\mu = 3$. Thus the matrix chain $\mathbf{E}_3$ must be nonsingular. Substituting $\mu = 3$ into (6.0.2), we obtain

$$\mathbf{E}_3\Big[\mathbf{P}_2\mathbf{P}_1\mathbf{P}_0\boldsymbol{x}' + \mathbf{Q}_2\boldsymbol{x} + \mathbf{Q}_1\boldsymbol{x} + \mathbf{Q}_0\boldsymbol{x}\Big] = \mathbf{A}_3\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2\boldsymbol{x} + \mathbf{B}\boldsymbol{u}. \tag{6.3.1}$$

Projectors $\mathbf{Q}_i$ is chosen such that $\operatorname{Im}\mathbf{Q}_i = \operatorname{Ker}\mathbf{E}_i$ and $\mathbf{P}_i = \mathrm{I} - \mathbf{Q}_i$, $i = 0, 1, 2$ is its complementary projectors. Also for this case , we assume that $\mathbf{Q}_j\mathbf{Q}_i = 0,\ j > i$ holds true. For the case of index-3 DAEs we can also have two possibilities depending on the spectrum of the matrix pencil $(\mathbf{E}, \mathbf{A})$ as follows:

6.3.1 Index-3 DAEs with a differential part

Assume that the matrix pencil $(\mathbf{E}, \mathbf{A})$ of (6.0.1a) has at least one finite eigenvalues, i.e, $\sigma_f(\mathbf{E}, \mathbf{A}) \neq \emptyset$. Thus from Section 5.4.3 for the case of index-3 DAE with differential part, $\boldsymbol{x}$ can be decomposed as

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{p}_1} & \boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{q}_1} & \boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0} & \boldsymbol{q} \end{pmatrix} \begin{pmatrix} \xi_p \\ \xi_{q,2} \\ \xi_{q,1} \\ \xi_{q,0} \end{pmatrix},\ \xi_p \in \mathbb{R}^{n_p},\ \xi_{q,2} \in \mathbb{R}^{k_2},\ \xi_{q,1} \in \mathbb{R}^{k_1},\ \xi_{q,0} \in \mathbb{R}^{k_0}.$$

Substituting $\boldsymbol{x}$ into (6.3.1) and simplifying, we obtain

$$\begin{pmatrix}\mathbf{E}_3\mathbf{P}_2\mathbf{P}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{p}_1} & \mathbf{E}_3\mathbf{P}_2\mathbf{P}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{q}_1} & -\mathbf{E}_3\mathbf{Q}_0\mathbf{Q}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0} & 0\end{pmatrix}\begin{pmatrix}\xi_p\\ \xi_{q,2}\\ \xi_{q,1}\\ \xi_{q,0}\end{pmatrix}'$$

$$= \begin{pmatrix}0 & \mathbf{A}_3\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{p}_1} & -\mathbf{E}_3\mathbf{Q}_2\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{q}_1} & -\mathbf{E}_3\mathbf{Q}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0} & -\mathbf{E}_3\boldsymbol{q}_0\end{pmatrix}\begin{pmatrix}\xi_p\\ \xi_{q,2}\\ \xi_{q,1}\\ \xi_{q,0}\end{pmatrix} + \mathbf{B}\boldsymbol{u} \tag{6.3.2}$$

In order to decouple (6.3.2), we construct full column matrices in three steps below. (i) First construct $\hat{\boldsymbol{p}}^{\mathrm{T}} \in \mathbb{R}^{n_0\times n}$, $\hat{\boldsymbol{q}}^{\mathrm{T}} \in \mathbb{R}^{k_0\times n}$ such that $\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\boldsymbol{q} = 0$ and $\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_3\boldsymbol{p} = 0$. This implies that $\mathrm{Span}(\hat{\boldsymbol{p}}) = \mathrm{Ker}\,(\mathbf{E}_3\boldsymbol{q})^{\mathrm{T}}$ and $\mathrm{Span}(\hat{\boldsymbol{q}}) = \mathrm{Ker}\,(\mathbf{E}_3\boldsymbol{p}_0)^{\mathrm{T}} = 0$. (ii) Next,we construct column matrices $\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} \in \mathbb{R}^{n_1\times n_0}$, $\hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}} \in \mathbb{R}^{k_1\times n_0}$ such that $\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0} = 0$ and $\hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0} = 0$. This implies that $\mathrm{Span}(\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}) = \mathrm{Ker}\,(\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0})^{\mathrm{T}}$ and $\mathrm{Span}(\hat{\boldsymbol{z}}_{\boldsymbol{q}_0}) = \mathrm{Ker}\,(\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0})^{\mathrm{T}}$. (iii) Finally, we construct column matrices $\hat{\boldsymbol{z}}_{\boldsymbol{p}_1}^{\mathrm{T}} \in \mathbb{R}^{n_p\times n_1}$, $\hat{\boldsymbol{z}}_{\boldsymbol{q}_1}^{\mathrm{T}} \in \mathbb{R}^{k_2\times n_1}$ such that $\hat{\boldsymbol{z}}_{\boldsymbol{p}_1}^{\mathrm{T}}\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{q}_1} = 0$ and $\hat{\boldsymbol{z}}_{\boldsymbol{q}_1}^{\mathrm{T}}\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{p}_1} = 0$. This also implies that $\mathrm{Span}(\hat{\boldsymbol{z}}_{\boldsymbol{p}_1}) = \mathrm{Ker}\,(\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{q}_1})^{\mathrm{T}}$ and $\mathrm{Span}(\hat{\boldsymbol{z}}_{\boldsymbol{q}_1}) = \mathrm{Ker}\,(\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{p}_1})^{\mathrm{T}}$. Thus left multiplying (6.3.2) by $\begin{pmatrix}\hat{\boldsymbol{z}}_{\boldsymbol{p}_1}^{\mathrm{T}}\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}} & \hat{\boldsymbol{z}}_{\boldsymbol{q}_1}^{\mathrm{T}}\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}} & \hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}} & \hat{\boldsymbol{q}}^{\mathrm{T}}\end{pmatrix}^{\mathrm{T}}$ and simplifying leads to

$$\begin{pmatrix}\hat{\boldsymbol{z}}_{\boldsymbol{p}_1}^{\mathrm{T}}\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\mathbf{P}_2\mathbf{P}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{p}_1} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & \hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\mathbf{P}_2\mathbf{P}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{q}_1} & 0 & 0\\ 0 & \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_3\mathbf{P}_2\mathbf{P}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{q}_1} & -\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_0\mathbf{Q}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0} & 0\end{pmatrix}\begin{pmatrix}\xi_p\\ \xi_{q,2}\\ \xi_{q,1}\\ \xi_{q,0}\end{pmatrix}' =$$

$$\begin{pmatrix}\hat{\boldsymbol{z}}_{\boldsymbol{p}_1}^{\mathrm{T}}\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{A}_3\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{p}_1} & 0 & 0 & 0\\ \hat{\boldsymbol{z}}_{\boldsymbol{q}_1}^{\mathrm{T}}\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{A}_3\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{p}_1} & -\hat{\boldsymbol{z}}_{\boldsymbol{q}_1}^{\mathrm{T}}\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_2\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{q}_1} & 0 & 0\\ \hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{A}_3\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{p}_1} & -\hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_2\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{q}_1} & -\hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0} & 0\\ \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{A}_3\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{p}_1} & -\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_2\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0}\boldsymbol{z}_{\boldsymbol{q}_1} & -\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0} & -\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_3\boldsymbol{q}_0\end{pmatrix}\begin{pmatrix}\xi_p\\ \xi_{q,2}\\ \xi_{q,1}\\ \xi_{q,0}\end{pmatrix} + \begin{pmatrix}\hat{\boldsymbol{z}}_{\boldsymbol{p}_1}^{\mathrm{T}}\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{B}\\ \hat{\boldsymbol{z}}_{\boldsymbol{q}_1}^{\mathrm{T}}\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{B}\\ \hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{B}\\ \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{B}\end{pmatrix}\boldsymbol{u}.$$

Using the above system, without loss of generality the implicit decoupled system of (6.0.1) is given by

$$\begin{aligned}
\mathbf{E}_p\xi_p' &= \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u},\\
\mathbf{E}_{q,2}\xi_{q,2} &= \mathbf{A}_{q,2}\xi_p + \mathbf{B}_{q,2}\boldsymbol{u},\\
\mathbf{E}_{q,1}\xi_{q,1} &= \mathbf{A}_{q,1}\xi_p + \mathbf{B}_{q,1}\boldsymbol{u} + \mathbf{E}_{\boldsymbol{q}_{1,2}}\xi_{q,2} + \mathbf{A}_{\boldsymbol{q}_{1,2}}\xi_{q,2}',\\
\mathbf{E}_{q,0}\xi_{q,0} &= \mathbf{A}_{q,0}\xi_p + \mathbf{B}_{q,0}\boldsymbol{u} + \mathbf{E}_{\boldsymbol{q}_{0,2}}\xi_{q,2} + \mathbf{E}_{\boldsymbol{q}_{0,1}}\xi_{q,1} + \mathbf{A}_{\boldsymbol{q}_{0,2}}\xi_{q,2}' + \mathbf{A}_{\boldsymbol{q}_{0,1}}\xi_{q,1}',\\
\boldsymbol{y} &= \mathbf{C}_p^{\mathrm{T}}\xi_p + \mathbf{C}_{q,2}^{\mathrm{T}}\xi_{q,2} + \mathbf{C}_{q,1}^{\mathrm{T}}\xi_{q,1} + \mathbf{C}_{q,0}^{\mathrm{T}}\xi_{q,0},
\end{aligned} \tag{6.3.3}$$

where

$$\begin{aligned}
&\mathbf{E}_p = \hat{\boldsymbol{z}}_{\boldsymbol{p}_1}^{\mathrm{T}} \hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{E}_3 \mathbf{P}_2 \mathbf{P}_1 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{p}_1} \in \mathbb{R}^{n_p \times n_p}, && \mathbf{A}_p = \hat{\boldsymbol{z}}_{\boldsymbol{p}_1}^{\mathrm{T}} \hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{A}_3 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{p}_1} \in \mathbb{R}^{n_p \times n_p}, && \mathbf{B}_p = \hat{\boldsymbol{z}}_{\boldsymbol{p}_1}^{\mathrm{T}} \hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{B} \in \mathbb{R}^{n_p \times m}, \\
&\mathbf{E}_{q,2} = \hat{\boldsymbol{z}}_{\boldsymbol{q}_1}^{\mathrm{T}} \hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{E}_3 \mathbf{Q}_2 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{q}_1} \in \mathbb{R}^{k_2 \times k_2}, && \mathbf{A}_{q,2} = \hat{\boldsymbol{z}}_{\boldsymbol{q}_1}^{\mathrm{T}} \hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{A}_3 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{p}_1} \in \mathbb{R}^{k_2 \times n_p}, && \mathbf{B}_{q,2} = \hat{\boldsymbol{z}}_{\boldsymbol{q}_1}^{\mathrm{T}} \hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{B} \in \mathbb{R}^{k_2 \times m}, \\
&\mathbf{E}_{q,1} = \hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{E}_3 \mathbf{Q}_1 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{k_1 \times k_1}, && \mathbf{A}_{q,1} = \hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{A}_3 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{p}_1} \in \mathbb{R}^{k_1 \times n_p}, && \mathbf{B}_{q,1} = \hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{B} \in \mathbb{R}^{k_1 \times m}, \\
&\mathbf{E}_{\boldsymbol{q}_{1,2}} = -\hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{E}_3 \mathbf{Q}_2 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{q}_1} \in \mathbb{R}^{k_1 \times k_2}, && \mathbf{A}_{\boldsymbol{q}_{1,2}} = -\hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}} \hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{E}_3 \mathbf{P}_2 \mathbf{P}_1 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{q}_1} \in \mathbb{R}^{k_1 \times k_2}, && \mathbf{E}_{q,0} = \hat{\boldsymbol{q}}^{\mathrm{T}} \mathbf{E}_3 \boldsymbol{q} \in \mathbb{R}^{k_0 \times k_0}, \\
&\mathbf{A}_{q,0} = \hat{\boldsymbol{q}}^{\mathrm{T}} \mathbf{A}_3 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{p}_1} \in \mathbb{R}^{k_0 \times n_p}, && \mathbf{B}_{q,0} = \hat{\boldsymbol{q}}^{\mathrm{T}} \mathbf{B} \in \mathbb{R}^{k_0 \times m}, && \mathbf{E}_{\boldsymbol{q}_{0,2}} = -\hat{\boldsymbol{q}}^{\mathrm{T}} \mathbf{E}_3 \mathbf{Q}_2 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{q}_1} \in \mathbb{R}^{k_0 \times k_2}, \\
&\mathbf{E}_{\boldsymbol{q}_{0,1}} = -\hat{\boldsymbol{q}}^{\mathrm{T}} \mathbf{E}_3 \mathbf{Q}_1 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{k_0 \times k_1}, && \mathbf{A}_{\boldsymbol{q}_{0,2}} = -\hat{\boldsymbol{q}}^{\mathrm{T}} \mathbf{E}_3 \mathbf{P}_2 \mathbf{P}_1 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{q}_1} \in \mathbb{R}^{k_0 \times k_2}, && \mathbf{A}_{\boldsymbol{q}_{0,1}} = \hat{\boldsymbol{q}}^{\mathrm{T}} \mathbf{E}_3 \mathbf{Q}_0 \mathbf{Q}_1 \boldsymbol{p} \boldsymbol{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{k_0 \times k_1}, \\
&\mathbf{C}_{q,2}^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{p}_0} \boldsymbol{z}_{\boldsymbol{q}_1} \in \mathbb{R}^{k_2 \times \ell}, && \mathbf{C}_{q,1}^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}} \boldsymbol{p}_0 \boldsymbol{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{k_1 \times \ell}, && \mathbf{C}_{q,0}^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}} \boldsymbol{q}_0 \in \mathbb{R}^{k_0 \times \ell}.
\end{aligned}$$

6.3.2 Index-3 DAEs without a differential part

Here, we assume that the matrix pencil $(\mathbf{E}, \mathbf{A})$ of (6.0.1a) has no finite eigenvalues, i.e., $\sigma_f(\mathbf{E}, \mathbf{A}) = \emptyset$. This implies that $\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2 = 0$, thus (6.3.1) simplifies to,

$$\mathbf{E}_3\Big[\mathbf{P}_2\mathbf{P}_1\mathbf{P}_0\boldsymbol{x}' + \mathbf{Q}_2\boldsymbol{x} + \mathbf{Q}_1\boldsymbol{x} + \mathbf{Q}_0\boldsymbol{x}\Big] = \mathbf{B}\boldsymbol{u}, \tag{6.3.4}$$

Also from Section 5.4.3 for the case of index-3 DAEs without differential part, $\boldsymbol{x}$ can be decomposed as $\boldsymbol{x} = \begin{pmatrix}\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0} & \boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0} & \boldsymbol{q}\end{pmatrix}\begin{pmatrix}\xi_{q,2}\\ \xi_{q,1}\\ \xi_{q,0}\end{pmatrix}$, $\xi_{q,2} \in \mathbb{R}^{k_2}$, $\xi_{q,1} \in \mathbb{R}^{k_1}$, $\xi_{q,0} \in \mathbb{R}^{k_0}$. Then substituting $\boldsymbol{x}$ into (6.3.4) and simplifying we obtain

$$\begin{pmatrix}\mathbf{E}_3\mathbf{P}_2\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0} & -\mathbf{E}_3\mathbf{Q}_0\mathbf{Q}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0} & 0\end{pmatrix}\begin{pmatrix}\xi_{q,2}\\ \xi_{q,1}\\ \xi_{q,0}\end{pmatrix}' = \begin{pmatrix}-\mathbf{E}_3\mathbf{Q}_2\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0} & -\mathbf{E}_3\mathbf{Q}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0} & -\mathbf{E}_3\boldsymbol{q}_0\end{pmatrix}\begin{pmatrix}\xi_{q,2}\\ \xi_{q,1}\\ \xi_{q,0}\end{pmatrix} + \mathbf{B}\boldsymbol{u}. \tag{6.3.5}$$

In order to decouple (6.3.5), we need to construct two pairs of full column rank matrices. This is done as follows: We first construct $\hat{\boldsymbol{p}}^{\mathrm{T}} \in \mathbb{R}^{n_0 \times n}$, $\hat{\boldsymbol{q}}^{\mathrm{T}} \in \mathbb{R}^{k_0 \times n}$ such that $\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\boldsymbol{q} = 0$ and $\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_3\boldsymbol{p} = 0$. This implies that $\mathrm{Span}(\hat{\boldsymbol{p}}) = \mathrm{Ker}\,(\mathbf{E}_3\boldsymbol{q})^{\mathrm{T}}$ and $\mathrm{Span}(\hat{\boldsymbol{q}}) = \mathrm{Ker}\,(\mathbf{E}_3\boldsymbol{p}_0)^{\mathrm{T}}$. We then construct $\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} \in \mathbb{R}^{n_1 \times n_0}$, $\hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}} \in \mathbb{R}^{k_1 \times n_0}$ such that $\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0} = 0$ and $\hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_2\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0} = 0$. This implies that $\mathrm{Span}(\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}) = \mathrm{Ker}\,(\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0})^{\mathrm{T}}$ and $\mathrm{Span}(\hat{\boldsymbol{z}}_{\boldsymbol{q}_0}) = \mathrm{Ker}\,(\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_2\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0})^{\mathrm{T}}$. Thus left multiplying (6.3.5) by $\begin{pmatrix}\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}} & \hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}} & \hat{\boldsymbol{q}}^{\mathrm{T}}\end{pmatrix}^{\mathrm{T}}$ and simplifying, we obtain

$$\begin{pmatrix}0 & 0 & 0\\ \hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\mathbf{P}_2\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0} & 0 & 0\\ \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_3\mathbf{P}_2\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0} & -\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_0\mathbf{Q}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0} & 0\end{pmatrix}\begin{pmatrix}\xi_{q,2}\\ \xi_{q,1}\\ \xi_{q,0}\end{pmatrix}' = \begin{pmatrix}-\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_2\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0} & 0 & 0\\ 0 & -\hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0} & 0\\ -\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_2\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0} & -\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0} & -\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_3\boldsymbol{q}\end{pmatrix}\begin{pmatrix}\xi_{q,2}\\ \xi_{q,1}\\ \xi_{q,0}\end{pmatrix} + \begin{pmatrix}\hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{B}\\ \hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{*T}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{B}\\ \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{B}\end{pmatrix}\boldsymbol{u}.$$

From the above system, without loss of generality the implicit decoupled system of (6.0.1) is given by

$$
\begin{aligned}
\mathbf{E}_{q,2}\xi_{q,2} &= \mathbf{B}_{q,2}\boldsymbol{u},\\
\mathbf{E}_{q,1}\xi_{q,1} &= \mathbf{B}_{q,1}\boldsymbol{u} + \mathbf{A}_{\boldsymbol{q}_{1,2}}\xi'_{q,2},\\
\mathbf{E}_{q,0}\xi_{q,0} &= \mathbf{B}_{q,0}\boldsymbol{u} + \mathbf{E}_{\boldsymbol{q}_{0,2}}\xi_{q,2} + \mathbf{E}_{\boldsymbol{q}_{0,1}}\xi_{q,1} + \mathbf{A}_{\boldsymbol{q}_{0,2}}\xi'_{q,2} + \mathbf{A}_{\boldsymbol{q}_{0,1}}\xi'_{q,1},\\
\boldsymbol{y} &= \mathbf{C}^{\mathrm{T}}_{q,2}\xi_{q,2} + \mathbf{C}^{\mathrm{T}}_{q,1}\xi_{q,1} + \mathbf{C}^{\mathrm{T}}_{q,0}\xi_{q,0},
\end{aligned}
\tag{6.3.6}
$$

where

$\mathbf{E}_{q,2} = \hat{\boldsymbol{z}}^{\mathrm{T}}_{\boldsymbol{p}_0}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_2\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0} \in \mathbb{R}^{k_2\times k_2}$, $\mathbf{B}_{q,2} = \hat{\boldsymbol{z}}^{\mathrm{T}}_{\boldsymbol{p}_0}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{B} \in \mathbb{R}^{k_2\times m}$, $\mathbf{E}_{q,1} = \hat{\boldsymbol{z}}^{\mathrm{T}}_{\boldsymbol{q}_0}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{k_1\times k_1}$,
$\mathbf{B}_{q,1} = \hat{\boldsymbol{z}}^{\mathrm{T}}_{\boldsymbol{q}_0}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{B} \in \mathbb{R}^{k_1\times m}$, $\mathbf{A}_{\boldsymbol{q}_{1,2}} = -\hat{\boldsymbol{z}}^{\mathrm{T}}_{\boldsymbol{q}_0}\hat{\boldsymbol{p}}^{\mathrm{T}}\mathbf{E}_3\mathbf{P}_2\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0} \in \mathbb{R}^{k_1\times k_2}$, $\mathbf{E}_{q,0} = \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_3\boldsymbol{q}_0 \in \mathbb{R}^{k_0,k_0}$,
$\mathbf{B}_{q,0} = \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{B} \in \mathbb{R}^{k_0\times m}$, $\mathbf{A}_{\boldsymbol{q}_{0,2}} = -\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_3\mathbf{P}_2\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0} \in \mathbb{R}^{k_0\times k_2}$ $\mathbf{A}_{\boldsymbol{q}_{0,1}} = \hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_0\mathbf{Q}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{k_0\times k_1}$
$\mathbf{E}_{\boldsymbol{q}_{0,1}} = -\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_1\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{k_0\times k_1}$, $\mathbf{E}_{\boldsymbol{q}_{0,2}} = -\hat{\boldsymbol{q}}^{\mathrm{T}}\mathbf{E}_3\mathbf{Q}_2\boldsymbol{p}\boldsymbol{z}_{\boldsymbol{p}_0} \in \mathbb{R}^{k_0\times k_2}$, $\mathbf{C}^{\mathrm{T}}_{q,2} = \mathbf{C}^{\mathrm{T}}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{p}_0} \in \mathbb{R}^{k_2\times \ell}$,
$\mathbf{C}^{\mathrm{T}}_{q,1} = \mathbf{C}^{\mathrm{T}}\boldsymbol{p}_0\boldsymbol{z}_{\boldsymbol{q}_0} \in \mathbb{R}^{k_1\times \ell}$ and $\mathbf{C}^{\mathrm{T}}_{q,0} = \mathbf{C}^{\mathrm{T}}\boldsymbol{q}_0 \in \mathbb{R}^{k_0\times \ell}$.

Example 6.3.1 In this example, we use matrices from Example 5.4.1. This DAE is of index-3 and its matrix has at least one finite eigenvalue. Thus following the procedure in Section 6.3.1, we can construct the decoupling column matrices given by

$$
\hat{\boldsymbol{p}}^{\mathrm{T}} = \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \quad
\hat{\boldsymbol{q}}^{\mathrm{T}} = \begin{pmatrix} 0\\ 0\\ -\frac{4L^2}{3L+g}\\ 0\\ 0\\ -\frac{2L^2}{m_2(3L+g)}\\ 1 \end{pmatrix}^{\mathrm{T}}, \quad
\hat{\boldsymbol{z}}^{\mathrm{T}}_{\boldsymbol{p}_0} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix},
$$

$$
\hat{\boldsymbol{z}}_{\boldsymbol{q}_0} = \begin{pmatrix} 0\\ 0\\ -L\\ 0\\ 0\\ 1 \end{pmatrix}, \quad
\hat{\boldsymbol{z}}^{\mathrm{T}}_{\boldsymbol{p}_1} = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0 \end{pmatrix}, \quad
\hat{\boldsymbol{z}}_{\boldsymbol{q}_1} = \begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 1 \end{pmatrix}.
$$

Substituting the above column matrices and those from Example 5.4.1 into (6.3.3) , we obtain an implicit decoupled system with system matrices given by

$$
\mathbf{E}_p = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&m_1&0\\ 0&0&0&m_2 \end{pmatrix}, \mathbf{A}_p = \begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ -\frac{gm_2}{L}&\frac{gm_2}{L}&0&0\\ \frac{gm_2}{L}&-\frac{gm_2}{L}&0&0 \end{pmatrix}, \mathbf{B}_p = \begin{pmatrix} 0\\ 0\\ 1\\ 0 \end{pmatrix}, \mathbf{E}_{q,2} = 0,\ \mathbf{A}_{q,2} = \begin{pmatrix} 0&0&0&0 \end{pmatrix}, \mathbf{B}_{q,2} = 0,\ \mathbf{E}_{q,1} = L,
$$

$\mathbf{A}_{q,1} = \begin{pmatrix}0 & 0 & 0 & 0\end{pmatrix}$, $\mathbf{B}_{q,1} = 0$, $\mathbf{E}_{\boldsymbol{q}_{1,2}} = 0$, $\mathbf{A}_{\boldsymbol{q}_{1,2}} = 0$, $\mathbf{E}_{q,0} = \begin{pmatrix}\frac{4L^3}{m_2(3L+g)}\end{pmatrix}$, $\mathbf{A}_{q,0} = \begin{pmatrix}0 & 0 & 0 & 0\end{pmatrix}$, $\mathbf{B}_{q,0} = 0$, $\mathbf{E}_{\boldsymbol{q}_{0,2}} = 0$, $\mathbf{E}_{\boldsymbol{q}_{0,1}} = -\frac{2L^2}{3L+g}$, $\mathbf{A}_{\boldsymbol{q}_{0,2}} = 0$, $\mathbf{A}_{\boldsymbol{q}_{0,1}} = -\frac{2L^2}{3L+g}$, $\mathbf{C}_p^{\mathrm{T}} = \begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$, $\mathbf{C}_{q,2}^{\mathrm{T}} = \begin{pmatrix}0\\1\end{pmatrix}$, $\mathbf{C}_{q,1}^{\mathrm{T}} = \begin{pmatrix}0\\0\end{pmatrix}$, $\mathbf{C}_{q,0}^{\mathrm{T}} = \begin{pmatrix}0\\0\end{pmatrix}$. We can observe that the system can be simplified to an ODE system given by

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & m_1 & 0\\ 0 & 0 & 0 & m_2\end{pmatrix}\xi_p' = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\frac{gm_2}{L} & \frac{gm_2}{L} & 0 & 0\\ \frac{gm_2}{L} & -\frac{gm_2}{L} & 0 & 0\end{pmatrix}\xi_p + \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\boldsymbol{u},$$
$$\boldsymbol{y} = \begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\xi_p, \tag{6.3.7}$$

We can observe (6.3.7) is an implicit version of (5.3.20) and their solutions coincide.

Example 6.3.2 Using matrices from Example 5.4.2 and following the procedure for the case of index-3 DAEs without a differential part, we construct the decoupling column matrices given by

$$\hat{\boldsymbol{p}}^{\mathrm{T}} = \begin{pmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}^{\mathrm{T}}, \ \hat{\boldsymbol{q}}^{\mathrm{T}} = \begin{pmatrix}\frac{1}{3}\\ \frac{2}{3}\\ 1\end{pmatrix}^{\mathrm{T}}, \ \hat{\boldsymbol{z}}_{\boldsymbol{p}_0}^{\mathrm{T}} = \begin{pmatrix}0\\1\end{pmatrix}^{\mathrm{T}}, \ \hat{\boldsymbol{z}}_{\boldsymbol{q}_0}^{\mathrm{T}} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}.$$

Substituting these column matrices and those from Example 5.4.2 into (6.3.6), we obtain an implicit decoupled system with system matrices given by:

$$\mathbf{E}_{q,2} = -1,\ \mathbf{B}_{q,2} = 0,\ \mathbf{E}_{q,1} = -\frac{1}{\sqrt{2}},\ \mathbf{B}_{q,1} = \frac{1}{10\sqrt{2}},\ \mathbf{A}_{\boldsymbol{q}_{1,2}} = -\frac{1}{\sqrt{2}},\ \mathbf{E}_{q,0} = -\frac{1}{3},$$
$$\mathbf{B}_{q,0} = \frac{17}{5},\ \mathbf{E}_{\boldsymbol{q}_{0,2}} = \frac{1}{3},\ \mathbf{E}_{\boldsymbol{q}_{0,1}} = \frac{1}{3},\ \mathbf{A}_{\boldsymbol{q}_{0,2}} = -\frac{1}{3},\ \mathbf{A}_{\boldsymbol{q}_{0,1}} = -\frac{1}{3}.$$

Hence the implicit decoupled system of this DAE is given by

$$\begin{aligned}
-\xi_{q,2} &= 0\boldsymbol{u},\\
-\frac{1}{\sqrt{2}}\xi_{q,1} &= \frac{1}{10\sqrt{2}}\boldsymbol{u} - \frac{1}{\sqrt{2}}\xi_{q,2}',\\
-\frac{1}{3}\xi_{q,0} &= \frac{17}{5}\boldsymbol{u} + \frac{1}{3}\xi_{q,2} + \frac{1}{3}\xi_{q,1} - \frac{1}{3}\xi_{q,2}' - \frac{1}{3}\xi_{q,1}',\\
\boldsymbol{y} &= -29\xi_{q,2} + 29.96\xi_{q,1} + 0.04\xi_{q,0}.
\end{aligned}$$

After solving the above system leads to an output solution $\boldsymbol{y} = -3.4\boldsymbol{u} - 0.004\boldsymbol{u}'$ which coincides with that obtained in Example 5.4.2.

6.4 Comparison of implicit and explicit decoupling methods

In this Section, we compare the no-inversion decoupling procedure discussed in this Chapter and the inversion decoupling procedure presented in Chapter 5. We call the no-inversion and inversion procedures, the implicit and explicit decoupling procedures, respectively. Let us first generalize the implicit decoupled system as follows. Assume (6.0.1) is an index-μ DAE. If the spectrum of the matrix pencil has at least one finite eigenvalue, then the DAE can be decoupled implicitly as

$$\mathbf{E}_p\xi_p' = \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u} \tag{6.4.1a}$$
$$-\mathcal{L}\xi_q' = \mathbf{A}_q\xi_p - \mathcal{L}_q\xi_q + \mathbf{B}_q\boldsymbol{u}, \tag{6.4.1b}$$
$$\boldsymbol{y} = \mathbf{C}_p^{\mathrm{T}}\xi_p + \mathbf{C}_q^{\mathrm{T}}\xi_q, \tag{6.4.1c}$$

where $\mathcal{L}$ is a nilpotent matrix of index μ. $\mathcal{L}_q$ is a non-singular lower triangular matrix with block diagonal matrices for $\mu > 1$. $\xi_p \in \mathbb{R}^{n_p}$, $\xi_q \in \mathbb{R}^{n_q}$, $\mathbf{A}_q \in \mathbb{R}^{n_q\times n_p}$, $\mathbf{B}_q \in \mathbb{R}^{n_q\times m}$ and $\mathbf{C}_q \in \mathbb{R}^{n_q\times\ell}$, $\mathbf{C}_p \in \mathbb{R}^{n_p\times\ell}$. And, if spectrum of the matrix pencil of (6.0.1) has only infinite eigenvalues then (6.4.1) simplifies to

$$-\mathcal{L}\xi_q' = -\mathcal{L}_q\xi_q + \mathbf{B}_q\boldsymbol{u}, \tag{6.4.2a}$$
$$\boldsymbol{y} = \mathbf{C}_q^{\mathrm{T}}\,\xi_q. \tag{6.4.2b}$$

For comparison with the DAE (6.0.1), we can rewrite the implicit decoupled systems either (6.4.1) or (6.4.2) in the descriptor form given by

$$\tilde{\mathbf{E}}\xi' = \tilde{\mathbf{A}}\xi + \tilde{\mathbf{B}}\boldsymbol{u}, \tag{6.4.3a}$$
$$\boldsymbol{y} = \tilde{\mathbf{C}}^{\mathrm{T}}\xi, \tag{6.4.3b}$$

where $\tilde{\mathbf{E}} = \begin{pmatrix}\mathbf{E}_p & 0\\ 0 & -\mathcal{L}\end{pmatrix} \in \mathbb{R}^{n\times n}$, $\tilde{\mathbf{A}} = \begin{pmatrix}\mathbf{A}_p & 0\\ \mathbf{A}_q & -\mathcal{L}_q\end{pmatrix} \in \mathbb{R}^{n\times n}$, $\tilde{\mathbf{B}} = \begin{pmatrix}\mathbf{B}_p\\ \mathbf{B}_q\end{pmatrix} \in \mathbb{R}^{n\times m}$, $\tilde{\mathbf{C}} = \begin{pmatrix}\mathbf{C}_p\\ \mathbf{C}_q\end{pmatrix} \in \mathbb{R}^{n\times\ell}$. If the spectrum of the matrix pencil $(\mathbf{E}, \mathbf{A})$ has at least one finite eigenvalue and $\tilde{\mathbf{E}} = -\mathcal{L} \in \mathbb{R}^{n\times n}$, $\tilde{\mathbf{A}} = -\mathcal{L}_q \in \mathbb{R}^{n\times n}$, $\tilde{\mathbf{B}} = \mathbf{B}_q \in \mathbb{R}^{n\times m}$, $\tilde{\mathbf{C}} = \mathbf{C}_q \in \mathbb{R}^{n\times\ell}$, if the spectrum of the matrix pencil $(\mathbf{E}, \mathbf{A})$ has no finite eigenvalues. We can observe that this form also reveals the interconnection structure of the DAE (6.0.1). Moreover it can also be proved that systems (6.0.1) and (6.4.3) are equivalent. This implies that also their matrix pencils $(\mathbf{E}, \mathbf{A})$ and $(\tilde{\mathbf{E}}, \tilde{\mathbf{A}})$ are equivalent, thus they must have the same spectrum. If we consider

DAEs whose matrix pencil $(\mathbf{E}, \mathbf{A})$ has at least one finite eigenvalue, we can show that they have same spectrum, since we can easily show that $\det(\lambda\tilde{\mathbf{E}} - \tilde{\mathbf{A}}) = \det(\lambda\mathbf{E}_p - \mathbf{A}_p)$, since $\det(\mathcal{L}_q - \lambda\mathcal{L}) = (1)^{n_q}$. This identity shows that the finite eigenvalues of the matrix pencil $(\mathbf{E}, \mathbf{A})$ coincide with the (possibly complex) eigenvalues of the matrix $\mathbf{E}_p^{-1}\mathbf{A}_p$ of the differential part, which are exactly n_p, counting their multiplicity. This implies that the differential part of the implicit decoupled system also inherits the stability properties of the DAE (6.0.1a). Hence both the implicit and explicit decoupling procedure preserves the dimension and stability of the DAE. If we compare the descriptor forms (5.4.34) and (6.4.3), they coincide if $\mathbf{E}_p = \mathrm{I}$ and $\mathcal{L}_q = \mathrm{I}$. The main difference between these two procedures is the computational cost involved in deriving the respective decoupled systems. The explicit decoupling is the most expensive since its decoupling procedure involves inversion of matrix $\mathbf{E}_\mu$ which can be computationally very expensive. We note that both decoupling procedure can lead to a complete decoupling, that is, when matrix $\mathbf{A}_q$ vanishes if one uses the so called canonical projectors proposed by März [42]. In Example 6.4.1, we compare the computational cost of the explicit and implicit decoupling procedure using index-1 power system models.

Example 6.4.1 In this example, we use index-1 power system models obtained from [54–57] to compare the computational cost of the explicit and implicit decoupling methods of DAEs. They are all index-1 DAEs of the form (6.0.1). We were able to decouple them into n_p differential equations and n_q algebraic equations, where $n = n_q + n_q$ is the dimension of the DAE using both the explicit and the implicit decoupling methods. We compared the time both methods took to generate the matrices of their respective decoupled system as shown in Table 6.1. The experiments were done using Matlab2012b on a laptop of 6.00GB of RAM with 64 bit operating system. From Table 6.1, we can observe that the implicit decoupling procedure takes far less time to decouple the DAE than the explicit decoupling method. We in fact gain more than 85% times reduction for all the power systems. We can also observe that for the large examples the explicit method fail to decouple the system in the allowed computational time.

Table 6.1: Computational cost of implicit and explicit decoupling

Systems	# inputs/# outputs		Decoupled model		Comp. Cost (Seconds)		% Time Reduction
n	# inputs	# outputs	n_p	n_q	Explicit method	Implicit method	
40366	2	2	5727	34639	-	20.9	100.0
40337	2	1	5723	34614	-	20.5	100.0
21476	1	1	3172	18304	98.5	5.5	89.4
21128	4	4	3078	18050	79.4	5.0	88.1
20944	2	2	3012	17932	76.9	4.6	88.8
20738	1	6	2940	17798	82.5	4.9	88.7
16861	4	4	2476	14385	58.7	4.4	86.0
15066	4	4	1998	13068	53.8	3.9	86.6
13309	8	8	1676	11633	32.4	1.7	89.9
13296	46	46	1664	11632	29.0	1.9	88.0
13275	4	4	1693	11582	29.2	1.9	88.0
13250	1	1	1664	11586	28.8	1.7	88.7
13250	46	46	1664	11586	28.7	1.9	87.9
13251	28	28	1664	11587	28.6	1.7	88.7
13251	1	1	1664	11587	28.4	1.8	88.3
11685	1	1	1257	10428	23.8	1.3	89.7
11305	4	4	1450	9855	24.1	1.7	86.9
9735	4	4	1142	8593	20.8	1.3	88.0
7135	4	4	606	6529	14.4	1.0	86.5

From this experiment, we can conclude that the implicit decoupling method is computationally cheaper to use than the explicit decoupling method. In Chapter 5 and 6, we have derived two decoupling procedures for decoupling linear constant DAEs into differential and algebraic parts using special bases of projectors. Both procedures preserve dimension of the DAE and the spectrum of the matrix pencil $(\mathbf{E}, \mathbf{A})$. One may wonder wether the construction of the special bases of projectors is numerically feasible especially with large-scale problems. Fortunately, the same procedure proposed in [66] to construct projector onto the nullspace of a singular large sparse matrix can also be used to construct these bases more efficiently. Also most of the applications that lead to DAEs, have special structures of matrix $\mathbf{E}$ and $\mathbf{A}$, thus one can easily be able to construct these bases explicitly. However some applications such as the circuits problems which are modeled using the incidence matrices, we recommend to use the incidence matrices to construct these bases instead of using singular matrix $\mathbf{E}$ and $\mathbf{A}$ since these matrices may be ill-conditioned. Since the main objective of this thesis is to develop robust MOR methods for DAEs. We use the explicit and implicit decoupled systems derived in Chapter 5 and 6, to develop MOR methods for DAEs. This is discussed in Chapter 7 and 8, respectively.

Chapter 7

Index-aware Model Order Reduction (IMOR) method

Some of the content in this Chapter can also be found in our papers [1, 2, 6]. In this Chapter, we introduce the Index-aware model order reduction method which can be abbreviated as IMOR method. We use the decoupled systems derived in Chapter 5 to develop the IMOR method. Consider DAEs of the form

$$\mathbf{E}\dot{\boldsymbol{x}}(t) = \mathbf{A}\boldsymbol{x}(t) + \mathbf{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{7.0.1a}$$

$$\boldsymbol{y}(t) = \mathbf{C}^{\mathrm{T}}\boldsymbol{x}(t), \tag{7.0.1b}$$

where $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{n\times \ell}$, the input vector $\boldsymbol{u}(t) \in \mathbb{R}^m$ and output vector $\boldsymbol{y}(t) \in \mathbb{R}^\ell$ of the system. $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector and $\boldsymbol{x}_0$ is the initial value. The number of state variables n is called the order of system or the state-space dimension. m and ℓ are the number of inputs and outputs, respectively. Before deriving the IMOR method we propose a method which can be used to reduce the algebraic parts of the decoupled system. This method is presented in the next Section. We call this method the

Algebraic Elimination (AE) method. The main idea of the AE method is to eliminate algebraic variables of a given DAE which do not contribute to the output solution.

7.1 Algebraic Elimination MOR method

In this Section, we discuss the reduction of the algebraic parts of the decoupled system, if the decoupled systems are derived from Chapter 5. This is done using reordering techniques and then eliminate algebraic variables which do not contribute to the output solutions.

7.1.1 Index-1 DAEs

Assume (7.0.1a) is of index-1 then it can be decoupled into the form (5.2.4) given by

$$\xi_p' = \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u}, \tag{7.1.1a}$$
$$\xi_q = \mathbf{A}_q\xi_p + \mathbf{B}_q\boldsymbol{u}, \tag{7.1.1b}$$
$$\boldsymbol{y} = \mathbf{C}_p^{\mathrm{T}}\xi_p + \mathbf{C}_q^{\mathrm{T}}\xi_q, \tag{7.1.1c}$$

where $\mathbf{A}_p \in \mathbb{R}^{n_p\times n_p}$, $\mathbf{B}_p \in \mathbb{R}^{n_p\times m}$, $\mathbf{A}_q \in \mathbb{R}^{n_q\times n_p}$, $\mathbf{B}_q \in \mathbb{R}^{n_q\times m}$ and $\mathbf{C}_p \in \mathbb{R}^{n_p\times \ell}$, $\mathbf{C}_q \in \mathbb{R}^{n_q\times \ell}$. Let us assume $\mathbf{C}_q \neq 0$ otherwise the DAE can just be reduced to an ODE (7.1.1a) of dimension n_p. Consider the algebraic subsystem of (7.1.1) given by

$$\xi_q = \mathbf{A}_q\xi_p + \mathbf{B}_q\boldsymbol{u}, \tag{7.1.2a}$$
$$\boldsymbol{y}_q = \mathbf{C}_q^{\mathrm{T}}\xi_q. \tag{7.1.2b}$$

The algebraic reduction of (7.1.2) can be done as follows. Assume $\mathbf{C}_q$ has at least one zero row, i.e., the row rank of $\mathbf{C}_q$ is less than n_q. Let $\mathbf{P}_{\pi_1} \in \mathbb{R}^{n_q\times n_q}$ be a permutation matrix such that $\mathbf{P}_{\pi_1}\mathbf{C}_q = \begin{pmatrix}\tilde{\mathbf{C}}_{q_1}\\ \mathbf{0}\end{pmatrix}$, where $\tilde{\mathbf{C}}_{q_1} \in \mathbb{R}^{\tau\times\ell}$ and also let $\xi_q = \mathbf{P}_{\pi_1}^{\mathrm{T}}\tilde{\xi}_q = \begin{pmatrix}\tilde{\xi}_{q_1}\\ \tilde{\xi}_{q_2}\end{pmatrix}$, where $\tilde{\xi}_{q_1} \in \mathbb{R}^{\tau}$, $\tilde{\xi}_{q_2} \in \mathbb{R}^{n_q-\tau}$ and $\mathbf{Q}_{\pi_1} = \mathbf{P}_{\pi_1}^{-\mathrm{T}}$. Then $\mathbf{A}_q$ and $\mathbf{B}_q$ can be partitioned as $\mathbf{Q}_{\pi_1}\mathbf{A}_q = \begin{pmatrix}\tilde{\mathbf{A}}_{q_1}\\ \tilde{\mathbf{A}}_{q_2}\end{pmatrix}$ and $\mathbf{Q}_{\pi_1}\mathbf{B}_q = \begin{pmatrix}\tilde{\mathbf{B}}_{q_1}\\ \tilde{\mathbf{B}}_{q_2}\end{pmatrix}$, where $\tilde{\mathbf{A}}_{q_1} \in \mathbb{R}^{\tau\times n_p}$, $\tilde{\mathbf{A}}_{q_2} \in \mathbb{R}^{(n_q-\tau)\times n_p}$, $\tilde{\mathbf{B}}_{q_1} \in \mathbb{R}^{\tau\times m}$, $\tilde{\mathbf{B}}_{q_2} \in \mathbb{R}^{(n_q-\tau)\times m}$. Hence (7.1.2) can be reduced to a reduced-order algebraic system of dimension $\tau \ll n_q$ given by

$$\begin{aligned}\tilde{\xi}_{q_1} &= \tilde{\mathbf{A}}_{q_1}\xi_p + \tilde{\mathbf{B}}_{q_1}\boldsymbol{u},\\ \boldsymbol{y}_p &= \tilde{\mathbf{C}}_{q_1}^{\mathrm{T}}\tilde{\xi}_{q_1}.\end{aligned} \tag{7.1.3}$$

Thus, system (7.1.1) can be reduced to a reduced-order model of dimension $n_p + \tau < n$ given by

$$\begin{aligned}\xi_p' &= \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u},\\ \tilde{\xi}_{q_1} &= \tilde{\mathbf{A}}_{q_1}\xi_p + \tilde{\mathbf{B}}_{q_1}\boldsymbol{u},\\ \boldsymbol{y} &= \mathbf{C}_p^{\mathrm{T}}\xi_p + \tilde{\mathbf{C}}_{q_1}^{\mathrm{T}}\tilde{\xi}_{q_1}.\end{aligned} \tag{7.1.4}$$

Hence (7.1.4) is AE reduced-order model of (7.0.1).

7.1.2 Index-2 DAEs

Assume (7.0.1a) is a DAE of index-2. If we consider the case of the index-2 DAEs whose matrix pencil with spectrum having at least one finite eigenvalue. Then, the DAE (7.0.1) can be decoupled into the form (5.3.15) given by

$$\begin{aligned}\xi_p' &= \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u},\\ \xi_{q,1} &= \mathbf{A}_{q,1}\xi_p + \mathbf{B}_{q,1}\boldsymbol{u},\\ \xi_{q,0} &= \mathbf{A}_{q,0}\xi_p + \mathbf{B}_{q,0}\boldsymbol{u} + \mathbf{A}_{\boldsymbol{q}_{0,1}}\xi_{q,1}',\\ \boldsymbol{y} &= \mathbf{C}_p^{\mathrm{T}}\xi_p + \mathbf{C}_{q,1}^{\mathrm{T}}\xi_{q,1} + \mathbf{C}_{q,0}^{\mathrm{T}}\xi_{q,0},\end{aligned} \tag{7.1.5}$$

where $\mathbf{A}_p \in \mathbb{R}^{n_p\times n_p}$, $\mathbf{B}_p \in \mathbb{R}^{n_p\times m}$, $\mathbf{A}_{q,1} \in \mathbb{R}^{k_1\times n_p}$, $\mathbf{B}_{q,1} \in \mathbb{R}^{k_1\times m}$, $\mathbf{A}_{q,0} \in \mathbb{R}^{k_2\times n_p}$, $\mathbf{B}_{q,0} \in \mathbb{R}^{k_2\times m}$, $\mathbf{A}_{\boldsymbol{q}_{0,1}} \in \mathbb{R}^{k_2\times k_1}$ and $\mathbf{C}_p \in \mathbb{R}^{n_p\times \ell}$, $\mathbf{C}_{q,1} \in \mathbb{R}^{k_1\times \ell}$, $\mathbf{C}_{q,0} \in \mathbb{R}^{k_2\times \ell}$. If we consider only algebraic parts of the system (7.1.5), we obtain an algebraic subsystem given by

$$\xi_{q,1} = \mathbf{A}_{q,1}\xi_p + \mathbf{B}_{q,1}\boldsymbol{u}, \tag{7.1.6a}$$

$$\xi_{q,0} = \mathbf{A}_{q,0}\xi_p + \mathbf{B}_{q,0}\boldsymbol{u} + \mathbf{A}_{\boldsymbol{q}_{0,1}}\xi_{q,1}', \tag{7.1.6b}$$

$$\boldsymbol{y}_q = \mathbf{C}_{q,1}^{\mathrm{T}}\xi_{q,1} + \mathbf{C}_{q,0}^{\mathrm{T}}\xi_{q,0}. \tag{7.1.6c}$$

We can observe that, if $\mathbf{C}_{q,1} = \mathbf{0}$ and $\mathbf{C}_{q,0} = \mathbf{0}$, then the DAE (7.0.1) can be reduced to an ODE (7.1.6a) of dimension n_p. If $\mathbf{C}_{q,1} \neq 0$ and $\mathbf{C}_{q,0} = \mathbf{0}$ or $\mathbf{C}_{q,1} = \mathbf{0}$, $\mathbf{A}_{\boldsymbol{q}_{0,1}} = \mathbf{0}$ and

$\mathbf{C}_{q,0} \neq \mathbf{0}$ then the DAE (7.0.1) can be reduced to an index-1 DAE of dimension $n_p + k_1$ and $n_p + k_0$, respectively. If (7.1.6) is reduced to an index-1 DAE. Then, we use the same procedure in the previous Section to further reduce the algebraic parts. Assume the above conditions are not satisfied then the algebraic subsystem (7.1.6) can be reduced as follows. We assume $\mathbf{C}_{q,1} \in \mathbb{R}^{k_1 \times \ell}$ and $\mathbf{C}_{q_0}\mathbb{R}^{k_0 \times \ell}$ have at least one zero row, i.e., the row rank of $\mathbf{C}_{q,1}$ and $\mathbf{C}_{q_0}$ is less than k_1 and k_2, respectively. First, we compute the permutation matrices $\mathbf{V}_\pi, \mathbf{W}_\pi \in \mathbb{R}^{k_2 \times k_2}$ such that $\mathbf{W}_\pi^{\mathrm{T}}\mathbf{C}_{q,0} = \begin{pmatrix} \mathbf{C}_{q_1,0} \\ \mathbf{0} \end{pmatrix}$. Then, we have partitions $\mathbf{V}_\pi \mathbf{B}_{q,0} = \begin{pmatrix} \mathbf{B}_{q_1,0} \\ \mathbf{B}_{q_2,0} \end{pmatrix}$, $\mathbf{V}_\pi \mathbf{A}_{q,0} = \begin{pmatrix} \mathbf{A}_{q_1} \\ \mathbf{A}_{q_2} \end{pmatrix}$. Next, we construct another set of permutation matrices $\mathbf{P}_\pi, \mathbf{Q}_\pi \in \mathbb{R}^{k_1 \times k_1}$ such that $\mathbf{V}_\pi \mathbf{A}_{\boldsymbol{q}_{0,1}} \mathbf{Q}_\pi = \begin{pmatrix} \mathbf{A}_{\boldsymbol{q}_{0_{11},1}} & \mathbf{0} \\ \mathbf{A}_{\boldsymbol{q}_{0_{21},1}} & \mathbf{0} \end{pmatrix}$, $\mathbf{Q}_\pi^{\mathrm{T}}\mathbf{C}_{q,1} = \begin{pmatrix} \mathbf{C}_{q_1,1} \\ \mathbf{0} \end{pmatrix}$. Then, we have a partition $\mathbf{P}_\pi \mathbf{A}_{q,1} = \begin{pmatrix} \mathbf{A}_{q_1,1} \\ \mathbf{A}_{q_1,2} \end{pmatrix}$, $\mathbf{P}_\pi \mathbf{B}_{q,1} = \begin{pmatrix} \mathbf{B}_{q_1,1} \\ \mathbf{B}_{q_2,1} \end{pmatrix}$. If we let $\xi_{q,1} = \mathbf{Q}_\pi \tilde{\xi}_{q,1} = \begin{pmatrix} \tilde{\xi}_{q_1,1} \\ \tilde{\xi}_{q_2,1} \end{pmatrix}$, where $\tilde{\xi}_{q_1,1} \in \mathbb{R}^{\tau_1}$, $\tilde{\xi}_{q_2,1} \in \mathbb{R}^{k_1-\tau_1}$ and $\xi_{q,0} = \mathbf{W}_\pi \tilde{\xi}_{q,0} = \begin{pmatrix} \tilde{\xi}_{q_1,0} \\ \tilde{\xi}_{q_2,0} \end{pmatrix}$, where $\tilde{\xi}_{q_1,0} \in \mathbb{R}^{\tau_2}$

$\tilde{\xi}_{q_2,0} \in \mathbb{R}^{k_2-\tau_2}$. Then left multiply (7.1.6a) and (7.1.6b) with $\mathbf{P}_\pi$ and $\mathbf{V}_\pi$, respectively. We obtain a partitioned system of (7.1.6). We can then eliminate the algebraic variables $\tilde{\xi}_{q_2,1}$ and $\tilde{\xi}_{q_2,0}$ which do not contribute to the output solution (7.1.6c). This leads to a reduced-order model of dimension $\tau_1 + \tau_2 < n_q$ given by

$$\xi_{q,1} = \mathbf{A}_{q_\tau,1}\xi_p + \mathbf{B}_{q_\tau,1}\boldsymbol{u}, \tag{7.1.7a}$$

$$\xi_{q_\tau,0} = \mathbf{A}_{q_\tau,0}\xi_p + \mathbf{B}_{q_\tau,0}\boldsymbol{u} + \mathbf{A}_{\boldsymbol{q}_{0_\tau,1}}\xi'_{q_\tau,1}, \tag{7.1.7b}$$

$$\boldsymbol{y}_{q_\tau} = \mathbf{C}^{\mathrm{T}}_{q_\tau,1}\xi_{q_\tau,1} + \mathbf{C}^{\mathrm{T}}_{q_\tau,0}\xi_{q_\tau,0}, \tag{7.1.7c}$$

where $\mathbf{B}_{q_\tau,1} = \mathbf{B}_{q_1,1} \in \mathbb{R}^{\tau_1 \times m}$, $\mathbf{A}_{q_\tau,0} = \mathbf{A}_{q_1} \in \mathbb{R}^{\tau_2 \times n_p}$, $\mathbf{A}_{\boldsymbol{q}_{0_\tau,1}} = \mathbf{A}_{\boldsymbol{q}_{0_{11},1}} \in \mathbb{R}^{\tau_2 \times \tau_1}$ and $\mathbf{C}_{q_\tau,1} = \mathbf{C}_{q_1,1} \in \mathbb{R}^{\tau_1 \times \ell}$, $\mathbf{C}_{q_\tau,0} = \mathbf{C}_{q_1,0} \in \mathbb{R}^{\tau_2 \times \ell}$. Thus the DAE (7.0.1) is reduced to a reduced-order model of dimension $\mathbf{n}_p + \tau_1 + \tau_2 < n$ given by

$$\begin{aligned} \xi'_p &= \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u}, \\ \xi_{q,1} &= \mathbf{A}_{q_\tau,1}\xi_p + \mathbf{B}_{q_\tau,1}\boldsymbol{u}, \\ \xi_{q_\tau,0} &= \mathbf{A}_{q_\tau,0}\xi_p + \mathbf{B}_{q_\tau,0}\boldsymbol{u} + \mathbf{A}_{\boldsymbol{q}_{0_\tau,1}}\xi'_{q_\tau,1}, \\ \boldsymbol{y} &= \mathbf{C}^{\mathrm{T}}_p\xi_p + \mathbf{C}^{\mathrm{T}}_{q_\tau,1}\xi_{q_\tau,1} + \mathbf{C}^{\mathrm{T}}_{q_\tau,0}\xi_{q_\tau,0}. \end{aligned}$$

Example 7.1.1 In this example, we consider the decoupled systems of index-1 DAE model generated in Table 6.1 using the explicit decoupling procedure. Recall, we were able to decouple these DAEs into n_p differential equations and n_q algebraic equations, where $n = n_q + n_q$ is the dimension of the DAE. We cam then apply the AE method to these decoupled systems in order to reduce the number of algebraic equations. In Table 7.1, we can observe that the algebraic equations of the decoupled power systems are greatly reduced. In fact, we obtain an overall reduction of more than 85% as shown in last column of Table 7.1.

Table 7.1: Algebraic Reduced models of power systems

Systems	# inputs/# outputs		Decoupled model		Alg. Reduced model		Reduced system	% Reduction
n	# inputs	# outputs	n_p	n_q	n_p	τ	$n_p + \tau$	
40366	2	2	5727	34639	5727	8	5735	85.8
40337	2	1	5723	34614	5723	6	5729	85.8
21476	1	1	3172	18304	3172	34	3206	85.1
21128	4	4	3078	18050	3078	16	3094	85.4
20944	2	2	3012	17932	3012	8	3020	85.6
20738	1	6	2940	17798	2940	0	1755	91.5
16861	4	4	2476	14385	2476	16	2492	85.2
15066	4	4	1998	13068	1998	16	2014	86.6
13309	8	8	1676	11633	1676	0	1676	87.4
13296	46	46	1664	11632	1664	92	1756	86.8
13275	4	4	1693	11582	1693	16	1709	87.1
13250	1	1	1664	11586	1664	1	1665	87.4
13250	46	46	1664	11586	1664	46	1710	87.1
13251	28	28	1664	11587	1664	0	1664	87.4
13251	1	1	1664	11587	1664	0	1664	87.4
11685	1	1	1257	10428	1257	4	1261	89.2
11305	4	4	1450	9855	1450	16	1466	87.0
9735	4	4	1142	8593	1142	16	1158	88.1
7135	4	4	606	6529	606	16	622	91.2

Example 7.1.2 In this example, we use RLC network descriptor models of electric power grids obtained from [17] and PEEC model of dimension $n = 480$ from [49]. These are all index-2 DAEs of the form (2.3.1). We were able to decouple these models into differential and algebraic parts using the procedure presented in Section 5 and then used the AE method to reduce the algebraic parts. The results are shown in Table 7.2. We can observe that most of the algebraic equations are eliminated, although we do not gain too much overall reduction since these DAEs have more differential equations than the algebraic equations.

The AE method does not reduce the differential part and the algebraic part is not always completely reduced. Thus, the AE reduced-order models must be further be reduced

Table 7.2: Algebraic Reduced models of RLC systems

Systems	# inputs/# outputs		Decoupled model			Alg. Reduced model			Reduced system	% Reduction
n	# inputs	# outputs	n_p	k_1	k_2	n_p	τ_1	τ_2	$n_p + \tau_1 + \tau_2$	
4182	1	1	4028	35	119	4028	0	1	4029	3.7
2182	1	1	2028	35	119	2028	0	1	2029	7.0
1182	1	1	1028	35	119	1028	0	1	1029	12.9
682	1	1	528	35	119	528	0	1	529	22.4
4182	3	3	4028	35	119	4028	0	3	4031	3.6
2182	3	3	2028	35	119	2028	0	3	2031	6.9
1182	3	3	1028	35	119	1028	0	3	1031	12.8
682	3	3	528	35	119	528	0	3	531	22.1
480	1	1	181	61	238	181	0	1	182	62.1

using the Index-aware MOR method.

7.2 Index-aware MOR method

In this Section, we present the Index-aware MOR method which can be abbreviated as the IMOR method. This MOR method was first proposed in [1, 2] for the case of index-1 and -2 DAEs, respectively and its generalization in [6] which we called the GIMOR method. The IMOR method uses the system matrices from the decoupled systems derived in Chapter 5. This is done by reducing both the differential and algebraic parts separately of the decoupled systems. We use the conventional MOR methods to reduce the differential part and we have developed new methods that reduces the algebraic part. For the algebraic part, we first apply the Algebraic Elimination (AE) method proposed in the previous Section so that we can eliminate some algebraic variables which do not contribute to the output solution. We note that this idea is new from what we proposed in [1, 2, 6], it greatly increases the efficiency of the IMOR method. The main motivation of the IMOR method is the need to develop computationally efficient methods which can reduce higher index DAEs. We are not the only people who have attempted to develop MOR methods specifically for DAEs by first splitting them into differential and algebraic parts. Some of the recently developed MOR methods for DAEs, have already been discussed in Section 3.3. The most successful and accurate MOR methods for DAEs are the balanced truncation and interpolatory projection methods for DAEs. However their splitting procedure is based on spectral projectors which may be numerically infeasible, see [25, 45]. Moreover, the spectral projectors are not sufficiently good tools on appropriate generalizations for variable coefficient linear DAEs and nonlinear DAEs, respectively [42]. This gives our decoupling procedure and the IMOR method an advantage over the existing MOR methods for DAEs since it is based on projector and matrix chain introduced by März [42] which are extendable to variable coefficient linear DAEs and nonlinear DAEs, respectively. Thus the IMOR method can also be extended to nonlinear DAEs.

7.2.1 Index-aware MOR for index-1 DAEs

Assume (7.0.1a) is an index-1 DAE, then its explicit decoupled system can be written in the form (7.1.1). Strictly separating the decoupled system (7.1.1) into differential and algebraic parts leads to

$$\begin{aligned} \xi_p' &= \mathbf{A}_p \xi_p + \mathbf{B}_p \boldsymbol{u}, \\ \boldsymbol{y}_p &= \mathbf{C}_p^{\mathrm{T}} \xi_p, \end{aligned} \tag{7.2.1}$$

and

$$\xi_q = \mathbf{A}_q \xi_p + \mathbf{B}_q \boldsymbol{u}, \tag{7.2.2a}$$

$$\boldsymbol{y}_q = \mathbf{C}_q^{\mathrm{T}} \xi_q, \tag{7.2.2b}$$

where the output equation of the DAE can be reconstructed as $\boldsymbol{y} = \boldsymbol{y}_p + \boldsymbol{y}_q$. Then, the IMOR method for index-1 DAE can be derived into the following two steps:

(a) **Reduction of the differential part:** Here we consider the differential subsystem (7.2.1). This subsystem can be reduced by convectional MOR methods such as those presented in [3,9,45,58]. In this Section, we restrict ourselves on the Krylov subspace based methods and the method of choice will be the PRIMA method [49]. Here we seek a reduced-order model of (7.2.1) given by

$$\begin{aligned} \xi_{p_{r_1}}' &= \mathbf{A}_{p_{r_1}} \xi_{p_{r_1}} + \mathbf{B}_{p_{r_1}} \boldsymbol{u}, \\ \boldsymbol{y}_{p_{r_1}} &= \mathbf{C}_{p_{r_1}}^{\mathrm{T}} \xi_{p_{r_1}}, \end{aligned} \tag{7.2.3}$$

where $\mathbf{A}_{p_{r_1}} \in \mathbb{R}^{r_1 \times r_1}$, $\mathbf{B}_{p_{r_1}} \in \mathbb{R}^{r_1 \times m}$ and $\mathbf{C}_p \in \mathbb{R}^{r_1 \times \ell}$, such that $r_1 \ll n_p$. The approximation error $\boldsymbol{y}_p - \boldsymbol{y}_{p_{r_1}}$ and $\mathbf{H}_p(s) - \mathbf{H}_{p_{r_1}}(s)$ must be small with respect to a specific norm. In the frequency domain, this means that the transfer function of (7.2.3) is given by $\mathbf{H}_{p_{r_1}}(s) = \mathbf{C}_{p_{r_1}}^{\mathrm{T}}(s\mathrm{I} - \mathbf{A}_{p_{r_1}})^{-1}\mathbf{B}_{p_{r_1}}$ approximates $\mathbf{H}_p(s)$ of (7.2.1) well. The reduced-order subsystem (7.2.3) can be obtained via projection as follows. We first construct $n_p \times r_1 m$ matrix $\mathbf{V}_p$ that approximates the original state-space $\xi_p(t)$ by $\mathbf{V}_p\xi_{p_{r_1}}$ and then enforce the Galerkin condition

$$\mathbf{V}_p^{\mathrm{T}}\left(\mathbf{V}_p\xi_p' - \mathbf{A}_p\mathbf{V}_p\xi_{p_{r_1}} - \mathbf{B}_p\boldsymbol{u}\right) = \mathbf{0}, \qquad \boldsymbol{y}_{p_{r_1}} = \mathbf{C}_p^{\mathrm{T}}\mathbf{V}_p\xi_{p_{r_1}}.$$

This leads to a reduced-order model (7.2.3) with the system matrices

$$\mathbf{A}_{p_{r_1}} = \mathbf{V}_p^{\mathrm{T}}\mathbf{A}_p\mathbf{V}_p, \quad \mathbf{B}_{p_{r_1}} = \mathbf{V}_p^{\mathrm{T}}\mathbf{B}_p, \quad \mathbf{C}_{p_{r_1}} = \mathbf{V}_p^{\mathrm{T}}\mathbf{C}_p, \text{ and } \mathbf{V}_p^{\mathrm{T}}\mathbf{V}_p = \mathrm{I}_{r_1 m}.$$

The projection matrix $\mathbf{V}_p$ determine the subspace of interest and can be constructed in many different ways, see [3, 9, 45, 58]. If, we apply the Arnoldi process, based on the Krylov subspace

$$\mathcal{K}_{r_1}(\mathbf{M}_p, \mathbf{R}_p) := \mathrm{Span}\{\mathbf{R}_p, \mathbf{M}_p\mathbf{R}_p, \cdots, \mathbf{M}_p^{r_1-1}\mathbf{R}_p\},\ r_1 \le n_p, \tag{7.2.4}$$

where $\mathbf{M}_p := (s_0\mathrm{I} - \mathbf{A}_p)^{-1}$, $\mathbf{R}_p := (s_0\mathrm{I} - \mathbf{A}_p)^{-1}\mathbf{B}_p$ and $s_0 \in \mathbb{C}\setminus\sigma(\mathbf{A}_p)$ which can be chosen arbitrary. We denote by $\mathbf{V}_p \in \mathbb{R}^{n_p \times r_1 m}$ the matrix of an orthonormal basis for $\mathcal{K}_{r_1}(\mathbf{M}_p, \mathbf{R}_p)$, so that $\mathbf{V}_p^{\mathrm{T}}\mathbf{V}_p = \mathrm{I}_{r_1 m}$.

(b) **Reduction of the algebraic part:** In this Section, we derive the reduction procedure for the algebraic subsystem (7.2.2). For this case, we seek a reduced-order algebraic subsystem of the form

$$\xi_{q_{r_2}} = \mathbf{A}_{q_{r_2}}\xi_{p_{r_1}} + \mathbf{B}_{q_{r_2}}\boldsymbol{u}, \tag{7.2.5a}$$

$$\boldsymbol{y}_{q_{r_2}} = \mathbf{C}_{q_{r_2}}^{\mathrm{T}}\xi_{q_{r_2}}, \tag{7.2.5b}$$

where $\mathbf{A}_{q_{r_2}} \in \mathbb{R}^{r_2 \times r_1}$, $\mathbf{B}_{q_{r_2}} \in \mathbb{R}^{r_2 \times m}$ and $\mathbf{C}_{q_{r_2}} \in \mathbb{R}^{r_2 \times \ell}$, $r_2 < n_q$. This can be done as follows. If we substitute $\xi_p = \mathbf{V}_p\xi_{p_{r_1}}$ into (7.2.2a), we obtain

$$\tilde{\xi}_q = \mathbf{A}_q\mathbf{V}_p\xi_{p_{r_1}} + \mathbf{B}_q\boldsymbol{u}. \tag{7.2.6}$$

From (7.2.6), we can observe that the reduction procedure for the differential variables induces a reduction for the algebraic variables, where $\tilde{\xi}_q$ is the approximation of ξ_q induced by the reduction of ξ_p. According to [1], this relations shows that $\tilde{\xi}_q$

lives in the subspace $\mathcal{V}_q$ given by

$$\mathcal{V}_q := \mathrm{Span}\{\mathbf{B}_q, \mathbf{A}_q\mathbf{V}_p\} = \mathrm{Span}\{\mathbf{B}_q\} + \mathbf{A}_q\mathcal{K}_{r_1}(\mathbf{M}_p, \mathbf{R}_p). \tag{7.2.7}$$

We denote by r_2 the dimension of $\mathcal{V}_q$ and by $\mathbf{V}_q \in \mathbb{R}^{n_q \times r_2 m}$ the matrix of an orthonormal basis for $\mathcal{V}_q$, so that $\mathbf{V}_q^{\mathrm{T}}\mathbf{V}_q = \mathrm{I} \in \mathbb{R}^{r_2 m \times r_2 m}$. Thus, we can represent the algebraic solution in the form $\tilde{\xi}_q = \mathbf{V}_q\xi_{q_{r_2}}$. Substituting $\tilde{\xi}_q = \mathbf{V}_q\xi_{q_{r_2}}$ into (7.2.6) and (7.2.2b) leads to a reduced-order algebraic subsystem of the form (7.2.5) with system matrices

$$\mathbf{A}_{q_{r_2}} = \mathbf{V}_q^{\mathrm{T}}\mathbf{A}_q\mathbf{V}_p, \quad \mathbf{B}_{q_{r_2}} = \mathbf{V}_q^{\mathrm{T}}\mathbf{B}_q \quad \text{and} \quad \mathbf{C}_{q_{r_2}} = \mathbf{V}_q^{\mathrm{T}}\mathbf{C}_q.$$

Hence the IMOR reduced-order model of (7.0.1) is of dimension $r_1 + r_2 \ll n$ and is given by

$$\xi'_{p_{r_1}} = \mathbf{A}_{p_{r_1}}\xi_{p_{r_1}} + \mathbf{B}_{p_{r_1}}\boldsymbol{u}, \tag{7.2.8a}$$

$$\xi_{q_{r_2}} = \mathbf{A}_{q_{r_2}}\xi_{p_{r_1}} + \mathbf{B}_{q_{r_2}}\boldsymbol{u}, \tag{7.2.8b}$$

$$\boldsymbol{y}_r = \mathbf{C}_{p_{r_1}}^{\mathrm{T}}\xi_{p_{r_1}} + \mathbf{C}_{q_{r_2}}^{\mathrm{T}}\xi_{q_{r_2}}. \tag{7.2.8c}$$

7.2.2 Index-aware MOR for higher index DAEs

Here, we generalize the IMOR method for higher index DAEs whose decoupled systems with or without a differential part. We also present a theoretical explanation why the conventional MOR method fail for higher index DAEs and under which conditions they can be used. We consider the two cases of decoupled systems as follows.

Decoupled systems with a differential part

Here, we assume that (7.0.1a) is of index-μ with the spectrum of its matrix pencil with at least one finite eigenvalue. This DAE can be decoupled in form (5.4.17) given by

$$\begin{aligned} \xi'_p &= \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u}, \\ -\mathcal{L}\xi'_q &= \mathbf{A}_q\xi_p - \xi_q + \mathbf{B}_q\boldsymbol{u}, \\ \boldsymbol{y} &= \mathbf{C}_p^{\mathrm{T}}\xi_p + \mathbf{C}_q^{\mathrm{T}}\xi_q, \end{aligned} \tag{7.2.9}$$

where $\xi_p \in \mathbb{R}^{n_p}$, $\mathbf{A}_p \in \mathbb{R}^{n_p \times n_p}$, $\mathbf{B}_p \in \mathbb{R}^{n_p}$, $\xi_q \in \mathbb{R}^{n_q}$, $\mathbf{A}_q \in \mathbb{R}^{n_q \times n_p}$, $\mathbf{B}_q \in \mathbb{R}^{n_q \times m}$, $\mathcal{L} \in \mathbb{R}^{n_q \times n_q}$ is a strictly lower triangular nilpotent matrix of index-μ. In Section 3.2.1, we discussed the limitation of the conventional MOR methods to DAEs using numerical examples. Here, we theoretically explains why these methods indeed fail and what are their limitation on reducing DAEs. This is done as follows. Taking the Laplace transform of (7.2.9) and simplifying, we obtain

$$\begin{aligned}\mathbf{Y}(s) &= \mathbf{C}_p^{\mathrm{T}}(s\mathrm{I} - \mathbf{A}_p)^{-1}\mathbf{B}_p\mathbf{U}(s) + \mathbf{C}_q^{\mathrm{T}}(\mathrm{I} - s\mathcal{L})^{-1}\Big[\mathbf{A}_q(s\mathrm{I} - \mathbf{A}_p)^{-1}\mathbf{B}_p + \mathbf{B}_q\Big]\mathbf{U}(s) \\ &\quad + C_q^{\mathrm{T}}(\mathrm{I} - s\mathcal{L})^{-1}\Big[\mathbf{A}_q(s\mathrm{I} - \mathbf{A}_p)^{-1}\xi_p(0) - \mathcal{L}\xi_q(0)\Big] + \mathbf{C}_p^{\mathrm{T}}(s\mathrm{I} - \mathbf{A}_p)^{-1}\xi_p(0).\end{aligned} \tag{7.2.10}$$

We already know that $\xi_p(0)$ can be chosen arbitrary while $\xi_q(0)$ has to satisfy certain hidden constraints. Thus setting $\xi_p(0) = 0$, then (7.2.10) simplifies to

$$\begin{aligned}\mathbf{Y}(s) &= \mathbf{C}_p^{\mathrm{T}}(s\mathrm{I} - \mathbf{A}_p)^{-1}\mathbf{B}_p\mathbf{U}(s) + \mathbf{C}_q^{\mathrm{T}}(\mathrm{I} - s\mathcal{L})^{-1}\Big[\mathbf{A}_q(s\mathrm{I} - \mathbf{A}_p)^{-1}\mathbf{B}_p + \mathbf{B}_q\Big]\mathbf{U}(s) \\ &\quad - \mathbf{C}_q^{\mathrm{T}}(\mathrm{I} - s\mathcal{L})^{-1}\mathcal{L}\xi_q(0).\end{aligned} \tag{7.2.11}$$

In order to derive the reduced-order model using the conventional MOR methods, we always assume vanishing initial conditions, i.e., $\xi(0) = 0$ which leads to the input-output relation $\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s)$, where $\mathbf{H}(s)$ is the transfer function. Then, $\mathbf{H}(s)$ is approximated such that $\mathbf{H}(s) - \mathbf{H}_r(s)$ is small in the suitable system norm, where $\mathbf{H}_r(s)$ is the transfer function of the reduced-order model. However, from (7.2.11), we can observe that, we can not always have this freedom for the case of DAEs since $\xi_q(0)$ does not always vanish to zero for higher index DAEs. This is only possible for the case of index-1 systems since $\mathcal{L} = 0$, which implies $\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s)$. Thus assuming vanishing initial condition does not affect index-1 DAEs. This the reason why conventional MOR methods lead to accurate reduced-order models for index-1 DAEs and fail for higher index DAEs. We further explain in depth what actually destroys the accuracy of the convention MOR methods and under which conditions can they be used. This is done as follows. If we let $\mathcal{P}(s) := -\mathbf{C}_q^{\mathrm{T}}(\mathrm{I} - s\mathcal{L})^{-1}\mathcal{L}\xi_q(0)$, then (7.2.11) can written as

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s) + \mathcal{P}(s), \tag{7.2.12}$$

where the traditional transfer function $\mathbf{H}(s)$ can be decomposed as $\mathbf{H}(s) = \mathbf{H}_p(s) + \mathbf{H}_q(s)$, where $\mathbf{H}_p(s) = \mathbf{C}_p^{\mathrm{T}}(s\mathrm{I} - \mathbf{A}_p)^{-1}\mathbf{B}_p$ and $\mathbf{H}_q(s) = \mathbf{C}_q^{\mathrm{T}}(\mathrm{I} - s\mathcal{L})^{-1}\Big[\mathbf{A}_q(s\mathrm{I} - \mathbf{A}_p)^{-1}\mathbf{B}_p + \mathbf{B}_q\Big]$, is the transfer of the differential and algebraic part, respectively. It can be proved that

$$(\mathrm{I} - s\mathcal{L})^{-1} = \sum_{j=0}^{\mu-1} \mathcal{L}^j s^j, \tag{7.2.13}$$

since $\mathcal{L}$ is a nilpotent matrix of index μ. Thus, using the identity (7.2.13), $\mathcal{P}(s)$ can be written as

$$\mathcal{P}(s) := -\mathbf{C}_q^{\mathrm{T}} \sum_{j=0}^{\mu-2} \mathcal{L}^{j+1} s^j \xi_q(0). \tag{7.2.14}$$

Using the definition of $\xi_q(0)$ from (5.4.19) and setting $\xi_p(0) = 0$, we obtain

$$\xi_q(0) = \sum_{i=1}^{\mu-1} \sum_{k=0}^{i-1} \mathcal{L}^i \mathbf{A}_q \mathbf{A}_p^k \mathbf{B}_p \boldsymbol{u}^{(i-k-1)}(0) + \sum_{i=0}^{\mu-1} \mathcal{L}^i \mathbf{B}_q \boldsymbol{u}^{(i)}(0). \tag{7.2.15}$$

Substituting (7.2.15) into (7.2.14) and simplifying, we can prove that $\mathcal{P}(s)$ is a polynomial of degree $\mu - 2$ given by

$$\mathcal{P}(s) := -\mathbf{C}_q^{\mathrm{T}} \sum_{j=0}^{\mu-2} \mathcal{L}^j s^j Q(\boldsymbol{u}(0)), \tag{7.2.16}$$

where

$$Q(\boldsymbol{u}(0)) := \mathcal{L}\xi_q(0) = \sum_{i=1}^{\mu-2} \sum_{k=0}^{i-1} \mathcal{L}^{i+1} \mathbf{A}_q \mathbf{A}_p^k \mathbf{B}_p \boldsymbol{u}^{(i-k-1)}(0) + \sum_{i=0}^{\mu-2} \mathcal{L}^{i+1} \mathbf{B}_q \boldsymbol{u}^{(i)}(0).$$

We can see that $Q(\boldsymbol{u}(0))$ is also a polynomial of degree $\mu - 2$ of the form

$$Q(\boldsymbol{u}(0)) = \zeta_0 \boldsymbol{u}^{(0)}(0) + \zeta_1 \boldsymbol{u}^{(1)}(0) + \zeta_2 \boldsymbol{u}^{(2)}(0) + \cdots + \zeta_{\mu-2} \boldsymbol{u}^{(\mu-2)}(0), \tag{7.2.17}$$

where ζ_j are constant matrices. We can observe that $\mathcal{P}(s)$ depends on the smoothness of $\boldsymbol{u}(0)$, i.e., $\boldsymbol{u}(0)$ must be at least $\mu - 2$ times differentiable. We observed that the conventional MOR methods fail if the polynomial $Q(\boldsymbol{u}(0))$ in (7.2.17) has nonzero coefficients. We note that even if $\mathcal{P}(s) = 0$ but the coefficients of $Q(\boldsymbol{u}(0))$ are nonzero, the conventional MOR methods will still lead to wrong reduced-order models or reduced-order models which are very difficult to solve. Hence $Q(\boldsymbol{u}(0))$ is the hidden polynomial that destroys the accuracy of the conventional MOR methods when applied on higher index DAEs. However there are some special cases where conventional MOR methods can be applied to higher index DAEs and lead to accurate reduced-order models. This happens when the coefficients of $Q(\boldsymbol{u}(0))$ are all zeros. This is can be illustrated using index-2 and -3 DAEs as follows.

(1) For index-2 DAEs, we substitute $\mu = 2$ into (7.2.16) and we obtain

$$\mathcal{P}(s) := -\mathbf{C}_q^{\mathrm{T}} Q(\boldsymbol{u}(0)), \tag{7.2.18}$$

where $Q(\boldsymbol{u}(0)) = \mathcal{L}\mathbf{B}_q\boldsymbol{u}(0)$. We can observe that index-2 DAEs cannot be reduced by conventional MOR methods if $\mathcal{L}\mathbf{B}_q \neq \mathbf{0}$ even if $\mathcal{P}(s) = \mathbf{0}$. But if $\mathcal{L}\mathbf{B}_q = \mathbf{0}$ conventional MOR methods can lead to accurate reduced-order models for index-2 DAEs. This is illustrated in the example below.

Example 7.2.1 In this example, we use the system matrices from Example 3.2.1 and the two cases of control input matrix $\mathbf{B}$. Both cases the DAEs are of index-2 since they have the same matrix pencil $(\mathbf{E}, \mathbf{A})$. Also since $\det(\lambda\mathbf{E}-\mathbf{A}) = 2\lambda+3 \neq 0$, thus both DAEs are solvable and their decoupled system have a differential part. These DAEs can be decoupled into the form (7.2.9). Below, we discuss the affect of conventional MOR method on the two DAEs.

(i) Here, we use control input matrix from Example 3.2.1(i), where the conventional MOR method lead to an accurate solution. This system is decoupled into 1 differential and 2 algebraic equations using the explicit decoupling method which lead to decoupled system of the form (5.3.15) with system matrices given by $\mathbf{A}_p = -\frac{3}{2}$, $\mathbf{B}_p = -\frac{3}{4}$, $\mathbf{A}_{q,1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{B}_{q,1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{A}_{q,0} = \begin{pmatrix} -\frac{4}{3} \\ -\frac{1}{3} \end{pmatrix}$, $\mathbf{B}_{q,0} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$, $\mathbf{A}_{q,01} = \begin{pmatrix} -1 & \frac{11}{30} \\ 0 & 1 \end{pmatrix}$, $\mathbf{C}_p = \frac{2}{3}$, $\mathbf{C}_{q,1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{C}_{q,0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Using these matrices, we can write the decoupled system into the form (7.2.9) with system matrices $\mathcal{L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & \frac{11}{30} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $\mathbf{A}_q = \begin{pmatrix} 0 \\ 0 \\ -\frac{4}{3} \\ -\frac{1}{3} \end{pmatrix}$ and

$\mathbf{B}_q = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$ and $\mathbf{C}_q = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$. Since, this is an index-2 DAE, the polynomial $\mathcal{P}(s)$ is of the form (7.2.18) and $Q(\boldsymbol{u}(0))$ is given by $Q(\boldsymbol{u}(0)) = \mathcal{L}\mathbf{B}_q\boldsymbol{u}(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}} \boldsymbol{u}(0)$. We can observe that $\mathcal{L}\mathbf{B}_q = \mathbf{0}$ thus, the conventional MOR methods can be used to reduce this system even if the DAE is of higher index. This is the reason why the PRIMA method lead to reduced-order model in Example 3.2.1(i) which was easier to solve numerically.

(ii) Here, we use control matrix from Example 3.2.1(ii), where the conventional MOR method lead to a reduced-order model which was very difficult to solve accurately since it needed big time steps. This system is also decoupled into 1 differential and 2 algebraic equations using the explicit decoupling method which leads to decoupled system of the form (5.3.15) with system matrices $\mathbf{A}_p = -\frac{3}{2}$, $\mathbf{B}_p = -\frac{3}{4}$, $\mathbf{A}_{q,1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{B}_{q,1} = \begin{pmatrix} \frac{11}{30} \\ 1 \end{pmatrix}$, $\mathbf{A}_{q,0} = \begin{pmatrix} -\frac{4}{3} \\ -\frac{1}{3} \end{pmatrix}$, $\mathbf{B}_{q,0} = \begin{pmatrix} \frac{3}{4} \\ 4 \end{pmatrix}$, $\mathbf{A}_{q,01} = \begin{pmatrix} -1 & \frac{11}{30} \\ 0 & 1 \end{pmatrix}$, $\mathbf{C}_p = \frac{2}{3}$, $\mathbf{C}_{q,1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{C}_{q,0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Using these matrices, we can write the decoupled system into the form (7.2.9). We observe that $\mathcal{L}$, $\mathbf{A}_q$ and $\mathbf{C}_q$ remains the same as in (i) but $\mathbf{B}_q$ changes to $\mathbf{B}_q = \begin{pmatrix} \frac{11}{30} & 1 & \frac{3}{4} & 4 \end{pmatrix}^{\mathrm{T}}$ since we just changed matrix $\mathbf{B}$. For this case $Q(\boldsymbol{u}(0))$ is given by $Q(\boldsymbol{u}(0)) = \mathcal{L}\mathbf{B}_q\boldsymbol{u}(0) = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}} \boldsymbol{u}(0)$. We can observe that $\mathcal{L}\mathbf{B}_q \neq \mathbf{0}$ but $\mathbf{C}_q\mathcal{L}\mathbf{B}_q = \mathbf{0}$. From Example 3.2.1(ii), we saw that the PRIMA reduced-order model was very difficult to solve since it needs bigger time steps even if $\mathcal{P}(s) = \mathbf{0}$. Thus, this agrees with our theory that conventional MOR methods cannot lead to accurate reduced-order systems for this class of DAEs even if $\mathcal{P}(s) = \mathbf{0}$ but $Q(\boldsymbol{u}(0)) \neq 0$.

(2) For index-3 DAEs, we substitute $\mu = 3$ into (7.2.16), we obtain

$$\mathcal{P}(s) := -\mathbf{C}_q^{\mathrm{T}}\Big[Q(\boldsymbol{u}(0)) + s\mathcal{L}Q(\boldsymbol{u}(0))\Big], \tag{7.2.19}$$

where $Q(\boldsymbol{u}(0)) = \Big[\mathcal{L}^2\mathbf{A}_q\mathbf{B}_p + \mathcal{L}\mathbf{B}_q\Big]\boldsymbol{u}(0) + \mathcal{L}^2\mathbf{B}_q\boldsymbol{u}'(0)$. We can observe that $Q(\boldsymbol{u}(0))$ has nonzero coefficients if either $\mathcal{L}\mathbf{B}_q \neq \mathbf{0}$ or $\mathcal{L}^2\mathbf{A}_q \neq \mathbf{0}$. Thus, these are the cases where the conventional MOR methods will fail for the case of index-3 systems.

Example 7.2.2 In this example, we use system matrices from Example 3.2.2, which is a generator model of index-3. This system can be decoupled into a decoupled system of the form (5.4.35) with system matrices given by
$\mathbf{A}_p = -2$, $\mathbf{B}_p = 2$, $\mathbf{A}_{q,2} = 0$, $\mathbf{B}_{q,2} = 1$, $\mathbf{A}_{q,1} = 0$, $\mathbf{B}_{q,1} = 1$, $\mathbf{C}_{q,2} = 0$,
$\mathbf{C}_{q,1} = 0$, $\mathbf{A}_{q_{1,2}} = 1$, $\mathbf{C}_p = 1$,

$$\mathbf{A}_{q,0} = \begin{pmatrix} 0\\0\\0\\0\\0\\0 \end{pmatrix}, \mathbf{B}_{q,0} = \begin{pmatrix} 0\\1\\-1\\1\\-1\\-1 \end{pmatrix}, \mathbf{A}_{q_{0,1}} = \begin{pmatrix} 1\\0\\0\\0\\0\\0 \end{pmatrix}, \mathbf{A}_{q_{0,2}} = \begin{pmatrix} 0\\0\\-1\\-1\\0\\0 \end{pmatrix} \text{ and } \mathbf{C}_{q,0} = \begin{pmatrix} 0\\0\\0\\0\\0\\1 \end{pmatrix}.$$

Thus this system can also written in the form (7.2.9) with system matrices given by

$$\mathcal{L} = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \quad \mathbf{A}_q = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \quad \mathbf{B}_q = \begin{pmatrix} 1\\1\\0\\1\\-1\\1\\-1\\-1 \end{pmatrix}, \quad \mathbf{C}_q = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\1 \end{pmatrix}.$$

Since this is an index-3 DAE its polynomial $\mathcal{P}(s)$ is of the form (7.2.19) and its hidden polynomial $\mathcal{Q}(\boldsymbol{u}(0))$ is given by

$$\mathcal{Q}(\boldsymbol{u}(0)) = \begin{pmatrix} 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}^{\mathrm{T}} \boldsymbol{u}(0) + \begin{pmatrix} 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}^{\mathrm{T}} \boldsymbol{u}'(0)$$

Thus the convectional MOR methods when applied on this system will fail since $\mathcal{Q}(\boldsymbol{u}(0))$ has nonzero coefficients for arbitrary $\boldsymbol{u}(0)$. This is the reason why the PRIMA method lead to unsolvable reduced order model in Example 3.2.2.

In conclusion, the limitations of conventional MOR methods to reduce DAEs can be summarized as follows. Almost all conventional MOR method aims at approximating the so called transfer function $\mathbf{H}(s)$ and ignore the polynomial $\mathcal{P}(s)$. As a result $\mathbf{H}_p(s)$ is well approximated but $\mathbf{H}_q(s)$ may be inaccurate since some of its information is in $\mathcal{P}(s)$. Thus most of the important information of the DAE is always lost. We have discussed that the conventional MOR methods fail if the hidden polynomial $\mathcal{Q}(\boldsymbol{u}(0))$ has nonzero coefficients. However, the conventional MOR method can be used to reduced index-1

DAEs since their polynomial $\mathcal{P}(s)$ does not exist and for some special cases of higher index DAEs, if hidden polynomial $\mathcal{Q}(\boldsymbol{u}(0))$ has zero coefficients. We can now see that it is extremely difficult to check wether the conventional MOR method will work on higher index DAEs or not. Hence, the best way to avoid this problem, is to first splitting the DAE system into differential and algebraic parts before applying any model order reduction method.

For the case of IMOR method. This can be done as follows. Separating (7.2.9) into

$$\begin{aligned} \xi_p' &= \mathbf{A}_p \xi_p + \mathbf{B}_p \boldsymbol{u}, \\ \boldsymbol{y}_p &= \mathbf{C}_p^{\mathrm{T}} \xi_p, \end{aligned} \tag{7.2.20}$$

and

$$-\mathcal{L}\xi_q' = \mathbf{A}_q \xi_p - \xi_q + \mathbf{B}_q \boldsymbol{u}, \tag{7.2.21a}$$

$$\boldsymbol{y}_q = \mathbf{C}_q^{\mathrm{T}} \xi_q, \tag{7.2.21b}$$

subsystems. Systems (7.2.20) and (7.2.21) are the differential and algebraic subsystems respectively. The differential subsystem (7.2.20) can be reduced by substituting $\xi_p = \mathbf{V}_p \xi_{p_r}$, where $\mathbf{V}_p$ can be constructed using the Arnoldi process. Thus, the reduced-order model of (7.2.20) is given by

$$\begin{aligned} \xi_{p_r}' &= \mathbf{A}_{p_r} \xi_{p_r} + \mathbf{B}_{p_r} \boldsymbol{u}, \\ \boldsymbol{y}_{p_r} &= \mathbf{C}_{p_r}^{\mathrm{T}} \xi_{p_r}, \end{aligned} \tag{7.2.22}$$

where $\mathbf{A}_{p_r} = \mathbf{V}_p^{\mathrm{T}} \mathbf{A}_p \mathbf{V}_p \in \mathbb{R}^{r \times r}$, $\mathbf{B}_{p_r} = \mathbf{V}_p^{\mathrm{T}} \mathbf{B}_p \in \mathbb{R}^{r \times m}$ and $\mathbf{C}_p = \mathbf{V}_p^{\mathrm{T}} \mathbf{C}_p \in \mathbb{R}^{r \times \ell}$, such that $r \ll n_p$. The transfer function of the reduced-order differential subsystem (7.2.22) is given by $\mathbf{H}_{p_r}(s) = \mathbf{C}_{p_r}^{\mathrm{T}} (s\mathrm{I} - \mathbf{A}_{p_r})^{-1} \mathbf{B}_{p_r}$. Next, we seek a reduced-order model of algebraic subsystem (7.2.21) which can be written as

$$-\mathcal{L}_r \xi_{q_r}' = \mathbf{A}_{q_r} \xi_{p_r} - \xi_{q_r} + \mathbf{B}_{q_r} \boldsymbol{u}, \tag{7.2.23a}$$

$$\boldsymbol{y}_{q_r} = \mathbf{C}_{q_r}^{\mathrm{T}} \xi_{q_r}, \tag{7.2.23b}$$

where $\mathbf{A}_{q_r} \in \mathbb{R}^{\tau \times r}$, $\mathbf{B}_{q_r} \in \mathbb{R}^{\tau \times m}$ and $\mathbf{C}_{q_r} \in \mathbb{R}^{\tau \times \ell}$, $\tau < n_q$ and its transfer function can be written as $\mathbf{H}_{q_r}(s) = \mathbf{C}_{q_r}^{\mathrm{T}} (\mathrm{I} - s\mathcal{L}_r)^{-1} \left[\mathbf{A}_{q_r} (s\mathrm{I} - \mathbf{A}_{p_r})^{-1} \mathbf{B}_{p_r} + \mathbf{B}_{q_r} \right]$. The matrices of the reduced-order model (7.2.23) can be constructed as follows. If we substitute $\xi_p = \mathbf{V}_p \xi_{p_r}$ into (7.2.2a) and rearranging, we obtain

$$\tilde{\xi}_q = \mathcal{L}\tilde{\xi}_q' + \mathbf{A}_q \mathbf{V}_p \xi_{p_r} + \mathbf{B}_q \boldsymbol{u}, \tag{7.2.24}$$

where $\tilde{\xi}_q$ is the approximation of ξ_q induced by the reduction of ξ_p. Equation (7.2.24) can be written as

$$\tilde{\xi}_q = \sum_{k=0}^{\mu-1} \mathcal{L}^k (\mathbf{V}_p \xi_{p_r}^{(k)} + \mathbf{B}_q \boldsymbol{u}^{(k)}), \tag{7.2.25}$$

where $\xi_{p_r}^{(k)} = \frac{\mathrm{d}^k \xi_{p_r}}{\mathrm{d}t^k}$ and $\boldsymbol{u}^{(k)} = \frac{\mathrm{d}^k \boldsymbol{u}}{\mathrm{d}t^k}$. We can observe, that the reduction of the differential part of the decoupled system, which confines ξ_p to the subspace $\mathcal{V}_p$, spanned by $\mathbf{V}_p$, then also $\xi_p^{(k)}$, $k = 1, \cdots, \mu - 1$, belongs to the same space. Thus from (7.2.25), we observe that for the algebraic variable ξ_q, we have the restriction

$$\xi_q \in \mathcal{V}_q = \mathcal{K}_\mu(\mathcal{L}, \mathbf{R}_q), \tag{7.2.26}$$

where $\mathbf{R}_q = \begin{bmatrix} \mathbf{B}_q & \mathbf{A}_q \mathbf{V}_p \end{bmatrix} \in \mathbb{R}^{n_q \times (r+1)m}$. We denote by $\mathbf{V}_q$ an orthonormal basis of $\mathcal{V}_q$ so that $\mathbf{V}_q^{\mathrm{T}} \mathbf{V}_q = \mathrm{I}$. We can then write $\xi_q = \mathbf{V}_q \xi_{q_r}$. Substituting $\xi_q = \mathbf{V}_q \xi_{q_r}$ and $\xi_p = \mathbf{V}_p \xi_{p_r}$ into (7.2.21) leads to a reduced-order algebraic subsystem of the form (7.2.23) with system matrices :

$$\mathcal{L}_r = \mathbf{V}_q^{\mathrm{T}} \mathcal{L} \mathbf{V}_q,\ \mathbf{A}_{q_r} = \mathbf{V}_q^{\mathrm{T}} \mathbf{A}_p \mathbf{V}_p,\ \mathbf{B}_{q_r} = \mathbf{V}_q^{\mathrm{T}} \mathbf{B}_q \quad \text{and} \quad \mathbf{C}_{q_r} = \mathbf{V}_q^{\mathrm{T}} \mathbf{C}_q.$$

Thus, the IMOR reduced-order model of (7.0.1) is given by

$$\begin{aligned} \xi_{p_r}' &= \mathbf{A}_{p_r} \xi_{p_r} + \mathbf{B}_{p_r} \boldsymbol{u}, \\ -\mathcal{L}_r \xi_{q_r}' &= \mathbf{A}_{q_r} \xi_{p_r} - \xi_{q_r} + \mathbf{B}_{q_r} \boldsymbol{u}, \\ \boldsymbol{y}_r &= \mathbf{C}_{p_r}^{\mathrm{T}} \xi_{p_r} + \mathbf{C}_{q_r}^{\mathrm{T}} \xi_{q_r}, \end{aligned} \tag{7.2.27}$$

with total dimension $r + \tau \ll n$, where r and $\tau = \dim(\mathcal{V}_q)$ are the dimension of the reduced-order differential and algebraic parts, respectively. The transfer function of the IMOR reduced model is equal to the sum of the transfer function of the differential and algebraic parts given by $\mathbf{H}_r(s) = \mathbf{H}_{p_r}(s) + \mathbf{H}_{q_r}(s)$.

Decoupled systems without differential part

Here, we consider the case of decoupled systems without a differential part. Assume (7.0.1a) is of index-μ with the spectrum of its matrix pencil with no finite eigenvalues. Thus, this DAE can be decoupled into the form (5.4.32) given by

$$\begin{aligned} -\mathcal{L}\xi_q' &= -\xi_q + \mathbf{B}_q \boldsymbol{u}, \\ \boldsymbol{y} &= \mathbf{C}_q^{\mathrm{T}} \xi_q, \end{aligned} \tag{7.2.28}$$

where $\xi_q \in \mathbb{R}^n$, $\mathbf{A}_q \in \mathbb{R}^{n\times n}\,\mathbf{B}_q \in \mathbb{R}^{n\times m}$, $\mathcal{L} \in \mathbb{R}^{n\times n}$ is a strictly lower triangular nilpotent matrix of index-μ. Taking the Laplace transform of (7.2.28) and simplifying, we obtain

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s) + \mathcal{P}(s),$$

where $\mathbf{H}(s) = \mathbf{C}_q^{\mathrm{T}}(\mathrm{I} - s\mathcal{L})^{-1}\mathbf{B}_q$ and $\mathcal{P}(s) = \mathbf{C}_q^{\mathrm{T}}(\mathrm{I} - s\mathcal{L})^{-1}\mathcal{L}\xi_q(0)$. Next, we discuss wether conventional MOR methods also fail for these class of DAEs. This is done by analyzing polynomial $\mathcal{P}(s)$ as follows. Using the identity (7.2.13), $\mathbf{H}(s)$ and $\mathcal{P}(s)$ can be written as

$$\mathbf{H}(s) = \mathbf{C}_q^{\mathrm{T}} \sum_{j=0}^{\mu-1} \mathcal{L}^j \mathbf{B}_q s^j, \quad \text{and} \quad \mathcal{P}(s) = -\mathbf{C}_q^{\mathrm{T}} \sum_{j=0}^{\mu-2} \mathcal{L}^{j+1} s^j \xi_q(0). \tag{7.2.29}$$

Using (7.2.15) and ignoring the differential part contribution, we can write $\xi_q(0)$ as

$$\xi_q(0) = \sum_{i=0}^{\mu-1} \mathcal{L}^i \mathbf{B}_q \boldsymbol{u}^{(i)}(0). \tag{7.2.30}$$

From (7.2.29) and (7.2.30), we observe that the moments of $\mathbf{H}(s)$ and coefficients of $\xi_q(0)$ lie in the same subspace $\mathcal{K}_\mu(\mathcal{L}, \mathbf{B}_q)$. Thus approximating $\mathbf{H}(s)$ is enough to approximate these class DAEs. Thus, the assumption of vanishing initial condition, used by conventional MOR methods does not affect the reduced-order models of the DAEs with only infinite spectrum. Hence conventional MOR methods can be used to reduce this class of DAEs. We can also observe that this subspace coincides with that in (7.2.14) if we ignore the differential components for the IMOR method. Thus, ξ_q lies in the subspace

$$\xi_q \in \mathcal{V}_q = \mathcal{K}_\mu(\mathcal{L}, \mathbf{B}_q),$$

we denote by $\mathbf{V}_q$ an orthonormal basis of $\mathcal{V}_q$ so that $\mathbf{V}_q^{\mathrm{T}}\mathbf{V}_q = \mathrm{I}$. Then, we can write $\xi_q = \mathbf{V}_q\xi_{q_r}$. We can also reduce the order of the algebraic part by substituting $\xi_q = \mathbf{V}_q\xi_{q_r}$ into (7.2.28). This leads to a IMOR reduced-order model of (7.0.1) which is given by

$$\begin{aligned} -\mathcal{L}_r\xi'_{q_r} &= -\xi_{q_r} + \mathbf{B}_{q_r}\boldsymbol{u}, \\ \boldsymbol{y}_r &= \mathbf{C}_{q_r}^{\mathrm{T}}\xi_{q_r}, \end{aligned} \tag{7.2.31}$$

with system matrices constructed as

$$\mathcal{L}_r = \mathbf{V}_q^{\mathrm{T}}\mathcal{L}\mathbf{V}_q,\ \mathbf{B}_{q_r} = \mathbf{V}_q^{\mathrm{T}}\mathbf{B}_q \quad \text{and} \quad \mathbf{C}_{q_r} = \mathbf{V}_q^{\mathrm{T}}\mathbf{C}_q$$

and its transfer function is given by $\mathbf{H}_r(s) = \mathbf{C}_{q_r}^{\mathrm{T}}(\mathrm{I} - s\mathcal{L}_r)^{-1}\mathbf{B}_{q_r}$. For comparison with other existing MOR methods, we can rewrite the IMOR reduced-order models of either (7.2.27) or (7.2.31), in descriptor form given by

$$\begin{aligned} \tilde{\mathbf{E}}_r\xi'_r &= \tilde{\mathbf{A}}_r\xi_r + \tilde{\mathbf{B}}_r\boldsymbol{u} \\ \tilde{\boldsymbol{y}}_r &= \tilde{\mathbf{C}}_r^{\mathrm{T}}\xi_r, \end{aligned}$$

where $\tilde{\mathbf{E}}_r = \begin{pmatrix} \mathrm{I} & \mathbf{0} \\ \mathbf{0} & -\mathcal{L}_r \end{pmatrix}$, $\tilde{\mathbf{A}}_r = \begin{pmatrix} \mathbf{A}_{p_r} & \mathbf{0} \\ \mathbf{A}_{q_r} & -\mathrm{I} \end{pmatrix}$, $\tilde{\mathbf{B}}_r = \begin{pmatrix} \mathbf{B}_{p_r} \\ \mathbf{B}_{q_r} \end{pmatrix}$, $\tilde{\mathbf{C}}_r = \begin{pmatrix} \mathbf{C}_{p_r} \\ \mathbf{C}_{q_r} \end{pmatrix}$ and $\tilde{\xi} = \begin{pmatrix} \xi_{p_r} \\ \xi_{q_r} \end{pmatrix}$ for DAEs with differential parts and $\tilde{\mathbf{E}}_r = -\mathcal{L}_r$, $\tilde{\mathbf{A}}_r = -\mathrm{I}$, $\tilde{\mathbf{B}}_r = \mathbf{B}_{q_r}$, $\tilde{\mathbf{C}}_r = \mathbf{C}_{q_r}$ and $\tilde{\xi} = \xi_{q_r}$ for DAEs without a differential part and the transfer function of the reduced-order model is given by $\tilde{\mathbf{H}}_r(s) = \tilde{\mathbf{C}}_r(s\tilde{\mathbf{E}}_r - \tilde{\mathbf{A}}_r)^{-1}\tilde{\mathbf{B}}_r$.

7.3 Simple examples

In this Section, we illustrate the IMOR method using small DAE examples with higher index.

Example 7.3.1 In this example, we use the decoupled system matrices from Example 7.2.1. We can recall that these decoupled systems are derived from system matrix of Example 3.2.1, for the two cases of control input matrix **B**.
(i) Using system matrices from Example 7.2.1(i), the decoupled system of the index-2

DAE can be written as

$$
\begin{aligned}
\xi_p' &= -\frac{3}{2}\xi_p - \frac{3}{4}\boldsymbol{u},\\
\xi_{q,1} &= \begin{pmatrix}0\\0\end{pmatrix}\xi_p + \begin{pmatrix}0\\0\end{pmatrix}\boldsymbol{u},\\
\xi_{q,0} &= \begin{pmatrix}-\frac{4}{3}\\-\frac{1}{3}\end{pmatrix}\xi_p - \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\boldsymbol{u} + \begin{pmatrix}-1 & \frac{11}{30}\\0 & 1\end{pmatrix}\xi_{q,1}',\\
\boldsymbol{y} &= \frac{2}{3}\xi_p + \begin{pmatrix}0 & 1\end{pmatrix}\xi_{q,1} + \begin{pmatrix}1 & 0\end{pmatrix}\xi_{q,0}.
\end{aligned}
\tag{7.3.1}
$$

The transfer function of this decoupled system can also be decompose as $\mathbf{H}(s) = \mathbf{H}_p(s) + \mathbf{H}_q(s)$, where $\mathbf{H}_p(s) = \frac{-1}{2s+3}$ and $\mathbf{H}_q(s) = \frac{2}{2s+3} - \frac{1}{2}$ which coincides with the transfer function of the DAE in Example 3.2.1(i) given by $\mathbf{H}(s) = \mathbf{C}^{\mathrm{T}}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$. The desired output solution also coincides with that of the original DAE given by $\boldsymbol{y}(t) = -\frac{1}{2}\boldsymbol{u}(t) - \frac{2}{3}e^{-\frac{3}{2}t}\Big[\xi_p(0) - \frac{3}{4}\int_0^{\mathrm{T}} \boldsymbol{u}(\tau)e^{\frac{3}{2}\tau}\,\mathrm{d}\tau\Big]$. Next, we need to reduce this system using the IMOR method. Before, we use the IMOR method, we need to first use the Algebraic Elimination method for index-2 DAEs. Thus using the AE method the decoupled system (7.3.1) can be reduced to

$$
\begin{aligned}
\xi_p' &= -\frac{3}{2}\xi_p - \frac{3}{4}\boldsymbol{u},\\
\xi_{q,1} &= \begin{pmatrix}0\\0\end{pmatrix}\xi_p + \begin{pmatrix}0\\0\end{pmatrix}\boldsymbol{u},\\
\tilde{\xi}_{q_1,0} &= -\frac{4}{3}\xi_p - \frac{1}{2}\boldsymbol{u} + \begin{pmatrix}-1 & \frac{11}{30}\end{pmatrix}\xi_{q,1}',\\
\boldsymbol{y}_r &= \frac{2}{3}\xi_p + \begin{pmatrix}0 & 1\end{pmatrix}\xi_{q,1} + \tilde{\xi}_{q_1,0}.
\end{aligned}
\tag{7.3.2}
$$

Then the decoupled system can be written in the form (7.2.9) with system matrices

$$
\mathbf{A}_p = -\frac{3}{2},\ \mathbf{B}_p = -\frac{3}{4},\ \mathbf{C}_p = \frac{2}{3},\ \mathcal{L} = \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\-1 & \frac{11}{30} & 0\end{pmatrix},\ \mathbf{A}_q = \begin{pmatrix}0\\0\\-\frac{4}{3}\end{pmatrix},\ \mathbf{B}_q = \begin{pmatrix}0\\0\\-\frac{1}{2}\end{pmatrix},\ \mathbf{C}_q = \begin{pmatrix}0\\1\\1\end{pmatrix}.
$$

We can observe that the differential part cannot be reduced any further, thus setting $\mathbf{V}_p = 1$. In order to further reduce the algebraic parts, we use (7.2.26) to construct the Krylov subspace of order-$\mu = 2$, $\mathcal{K}_2(\mathcal{L}, b_q) = \mathrm{Span}\{\mathbf{R}_q, \mathcal{L}\mathbf{R}_q\}$, where

$\mathbf{R}_q = \begin{bmatrix}\mathbf{B}_q & \mathbf{A}_q\mathbf{V}_p\end{bmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -\frac{1}{2} & -\frac{4}{3}\end{pmatrix}$. Then the orthonormal basis is given by $\mathbf{V}_q = \begin{pmatrix}0 & 0 & 1\end{pmatrix}^{\mathrm{T}}$. Then the reduced-order model is of the form (7.2.27) with system matrices given by $\mathbf{A}_{p_r} = -\frac{3}{2}$, $\mathbf{B}_{p_r} = -\frac{3}{4}$, $\mathbf{C}_{p_r} = \frac{2}{3}$, $\mathcal{L}_r = 0$, $\mathbf{A}_{q_r} = -\frac{4}{3}$, $\mathbf{B}_{q_r} = -\frac{1}{2}$, $\mathbf{C}_{q_r} = 1$. We can easily check that the transfer functions and output solutions of the reduced-order model and original DAE model coincide. Thus the reduced-order model of this DAE is given by

$$\tilde{\mathbf{E}}_r = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad \tilde{\mathbf{A}}_r = \begin{pmatrix}-\frac{3}{2} & 0\\ -\frac{4}{3} & -1\end{pmatrix}, \quad \tilde{\mathbf{B}}_r = \begin{pmatrix}-\frac{3}{4}\\ -\frac{1}{2}\end{pmatrix}, \quad \tilde{\mathbf{C}}_r = \begin{pmatrix}\frac{2}{3}\\ 1\end{pmatrix}. \tag{7.3.3}$$

Hence the DAE is reduced from dimension 5 to 2 leading to an accurate reduced-order model.

(ii) Using the matrices from Example 7.2.1(ii), the decoupled system of the index-2 DAE can be written as

$$\begin{aligned}
\xi_p' &= -\frac{3}{2}\xi_p - \frac{3}{4}\boldsymbol{u},\\
\xi_{q,1} &= \begin{pmatrix}0\\0\end{pmatrix}\xi_p + \begin{pmatrix}\frac{11}{30}\\ 1\end{pmatrix}\boldsymbol{u},\\
\xi_{q,0} &= \begin{pmatrix}-\frac{4}{3}\\ -\frac{1}{3}\end{pmatrix}\xi_p + \begin{pmatrix}\frac{3}{4}\\ 4\end{pmatrix}\boldsymbol{u} + \begin{pmatrix}-1 & \frac{11}{30}\\ 0 & 1\end{pmatrix}\xi_{q,1}',\\
\boldsymbol{y} &= \frac{2}{3}\xi_p + \begin{pmatrix}0 & 1\end{pmatrix}\xi_{q,1} + \begin{pmatrix}1 & 0\end{pmatrix}\xi_{q,0},
\end{aligned} \tag{7.3.4}$$

The transfer function of this decoupled system can also be decompose as $\mathbf{H}(s) = \mathbf{H}_p(s) + \mathbf{H}_q(s)$, where $\mathbf{H}_p(s) = \frac{-1}{2s+3}$ and $\mathbf{H}_q(s) = \frac{2}{2s+3} + \frac{7}{4}$ which coincides with the transfer function of the DAE in Example 3.2.1(ii) given by $\mathbf{H}(s) = \mathbf{C}^{\mathrm{T}}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$. The desired output solution also coincides with that of the original DAE system given by $\boldsymbol{y}(t) = \frac{7}{4}\boldsymbol{u}(t) - \frac{2}{3}e^{-\frac{3}{2}t}\Big[\xi_p(0) - \frac{3}{4}\int_0^{\mathrm{T}} \boldsymbol{u}(\tau)e^{\frac{3}{2}\tau}\,\mathrm{d}\tau\Big]$. Next, we need to reduce this system using the IMOR method. Before, we use the IMOR method, we need to first use the Algebraic Elimination method for index-2 DAEs. Thus, using the AE method the decoupled system (7.3.4) can be reduced to

$$\begin{aligned}
\xi_p' &= -\frac{3}{2}\xi_p - \frac{3}{4}\boldsymbol{u},\\
\xi_{q,1} &= \begin{pmatrix}0\\0\end{pmatrix}\xi_p + \begin{pmatrix}\frac{11}{30}\\ 1\end{pmatrix}\boldsymbol{u},\\
\tilde{\xi}_{q_1,0} &= -\frac{4}{3}\xi_p + \frac{3}{4}\boldsymbol{u} + \begin{pmatrix}-1 & \frac{11}{30}\end{pmatrix}\xi_{q,1}',\\
\boldsymbol{y} &= \frac{2}{3}\xi_p + \begin{pmatrix}0 & 1\end{pmatrix}\xi_{q,1} + \tilde{\xi}_{q_1,0}.
\end{aligned} \tag{7.3.5}$$

We can now apply the IMOR method as follow. The AE reduced-order model (7.3.5) can be written in form (7.2.9) with system matrices given by

$$\mathbf{A}_p = -\frac{3}{2},\ \mathbf{B}_p = -\frac{3}{4},\ \mathbf{C}_p = \frac{2}{3},\ \mathcal{L} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & \frac{11}{30} & 0 \end{pmatrix},\ \mathbf{A}_q = \begin{pmatrix} 0 \\ 0 \\ -\frac{4}{3} \end{pmatrix}, \mathbf{B}_q = \begin{pmatrix} \frac{11}{30} \\ 1 \\ \frac{3}{4} \end{pmatrix},\ \mathbf{C}_q = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

We can observe that the differential part cannot be reduced any further, thus we can just set $\mathbf{V}_p = 1$. In order to further reduce the algebraic parts, we use (7.2.26) to construct the Krylov subspace of order $\mu = 2$: $\mathcal{K}_2(\mathcal{L}, b_q) = \text{Span}\{\mathbf{R}_q, \mathcal{L}\mathbf{R}_q\}$, where $\mathbf{R}_q = \begin{bmatrix} \mathbf{B}_q & \mathbf{A}_q\mathbf{V}_p \end{bmatrix} = \begin{pmatrix} \frac{11}{30} & 0 \\ 1 & 0 \\ \frac{3}{4} & -\frac{4}{3} \end{pmatrix}$. Then the orthonormal basis is given by $\mathbf{V}_q = \begin{pmatrix} \frac{746}{2167} & 0 \\ \frac{2089}{2225} & 0 \\ 0 & 1 \end{pmatrix}$. Thus the IMOR reduced-order model is of the form (7.2.27) with system matrices given by

$$\mathbf{A}_{p_r} = -\frac{3}{2},\ \mathbf{B}_{p_r} = -\frac{3}{4},\ \mathbf{C}_{p_r} = \frac{2}{3},\ \mathcal{L}_r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ \mathbf{A}_{q_r} = \begin{pmatrix} 0 \\ -\frac{4}{3} \end{pmatrix}, \mathbf{B}_{q_r} = \begin{pmatrix} \frac{2225}{2089} \\ \frac{3}{4} \end{pmatrix},\ \mathbf{C}_{q_r} = \begin{pmatrix} \frac{2089}{2225} & 1 \end{pmatrix}.$$

We can observe the transfer functions and the output solutions of the original and reduced-order models coincide. This reduced-order model can be written in descriptor form with system matrices given by

$$\tilde{\mathbf{E}}_r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \tilde{\mathbf{A}}_r = \begin{pmatrix} -\frac{3}{2} & 0 & 0 \\ 0 & -1 & 0 \\ -\frac{4}{3} & 0 & -1 \end{pmatrix},\ \tilde{\mathbf{B}}_r = \begin{pmatrix} -\frac{3}{4} \\ \frac{2225}{2089} \\ \frac{3}{4} \end{pmatrix},\ \tilde{\mathbf{C}}_r = \begin{pmatrix} \frac{2}{3} \\ \frac{2089}{2225} \\ 1 \end{pmatrix}^{\mathrm{T}}. \tag{7.3.6}$$

Hence the DAE is reduced from dimension 5 to 3 leading to an accurate reduced-order model using the IMOR method.

Example 7.3.2 In this example, we used the same matrices from Example 7.2.2, which is an index-3 DAE problem. This system can be decoupled as

$$\begin{aligned}
\xi_p' &= -2\xi_p + 2\boldsymbol{u}, \\
\xi_{q,2} &= 0\xi_p + 1\boldsymbol{u}, \\
\xi_{q,1} &= 0\xi_p + 1\boldsymbol{u} + \xi_{q,2}', \\
\xi_{q,0} &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\xi_p + \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}\boldsymbol{u} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\xi_{q,1}' + \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}\xi_{q,2}',
\end{aligned} \tag{7.3.7}$$

$$\boldsymbol{y} = \xi_p + 0\xi_{q,2} + 0\xi_{q,1} + \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}\xi_{q,0}.$$

The transfer function of this decoupled system can also be decompose as $\mathbf{H}(s) = \mathbf{H}_p(s) + \mathbf{H}_q(s)$, where $\mathbf{H}_p(s) = \frac{2}{s+2}$ and $\mathbf{H}_q(s) = -1$ which coincides with the transfer function of the DAE in Example 3.2.2 given by $\mathbf{H}(s) = \mathbf{C}^{\mathrm{T}}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} = -\frac{s}{s+2}$. The desired output solution also coincides with that of the original DAE given by

$$\boldsymbol{y}(t) = -\boldsymbol{u}(t) + e^{-2t}\Big[\xi_p(0) - 2\int_0^{\mathrm{T}} \boldsymbol{u}(\tau)e^{2\tau}\,\mathrm{d}\tau\Big].$$

Next, we need to reduce this system using the IMOR method. Before, we use the IMOR method, we need to first use the Algebraic Elimination method for index-2 DAEs. Thus using the AE method the decoupled system (7.3.7) can be reduced to

$$\xi_p' = -2\xi_p + 2\boldsymbol{u}, \tag{7.3.8a}$$

$$\tilde{\xi}_{q,0} = 0\xi_p - \boldsymbol{u}, \tag{7.3.8b}$$

$$\tilde{\boldsymbol{y}} = \xi_p + \tilde{\xi}_{q,0} \tag{7.3.8c}$$

We can observe the transfer functions and the output solutions of the original and reduced-order models coincide. This reduced-order model can also be written in the descriptor form given by

$$\tilde{\mathbf{E}}_r = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad \tilde{\mathbf{A}}_r = \begin{pmatrix}-2 & 0\\ 0 & -1\end{pmatrix}, \quad \tilde{\mathbf{B}}_r = \begin{pmatrix}2\\ -1\end{pmatrix}, \quad \tilde{\mathbf{C}}_r = \begin{pmatrix}1\\ 1\end{pmatrix}. \tag{7.3.9}$$

Hence the DAE system is reduced from dimension 9 to 2 using the AE method. For this example we do not need to apply the IMOR method. This reduced-order system is solvable and its solutions coincides with the original model . We can recall from Example 3.2.2 that the PRIMA method lead to unsolvable reduced-order model. Hence IMOR method leads to reliable reduced-order models.

7.4 Extension of IMOR method to truncation methods

We have been more focused on moment matching methods while discussing the IMOR method. However, the IMOR method can be extended to SVD based methods such as the balancing and balanced approximations methods. These methods are well studied, see [3] and are well known to preserve stability and the existence of an a priori computable error bound. It used to be difficult to apply balanced truncation methods on large scale problems due to the fact that two matrix Lyapunov equations have to be solved which are computationally very expensive [45]. However, recent results on low rank approximations to the solution of the Lyapunov equations make the balanced truncation method attractive for large scale systems. The balanced truncation method has also been extended to reduce DAEs, see [45, 62]. We have already discussed this extension in Section 3.3.2 and its limitation. We observed that the balanced truncation method for descriptor systems involves solving four Lyapunov equations and it also relies on the spectral projectors which limits its applicability to large-size and general DAEs. However, the matrix and projector chain used in the IMOR method can be extended to variable coefficient DAEs and it requires only two Lyapunov equations. The extension of the IMOR method to truncation methods is done as follows. The main idea is that instead of using the moment matching method to reduce the differential part, we shall use the truncation methods and this will also induce a reduction in the algebraic parts as for the moment matching case. This implies that the reduction procedure for the algebraic part will remain unchanged. Let us consider the decoupled system (7.2.9) of the DAE given by

$$\begin{aligned} \xi_p' &= \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u}, \\ -\mathcal{L}\xi_q' &= \mathbf{A}_q\xi_p - \xi_q + \mathbf{B}_q\boldsymbol{u}, \\ \boldsymbol{y} &= \mathbf{C}_p^{\mathrm{T}}\xi_p + \mathbf{C}_q^{\mathrm{T}}\xi_q, \end{aligned} \tag{7.4.1}$$

where $\xi_p \in \mathbb{R}^{n_p}$, $\mathbf{A}_p \in \mathbb{R}^{n_p\times n_p}$, $\mathbf{B}_p \in \mathbb{R}^{n_p}$, $\xi_q \in \mathbb{R}^{n_q}$, $\mathbf{A}_q \in \mathbb{R}^{n_q\times n_p}$, $\mathbf{B}_q \in \mathbb{R}^{n_q\times m}$, $\mathcal{L} \in \mathbb{R}^{n_q\times n_q}$ is a strictly lower triangular nilpotent matrix of index-μ. We still first separate (7.4.1) into differential and algebraic parts as follows:

$$\begin{aligned} \xi_p' &= \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u}, \\ \boldsymbol{y}_p &= \mathbf{C}_p^{\mathrm{T}}\xi_p, \end{aligned} \tag{7.4.2}$$

and

$$-\mathcal{L}\xi_q' = \mathbf{A}_q\xi_p - \xi_q + \mathbf{B}_q\boldsymbol{u}, \tag{7.4.3a}$$

$$\boldsymbol{y}_q = \mathbf{C}_q^{\mathrm{T}}\xi_q, \tag{7.4.3b}$$

where the output equation can be reconstructed as $\boldsymbol{y} = \boldsymbol{y}_p + \boldsymbol{y}_q$.

7.4.1 Reduction of the differential part

Applying a state-space transformation $\xi_p = \mathbf{T}_p \tilde{\xi}_p$ does not affect the input-output behavior of the differential subsystem (7.4.2). Using this transformation system (7.4.2) can be transformed into

$$\begin{aligned} \tilde{\xi}_p' &= \mathbf{T}_p^{-1}\mathbf{A}_p\mathbf{T}_p\tilde{\xi}_p + \mathbf{T}_p^{-1}\mathbf{B}_p\boldsymbol{u}, \\ \tilde{\boldsymbol{y}}_p &= \mathbf{C}_p^{\mathrm{T}}\mathbf{T}_p\tilde{\xi}_p. \end{aligned} \tag{7.4.4}$$

The simplest transformation could be chosen to be based on the eigenvalue decomposition of the matrix $\mathbf{A}_p$ given by $\mathbf{A}_p\mathbf{T}_p = \mathbf{T}_p\Lambda_p$ which implies that $\Lambda_p = \mathbf{T}_p^{-1}\mathbf{A}_p\mathbf{T}_p$, where Λ_p is a diagonal matrix of the eigenvalues of $\mathbf{A}_p$. In order to truncate the transformed system (7.4.4), we can do a reordering such that the eigenvalues occur in the decreasing magnitude. Then the system can be truncated by restricting the matrix $\mathbf{T}_p$ to the dominant eigenvalues. This process is termed as modal truncation [58].

However, the commonly used truncation method is the balanced truncation method also known as Truncated Balanced Realization (TBR) method [3, 58]. For the case of balanced truncation method the transformed system (7.4.4) must be a balanced representation of the system (7.4.2), then we can truncate some of the state variables. A balanced realization of a system is one in which states that are difficult (easy) to reach are also difficult (easy) to observe. From a mathematical viewpoint, balancing methods consist of the simultaneous diagonalization of appropriate reachability and observability Gramians, which are positive definite matrices. Given a stable linear subsystem (7.4.2). The controllability and the observability Gramians associated to the linear subsystem (7.4.2) are defined as $\mathcal{P}_p = \int_0^\infty e^{\mathbf{A}_p t}\mathbf{B}_p\mathbf{B}_p^{\mathrm{T}}e^{\mathbf{A}_p^T t}\,\mathrm{d}t$ and $\mathcal{Q}_p = \int_0^\infty e^{\mathbf{A}_p t}\mathbf{C}_p\mathbf{C}_p^{\mathrm{T}}e^{\mathbf{A}_p^T t}\,\mathrm{d}t$, respectively. The matrices $\mathcal{P}_p$ and $\mathcal{Q}_p$ are the unique solutions of two Lyapunov equations:

$$\mathbf{A}_p\mathcal{P}_p + \mathcal{P}_p\mathbf{A}_p^{\mathrm{T}} = -\mathbf{B}_p\mathbf{B}_p^{\mathrm{T}}, \quad \mathbf{A}_p^{\mathrm{T}}\mathcal{Q}_p + \mathcal{Q}_p\mathbf{A}_p = -\mathbf{C}_p\mathbf{C}_p^{\mathrm{T}}. \tag{7.4.5}$$

After finding the Gramians, we look for a state space transformation $\mathbf{T}_p$ which balances the system (7.4.2). Model reduction by balanced truncation, requires balancing the whole system (7.4.2) followed by truncation of the state variables. This approach may turn out to be numerically inefficient and ill-conditioned, especially for large-scale problem. Hence it is much restricted on small problems. The reason is that often $\mathcal{P}_p$ and $\mathcal{Q}_p$ have numerically low rank compared to n_p. Thus, they are several developed algorithms for balancing and balanced truncation, which although in theory they are identical, in practice yield algorithms with quite different numerical properties [3]. Some of these algorithms are well discussed in [3]. One of the algorithm for constructing the balancing transformation $\mathbf{T}_p$ goes as follows. Since the (infinite) Gramians of a reachable, observable, and stable system (7.4.2) of dimension n_p are positive definite square matrices denoted by $\mathcal{P}_p, \mathcal{Q}_p \in \mathbb{R}^{n_p, n_p}$. Then, they can be decomposed using a Cholesky decomposition:

$$\mathcal{P}_p = \mathbf{U}_p \mathbf{U}_p^{\mathrm{T}} \quad \text{and} \quad \mathcal{Q}_p = \mathbf{R}_p \mathbf{R}_p^{\mathrm{T}}, \tag{7.4.6}$$

where $\mathbf{U}_p \in \mathbb{R}^{n_p, n_p}$ and $\mathbf{R}_p \in \mathbb{R}^{n_p, n_p}$ are upper and lower triangular matrices, respectively. The eigenvalue decomposition of $\mathbf{U}_p^{\mathrm{T}} \mathcal{Q}_p \mathbf{U}_p$ produces the orthogonal matrix $\mathbf{K}$ and the diagonal matrix Σ which is composed of the Hankel singular values of system (7.4.2). Then, we have

$$\mathbf{U}_p^{\mathrm{T}} \mathcal{Q}_p \mathbf{U}_p = \mathbf{K}_p \Sigma_p^2 \mathbf{K}_p^{\mathrm{T}}. \tag{7.4.7}$$

They are various balancing transformations that can be derived from (7.4.6) and (7.4.7), see [3] but we shall restrict ourselves on only one. Thus, the balancing transformation $\mathbf{T}_p \in \mathbb{R}^{n_p, n_p}$ and its inverse are

$$\mathbf{T}_p = \mathbf{U}_p \mathbf{K}_p \Sigma_p^{-1/2} \quad \text{and} \quad \mathbf{T}_p^{-1} = \Sigma_p^{1/2} \mathbf{K}_p^{\mathrm{T}} \mathbf{U}_p^{-1}. \tag{7.4.8}$$

The procedure (7.4.6)–(7.4.8) is called balancing [58]. It can easily be shown that $\mathbf{T}_p$ indeed balances the system (7.4.2) that is, $\mathbf{T}_p^{-1} \mathcal{P}_p \mathbf{T}_p^{-\mathrm{T}} = \Sigma_p$ and $\mathbf{T}_p^{\mathrm{T}} \mathcal{Q}_p \mathbf{T}_p = \Sigma_p$. Thus (7.4.4) is a balanced system. The next step is to truncate the system in order to obtain a reduced-order model of (7.4.2). The balance system (7.4.4) can be written as

$$\begin{aligned} \tilde{\xi}_p' &= \tilde{\mathbf{A}}_p \tilde{\xi}_p + \tilde{\mathbf{B}}_p \boldsymbol{u}, \\ \tilde{\boldsymbol{y}}_p &= \tilde{\mathbf{C}}_p^{\mathrm{T}} \tilde{\xi}_p. \end{aligned} \tag{7.4.9}$$

where $\tilde{\mathbf{A}}_p = \mathbf{T}_p^{-1} \mathbf{A}_p \mathbf{T}_p$, $\tilde{\mathbf{B}}_p = \mathbf{T}_p^{-1} \mathbf{B}_p$ and $\tilde{\mathbf{C}}_p = \mathbf{T}_p^{\mathrm{T}} \mathbf{C}_p$. Since a transformation was defined which transforms the system according to the Hankel singular values, now very easily a truncation can be defined [3]. Thus, the diagonal matrix Σ_p can be partitioned as $\Sigma_p = \begin{pmatrix} \Sigma_{p_1} & \mathbf{0} \\ \mathbf{0} & \Sigma_{p_2} \end{pmatrix}$, where Σ_{p_1} contains the largest Hankel singular values. This is the main advantage of this balanced truncation method over the moment matching methods, since now we can manually choose an appropriate value of the size of the reduction, instead of guessing one which can be time consuming. Thus $\tilde{\mathbf{A}}_p$, $\tilde{\mathbf{B}}_p$ and $\tilde{\mathbf{C}}_p$ can be partitioned in conformance with Σ_p:

$$\tilde{\mathbf{A}}_p = \begin{pmatrix} \tilde{\mathbf{A}}_{p_{11}} & \tilde{\mathbf{A}}_{p_{12}} \\ \tilde{\mathbf{A}}_{p_{21}} & \tilde{\mathbf{A}}_{p_{22}} \end{pmatrix}, \quad \tilde{\mathbf{B}}_p = \begin{pmatrix} \tilde{\mathbf{B}}_{p_1} \\ \tilde{\mathbf{B}}_{p_2} \end{pmatrix}, \quad \tilde{\mathbf{C}}_p = \begin{pmatrix} \tilde{\mathbf{C}}_{p_1} \\ \tilde{\mathbf{C}}_{p_2} \end{pmatrix} \tag{7.4.10}$$

and transformed variables can also be partitioned as $\tilde{\xi}_p = (\tilde{\xi}_{p_1}^{\mathrm{T}}, \tilde{\xi}_{p_2}^{\mathrm{T}})^{\mathrm{T}}$. If we truncate the state variable corresponding to the largest Hankel singular values, the reduced-order model of the differential subsystem (7.4.2) is given by

$$\begin{aligned} \xi'_{p_r} &= \mathbf{A}_{p_r}\tilde{\xi}_{p_r} + \mathbf{B}_{p_r}\boldsymbol{u}, \\ \boldsymbol{y}_{p_r} &= \mathbf{C}_{p_r}^{\mathrm{T}}\xi_{p_r}, \end{aligned} \tag{7.4.11}$$

where $\mathbf{A}_{p_r} = \tilde{\mathbf{A}}_{p_{11}} \in \mathbb{R}^{r\times r}$, $\mathbf{B}_{p_r} = \tilde{\mathbf{B}}_{p_1} \in \mathbb{R}^{r\times m}$ and $\mathbf{C}_{p_r} = \tilde{\mathbf{C}}_{p_1} \in \mathbb{R}^{r\times \ell}$ and $\xi_{p_r} = \tilde{\xi}_{p_1} \in \mathbb{R}^r$, $\boldsymbol{y}_{p_r} = \tilde{\boldsymbol{y}}_p \in \mathbb{R}^{\ell\times r}$. The reduced-order model (7.4.11) is also stable with Hankel singular values given by diagonal elements of $\Sigma_{p_1} = \mathrm{diag}(\sigma_1, \cdots, \sigma_r)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, where $r \ll n_p$ is the order of the reduced system (7.4.11). It is possible to choose r via computable error bound

$$\|\mathbf{H}_p - \mathbf{H}_{p_r}\|_2 \leq 2\|\boldsymbol{u}\|_2 \sum_{k=r+1}^{n_p} \sigma_k. \tag{7.4.12}$$

7.4.2 Reduction of the algebraic part

In the previous Section, we have just reduced the differential subsystem (7.4.2) but the algebraic subsystem (7.4.3) is unreduced. However, if we make a transformation $\xi_p = \mathbf{T}_p\tilde{\xi}_p$, where $\mathbf{T}_p$ is the balancing transformation of the differential part (7.4.2), it induces a balancing also in the algebraic part. Thus (7.4.3a) can be written as

$$-\mathcal{L}\tilde{\xi}'_q = \mathbf{A}_q\mathbf{T}_p\tilde{\xi}_p - \tilde{\xi}_q + \mathbf{B}_q\boldsymbol{u}, \tag{7.4.13}$$

where $\tilde{\xi}_q$ is the approximation of ξ_q induced by balancing of the differential subsystem (7.4.2). Thus, we can also partition $\tilde{\mathbf{A}}_q = \mathbf{A}_q\mathbf{T}_p$ as $\tilde{\mathbf{A}}_q = \begin{pmatrix}\tilde{\mathbf{A}}_{q_{11}} & \tilde{\mathbf{A}}_{q_{12}}\end{pmatrix}$ corresponding to the partition in (7.4.10). Thus, (7.4.13) can be written as

$$-\mathcal{L}\tilde{\xi}_q' = \tilde{\mathbf{A}}_{q_{11}}\tilde{\xi}_{p_r} - \tilde{\xi}_q + \mathbf{B}_q\boldsymbol{u}. \tag{7.4.14}$$

Further, (7.4.14) can be written as

$$\tilde{\xi}_q = \sum_{k=0}^{\mu-1}\mathcal{L}^k(\tilde{\mathbf{A}}_{q_{11}}\tilde{\xi}_{p_r}^{(k)} + \mathbf{B}_q\boldsymbol{u}^{(k)}), \tag{7.4.15}$$

where $\tilde{\xi}_{p_r}^{(k)} = \frac{\mathrm{d}^k\tilde{\xi}_{p_r}}{\mathrm{d}t^k}$ and $\boldsymbol{u}^{(k)} = \frac{\mathrm{d}^k\boldsymbol{u}}{\mathrm{d}t^k}$, since $\mathcal{L}$ is a nilpotent matrix of index-μ. Thus, from (7.4.15), we observe that for the algebraic variable $\tilde{\xi}_q$, we have the restriction

$$\tilde{\xi}_q \in \mathcal{T}_q = \mathcal{K}_\mu(\mathcal{L}, \mathbf{R}_q), \tag{7.4.16}$$

where $\mathbf{R}_q = \begin{bmatrix}\mathbf{B}_q & \tilde{\mathbf{A}}_{q_{11}}\end{bmatrix} \in \mathbb{R}^{n_q\times m(r+1)}$, we denote by $\mathbf{T}_q$ an orthonormal basis of $\mathcal{T}_q$ so that $\mathbf{T}_q^{\mathrm{T}}\mathbf{T}_q = \mathrm{I}$. We can then write $\tilde{\xi}_q = \mathbf{T}_q\xi_{q_r}$. Substituting $\tilde{\xi}_q = \mathbf{T}_q\xi_{q_r}$ into (7.4.14) and (7.4.3b), we obtain the reduced-order model of the algebraic part (7.4.3) is given by

$$-\mathcal{L}_r\xi_{q_r}' = \mathbf{A}_{q_r}\xi_{p_r} - \xi_{q_r} + \mathbf{B}_{q_r}\boldsymbol{u}, \tag{7.4.17a}$$
$$\boldsymbol{y}_{q_r} = \mathbf{C}_{q_r}^{\mathrm{T}}\xi_{q_r}. \tag{7.4.17b}$$

where

$$\mathcal{L}_r = \mathbf{T}_q^{\mathrm{T}}\mathcal{L}\mathbf{T}_q \in \mathbb{R}^{\tau\times\tau},\ \mathbf{A}_{q_r} = \mathbf{T}_q^{\mathrm{T}}\tilde{\mathbf{A}}_{q_{11}} \in \mathbb{R}^{\tau\times r},\ \mathbf{B}_{q_r} = \mathbf{T}_q^{\mathrm{T}}\mathbf{B}_q \in \mathbb{R}^{\tau\times m} \text{ and } \mathbf{C}_{q_r} = \mathbf{T}_q^{\mathrm{T}}\mathbf{C}_q \in \mathbb{R}^{\tau,\ell}.$$

$\tau = \dim(\mathcal{T}_q) \ll n_q$ is the dimension of the reduced-order algebraic system. Thus, the IMOR reduced-order model based on the balanced truncation method of a DAE is given by

$$\begin{aligned}\xi_{p_r}' &= \mathbf{A}_{p_r}\xi_{p_r} + \mathbf{B}_{p_r}\boldsymbol{u},\\ -\mathcal{L}_r\xi_{q_r}' &= \mathbf{A}_{q_r}\xi_{p_r} - \xi_{q_r} + \mathbf{B}_{q_r}\boldsymbol{u},\\ \boldsymbol{y}_r &= \mathbf{C}_{p_r}^{\mathrm{T}}\xi_{p_r} + \mathbf{C}_{q_r}^{\mathrm{T}}\xi_{q_r},\end{aligned} \tag{7.4.18}$$

with total dimension $r+\tau \ll n$, where r and τ are the dimension of the reduced differential and algebraic parts, respectively. We note that this reduced-order model will always be stable and have a computable error bound (7.4.12) for the differential part. This is illustrated in the example below.

Example 7.4.1 Consider a DAE with system matrices

$$\mathbf{E} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} -0.1 & 0 & 0.1 & 0 & -1 & 0 & 0 \\ 0 & -0.1 & 0 & 0 & 0 & -1 & 1 \\ 0.1 & 0 & -0.2 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & -0.1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}. \tag{7.4.19}$$

This DAE is solvable since $\det(\lambda\mathbf{E} - \mathbf{A}) \neq 0$ and its stable since $\sigma_f(\mathbf{E}, \mathbf{A}) = \{-\frac{1}{10}, \frac{-1 \pm \sqrt{2399}}{60}\} \in \mathbb{C}^-$. This system is of index-1 and can be decoupled into the form (7.4.1) with system matrices given by

$$\mathbf{A}_p = \begin{pmatrix} -\frac{1}{10} & \frac{1}{30} & -\frac{1}{3} \\ 0 & -\frac{1}{30} & -\frac{2}{3} \\ 0 & 1 & 0 \end{pmatrix}, \mathbf{B}_p = \begin{pmatrix} \frac{1}{15} \\ \frac{1}{30} \\ 0 \end{pmatrix}, \mathcal{L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{A}_q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{10} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{B}_q = \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{10} \\ 0 \end{pmatrix}, \mathbf{C}_p = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{C}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{7.4.20}$$

We can observe that the DAE system is decoupled into $n_p = 3$ differential equations and $n_q = 4$ algebraic equations. The system is still stable since it can easily be checked that $\sigma(\mathbf{A}_p) = \sigma_f(\mathbf{E}, \mathbf{A})$. Next, we use the balanced truncation method to reduce the differential part. This goes as follows. After substituting matrices $\mathbf{A}_p$, $\mathbf{B}_p$ and $\mathbf{C}_p$ into (7.4.5), we can solve for the Gramians given by

$$\mathcal{P}_p = \begin{pmatrix} 0.0169 & 0.0086 & 0.0025 \\ 0.0086 & 0.0167 & 0.0000 \\ 0.0025 & 0.0000 & 0.0250 \end{pmatrix} \quad \text{and} \quad \mathcal{Q}_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 22.5 & 0.75 \\ 0 & 0.75 & 15.025 \end{pmatrix}. \tag{7.4.21}$$

We then use these Gramians to construct the balancing transformation and its inverse using (7.4.8) obtaining

$$\mathbf{T}_p = \begin{pmatrix} 0 & -4.2857 & -3.5714 \\ 0 & -4.2857 & 3.4286 \\ 0.0004 & -0.0002 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{T}_p^{-1} = \begin{pmatrix} -0.072829 & -0.047269 & 2803.6 \\ -0.11429 & -0.11905 & 0 \\ -0.14286 & 0.14286 & 0 \end{pmatrix}. \quad (7.4.22)$$

Substituting (7.4.22) into (7.4.9), we obtain a balanced system. We can, then partition $\tilde{\mathbf{A}}_p$, $\tilde{\mathbf{B}}_p$ and $\tilde{\mathbf{C}}_p$ in conformance with Σ_p to obtain a reduced-order subsystem of the form (7.4.11) with coefficient matrices given by

$$\mathbf{A}_{p_r} = \begin{pmatrix} -0.016327 & 0.81633 \\ -0.81633 & -0.017007 \end{pmatrix}, \mathbf{B}_{p_r} = \begin{pmatrix} -0.14286 \\ -0.14286 \end{pmatrix}, \mathbf{C}_{p_r} = \begin{pmatrix} -0.14286 \\ 0.14286 \end{pmatrix}. \quad (7.4.23)$$

We observe that the reduced-order subsystem is of dimension $r = 2$ and the computable error bound is given by $\|\mathbf{H}_p - \mathbf{H}_{p_r}\|_2 \leq 3.2 \times 10^{-9} \|\boldsymbol{u}\|_2$. We can then compute the orthonormal basis $\mathbf{T}_q$ using (7.4.16) which reduces the algebraic system (7.4.3) given by

$$\mathbf{T}_q = \begin{pmatrix} -9.9504 \cdot 10^{-1} & -9.2464 \cdot 10^{-4} & -9.9507 \cdot 10^{-2} \\ 0 & 0 & 0 \\ 9.9511 \cdot 10^{-2} & -8.8689 \cdot 10^{-3} & -9.95 \cdot 10^{-1} \\ 3.7491 \cdot 10^{-5} & -9.9996 \cdot 10^{-1} & 8.9169 \cdot 10^{-3} \end{pmatrix}. \quad (7.4.24)$$

Substituting (7.4.24) into (7.4.17), we obtain reduced-order algebraic system given by

$$\mathcal{L}_r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{A}_{q_r} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -0.0072829 & -0.0047269 \\ -0.14286 & 0.14286 \end{pmatrix}, \mathbf{B}_{q_r} = \begin{pmatrix} -1.005 \\ -3.7743 \cdot 10^{-5} \\ -7.1046 \cdot 10^{-6} \end{pmatrix},$$

$$\mathbf{C}_{q_r} = \begin{pmatrix} 3.7491 \times 10^{-5} \\ -0.99996 \\ 0.0089169 \end{pmatrix}. \quad (7.4.25)$$

Thus the algebraic system is reduced to dimension $\tau = 3$. Substituting (7.4.23) and (7.4.25) into (7.4.18) we obtained a IMOR reduced-order model based on balanced truncation which we can write in the descriptor form given by

$$\mathbf{E}_r = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{A}_r = \begin{pmatrix} -0.016327 & 0.81633 & 0 & 0 & 0 \\ -0.81633 & -0.017007 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \mathbf{B}_r = \begin{pmatrix} -0.14286 \\ -0.14286 \\ -1.005 \\ -3.7743 \cdot 10^{-5} \\ -7.1046 \cdot 10^{-6} \end{pmatrix},$$

$$\mathbf{C}_r = \begin{pmatrix} -0.14286 & 0.14286 & 3.7491 \cdot 10^{-5} & -0.99996 & 0.0089169 \end{pmatrix}. \tag{7.4.26}$$

Hence the DAE (7.4.19) is reduced to a reduced-order model (7.4.26) of dimension $r + \tau = 5$. Figure 7.1, shows the comparison of the magnitude and phase angle of the

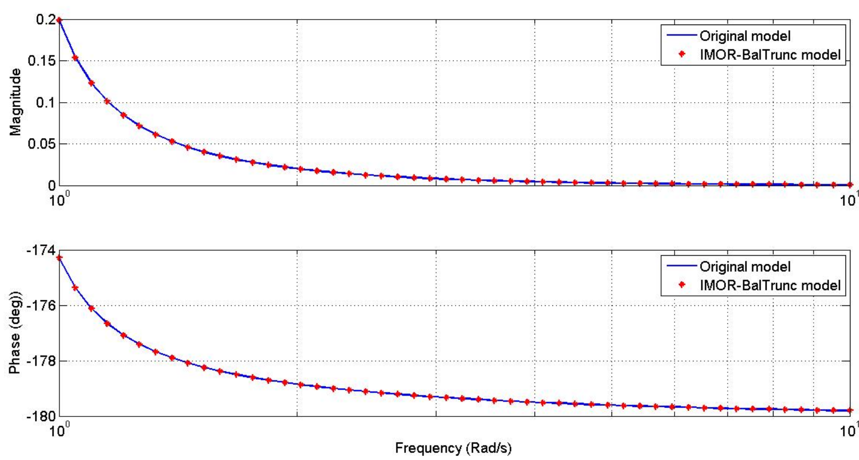

Figure 7.1: Comparison of the magnitude and phase angle of the transfer function.

transfer functions of the original model (7.4.19) and reduced-order model (7.4.26). We observed that the magnitude and phase angle of the transfer functions coincides and the approximate error $||\mathbf{H} - \mathbf{H}_r||_2 \approx 0$. Figure 7.2, shows the comparison of the output solutions and the approximation error using $\boldsymbol{u}(t) = \sin(\pi t)$ as the input function. We can observe that the solutions coincides with an acceptable approximation error.

We have discussed that IMOR method can also be extended to balance truncation method and lead to accurate reduced-order models. In the next section, we discuss the properties of the IMOR methods.

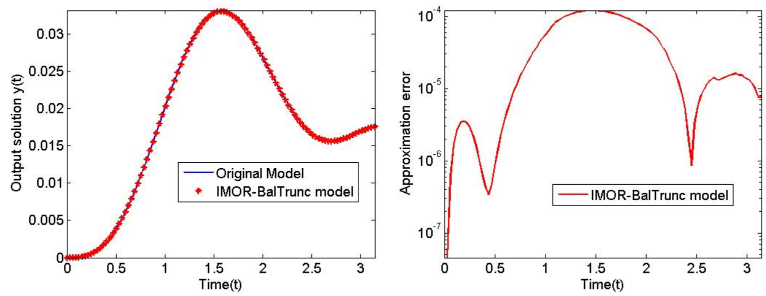

Figure 7.2: Output solution and the approximation error

7.5 Properties of the IMOR method

In this Section, we discuss the properties of the IMOR method. For convenience, we restrict ourselves on DAEs with a differential part. We note that the properties of the IMOR method depends on the properties of the conventional MOR method used to reduce the differential part. This is because the reduction of the algebraic part is induced by the reduction of the differential part. For instance, if we use the Arnoldi processes commonly known as the PRIMA method [49] to reduce the differential part. We can show that the IMOR method preserves the same properties as the PRIMA method such as moment matching property and passivity. This can be done as follows. Recall from Section 7.2.2 the transfer function of the DAE can be decomposed as

$$\mathbf{H}(s) = \begin{pmatrix}\mathbf{C}_p^{\mathrm{T}} & \mathbf{C}_q^{\mathrm{T}}\end{pmatrix}\begin{pmatrix} s\mathrm{I} - \mathbf{A}_p & 0 \\ -\mathbf{A}_q & s\mathcal{L} - \mathrm{I}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{B}_p \\ \mathbf{B}_q\end{pmatrix} = \mathbf{H}_p(s) + \mathbf{H}_q(s), \tag{7.5.1}$$

where $\mathbf{H}_p(s) = \mathbf{C}_p^{\mathrm{T}}(s\mathrm{I} - \mathbf{A}_p)^{-1}\mathbf{B}_p$ and $\mathbf{H}_q(s) = \mathbf{C}_q^{\mathrm{T}}(\mathrm{I} - s\mathcal{L})^{-1}\left[\mathbf{A}_q(s\mathrm{I} - \mathbf{A}_p)^{-1}\mathbf{B}_p + \mathbf{B}_q\right]$, is the transfer function of the differential and algebraic parts, respectively. We can use (7.5.1) to show that the IMOR methods also preserve the properties of model order reduction. The properties of the IMOR method are also presented in [7]. These properties are discussed as follows.

(i) Moment matching property.

This is one of the properties of MOR which must be fulfilled by any moment matching MOR method such as the PRIMA method. Thus, we need to check wether the IMOR methods fulfills the moment matching property as follows. If we use the block Arnoldi process (PRIMA method) to reduce the differential part of the decoupled system. It is well known that it will preserve the first r moments [49] of the differential component $\mathbf{H}_p(s)$ of the decomposed transfer function (7.5.1). This leads to the following theorem.

Theorem 7.5.1 *IMOR methods preserves the moment matching property if and only if the conventional MOR method applied on the differential part preserves the moment matching property.*

Proof 7.5.1 *The proof can be done following the same proof for moment matching property of the PRIMA method presented in [49]. If we choose the expansion point as $s_0 = 0$ and assume $\mathbf{A}_p$ is nonsingular. The transfer function $\mathbf{H}_p(s)$ of the differential part can be written as $\mathbf{H}_p(s) = \sum_{k=0}^{\infty} \mathbf{h}_p^{(k)} s^k$, where $\mathbf{h}_p^{(k)} = (-1)^k \mathbf{C}_p^{\mathrm{T}} \mathbf{M}_p^k \mathbf{R}_p$ are the (block) moments of $\mathbf{H}_p(s)$, $\mathbf{M}_p = -\mathbf{A}_p^{-1}$ and $\mathbf{R}_p = -\mathbf{A}_p^{-1}\mathbf{B}_p$. Likewise, the transfer function of the PRIMA reduced-order differential part can be written as $\tilde{\mathbf{H}}_p(s) = \sum_{k=0}^{r-1} \tilde{\mathbf{h}}_p^{(k)} s^k$, where $\tilde{\mathbf{h}}_p^{(k)} = (-1)^k \tilde{\mathbf{C}}_p^{\mathrm{T}} \tilde{\mathbf{M}}_p{}^k \tilde{\mathbf{R}}_p$, are the moments, $\tilde{\mathbf{M}}_p = -\tilde{\mathbf{A}}_p^{-1}$ and $\tilde{\mathbf{R}}_p = -\tilde{\mathbf{A}}_p^{-1}\tilde{\mathbf{B}}_p$. Then, $\tilde{\mathbf{C}}_p = \mathbf{V}_p^{\mathrm{T}}\mathbf{C}_p$, $\tilde{\mathbf{A}}_p = \mathbf{V}_p^{\mathrm{T}}\mathbf{A}_p\mathbf{V}_p$, $\tilde{\mathbf{B}}_p = \mathbf{V}_p^{\mathrm{T}}\mathbf{B}_p$. We can observe that $\tilde{\mathbf{h}}_p^{(k)}$ can be written as*

$$\tilde{\mathbf{h}}_p^{(k)} = -\mathbf{C}_p^{\mathrm{T}}\mathbf{V}_p\Big[(\mathbf{V}_p^{\mathrm{T}}\mathbf{A}_p\mathbf{V}_p)^{-1}\Big]^k(\mathbf{V}_p^{\mathrm{T}}\mathbf{A}_p\mathbf{V}_p)^{-1}\mathbf{V}_p^{\mathrm{T}}\mathbf{B}_p.$$

By construction $\mathbf{V}_p\mathbf{V}_p^{\mathrm{T}}$ is a projector onto $\mathcal{K}_r(\mathbf{M}_p, \mathbf{R}_p)$. Thus it holds

$$\mathbf{V}_p\mathbf{V}_p^{\mathrm{T}}\mathbf{M}_p^k\mathbf{R}_p = \mathbf{M}_p^k\mathbf{R}_p,\ k = 0, 1, \cdots, r-1.$$

This in turn implies that $\mathbf{V}_p^{\mathrm{T}}\mathbf{M}_p^{\mathrm{T}}\mathbf{R}_p = \tilde{\mathbf{M}}_p^k\tilde{\mathbf{R}}_p$, hence $\tilde{\mathbf{h}}_p^{(k)} = \mathbf{h}_p^{(k)}$, $k = 0, 1, \cdots, r-1$. Next, we can show that the induced reduction on the algebraic part of the DAE (7.0.1) *also preserves the first r moments of the algebraic component of the transfer function, $\mathbf{H}_q(s)$. The transfer function, $\mathbf{H}_q(s)$, of the algebraic part can be written as $\mathbf{H}_q(s) = \sum_{j=0}^{\mu-1} \mathbf{h}_q^{(j)} s^j$, where $\mathbf{h}_q^{(j)} = \mathbf{C}_q\mathcal{L}^j\Big[\mathbf{A}_q\mathbf{R}_p + \mathbf{B}_q\Big]$, $j = 0, \cdots, \mu-1$. Also, by construction $\mathbf{V}_q\mathbf{V}_q^{\mathrm{T}}$ is a projector onto $\mathcal{K}_\mu(\mathcal{L}, \mathbf{R}_q)$, where $\mathbf{R}_q = \Big[\mathbf{B}_q \quad \mathbf{A}_q\mathcal{K}_r(\mathbf{M}_p, \mathbf{R}_p)\Big]$. Thus it holds $\mathbf{V}_q\mathbf{V}_q^{\mathrm{T}}(\mathbf{A}_q\mathbf{M}_p^k\mathbf{R}_p + \mathbf{B}_q) = \mathbf{A}_q\mathbf{M}_p^k\mathbf{R}_p + \mathbf{B}_q$. Using the identity $\mathbf{V}_p^{\mathrm{T}}\mathbf{M}_p^{\mathrm{T}}\mathbf{R}_p = \tilde{\mathbf{M}}_p^k\tilde{\mathbf{R}}_p$, it is possible to show that $\tilde{\mathbf{h}}_q^{(k)} = \mathbf{h}_q^{(k)}$, $\quad k = 0, 1, \cdots, r-1$.*

The above discussion implies that the number of matching moments of the IMOR method depends on the MOR method used to reduce the differential part.

(ii) Passivity preservation property.
A passive system is one that does not generate energy internally. A strictly passive system is a dissipative system [66]. For an LTI system, (strict) passivity is equivalent to the transfer function being (strictly) positive real. According to [49] if we assume that $\mathbf{E}$ is symmetric and nonnegative definite the necessary and sufficient condition for the system admittance matrix $\mathbf{H}(s)$ to be passive has to satisfy the following theorem.

Theorem 7.5.2 (see [66]) *A rational matrix-valued transfer function $\mathbf{H}(s) \in \mathbb{C}^{m\times m}$ is positive real (strictly positive real) if and only if:*

1) $\mathbf{H}(s)$ *is analytic in* $\mathbb{C}^+ = \{s \in \mathbb{C} | \mathrm{Re}(s) > 0\}$;

2) $\Phi(j\omega) = \mathbf{H}(j\omega) + \mathbf{H}^*(j\omega)$ *is positive semi-definite (positive definite) for all* $\omega \in \mathbb{R}$ *such that* $j\omega$ *is not a pole of* $\mathbf{H}(s)$*, where* * *means the conjugate transpose operation;*

3) If $j\omega_0$ *or* ∞ *is a pole* $\mathbf{H}(s)$*, then it is a simple pole and the* $m \times m$ *residue matrix is positive semi-definite.*

Since systems (7.0.1) and (7.2.9) are equivalent. Thus their transfer function must coincide. Using (7.5.1) and (7.2.13) the transfer function of (7.0.1) can be rewritten as

$$
\begin{aligned}
\mathbf{H}(s) &= \mathbf{C}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B},\\
&= \mathbf{H}_p(s) + \mathbf{H}_q(s),\\
&= \mathbf{C}_p^T\mathbf{R}_p(s) + \mathbf{C}_q^T(\mathrm{I} - s\mathcal{L})^{-1}\left[\mathbf{A}_q\mathbf{R}_p(s) + \mathbf{B}_q\right],\\
&= \mathbf{C}_p^T\mathbf{R}_p(s) + \mathbf{C}_q^T\sum_{j=0}^{\mu-1}\mathcal{L}^j s^j\mathbf{N}(s), \text{ since } \mathbf{N}(s) = \mathbf{A}_q\mathbf{R}_p(s) + \mathbf{B}_q,\\
&= \underbrace{\mathbf{C}_p^T\mathbf{R}_p(s) + \mathbf{M}_0(s)}_{\mathbf{H}_{pr}(s)} + \underbrace{\sum_{j=1}^{\mu-1} s^j\mathbf{M}_j(s)}_{\mathbf{H}_{impr}(s)},
\end{aligned}
$$

where $\mathbf{R}_p(s) = (s\mathrm{I} - \mathbf{A}_p)^{-1}\mathbf{B}_p$, $\mathbf{M}_0(s) = \mathbf{C}_q^T\mathbf{N}(s)$ and $\mathbf{M}_j(s) = \mathbf{C}_q^T\mathcal{L}^j\mathbf{N}(s)$. $\mathbf{H}_{pr}(s)$ is the proper part (bounded as $s \to \infty$) and $\mathbf{H}_{impr}(s)$ the improper part (unbounded as

$s \to \infty$) of $\mathbf{H}(s)$. Thus, the transfer function $\mathbf{H}_p(s) = \mathbf{C}_p^T \mathbf{R}_p(s)$ of the differential part is a strictly proper part of $\mathbf{H}(s)$. Based on Theorem 7.5.2, $\mathbf{H}(s)$ is positive real if and only if $\mathbf{H}_{pr}(s)$ and $\mathbf{M}_j(s)$ are positive real. Consequently, a key to testing the passivity of DAEs is to first decouple it into its proper and improper parts [66]. Hence the matrices coefficients of the decoupled systems derived in Chapter 5, can be used to test the passivity of the DAEs using the passivity test for DAEs proposed in [66]. Following the proof for passivity preserving in [49], it can be proved that if the conventional MOR method applied on the differential part is passivity preserving then the differential part of the IMOR reduced-order model is also passive, i.e, $\tilde{\mathbf{H}}_{pr}(s)$ is positive real. However, in order to ensure that the IMOR methods are passivity preserving one need to also prove that $\tilde{\mathbf{M}}_j(s)$ is also positive real which is still an open question.

(iii) Approximation error.
In [7], they proposed that the approximation error of the reduced-order models for DAEs should be computed using the input-output transfer function (7.2.12) instead of just the transfer function. Thus, from (7.2.12) the approximation error of the IMOR method can be computed as

$$\|\mathbf{Y}(s) - \tilde{\mathbf{Y}}(s)\| \leq \|\mathbf{H}(s) - \tilde{\mathbf{H}}(s)\| \, \|\mathbf{U}(s)\| + \|\mathcal{P}(s) - \tilde{\mathcal{P}}(s)\|. \tag{7.5.2}$$

where $\mathcal{P}(s)$ is defined as in (7.2.16).Thus,
$\|\mathbf{H}(s) - \tilde{\mathbf{H}}(s)\| \leq \|\mathbf{H}_p(s) - \tilde{\mathbf{H}}_p(s)\| + \|\mathbf{H}_q(s) - \tilde{\mathbf{H}}_q(s)\|$ and
$\|\mathcal{P}(s) - \tilde{\mathcal{P}}(s)\| \leq \|\tilde{\mathbf{C}}_q \sum_{j=0}^{\mu-1} \tilde{\mathcal{L}}^j - \mathbf{C}_q \sum_{j=0}^{\mu-1} \mathcal{L}^j\| \, \|\tilde{Q}(\boldsymbol{u}(0)) - Q(\boldsymbol{u}(0))\|$. For example, if we consider the case of index-2 DAEs, from (7.2.18) $\mathcal{P}(s)$ is defined as
$\mathcal{P}(s) = -\mathbf{C}_q^{\mathrm{T}} \mathcal{L} \mathbf{B}_q \boldsymbol{u}(0)$. Then, we have

$$\|\mathcal{P}(s) - \tilde{\mathcal{P}}(s)\| = \|\tilde{\mathbf{C}}_q^{\mathrm{T}} \tilde{\mathcal{L}}_q \tilde{\mathbf{B}}_q - \mathbf{C}_q^{\mathrm{T}} \mathcal{L}_q \mathbf{B}_q\| \, \|\boldsymbol{u}(0)\|. \tag{7.5.3}$$

We note that for higher index DAEs the above inequality will depend on the derivatives of the input function $\boldsymbol{u}(t)$ at $t = 0$. Substituting (7.5.3) into (7.5.2), we obtain

$$\|\mathbf{Y}(s) - \tilde{\mathbf{Y}}(s)\| \leq \|\mathbf{H}_p(s) - \tilde{\mathbf{H}}_p(s)\| \, \|\mathbf{U}(s)\| + \|\mathbf{H}_q(s) - \tilde{\mathbf{H}}_q(s)\| \, \|\mathbf{U}(s)\| + \gamma \|\boldsymbol{u}(0)\|,$$

where $\gamma = \|\tilde{\mathbf{C}}_q^{\mathrm{T}} \tilde{\mathcal{L}}_q \tilde{\mathbf{B}}_q - \mathbf{C}_q^{\mathrm{T}} \mathcal{L}_q \mathbf{B}_q\|$. Hence the output-transfer function of the IMOR reduced-order model has a small approximation error if and only if

(a) $||\mathbf{H} - \tilde{\mathbf{H}}||$ is small

(b) and $||\mathcal{P}(s) - \tilde{\mathcal{P}}(s)||$ is also very small in a suitable norm $||.||$.

(iv) Stability

In Section 5.4, we already discussed that for the case of DAEs with a differential part the decoupled system inherits the stability properties of DAEs since $\sigma(\mathbf{A}_p) = \sigma_f(\mathbf{E}, \mathbf{A})$. Hence stability preservation of the IMOR method depends on the MOR method used to reduce the differential part.

We note that, if we use the balanced truncation method to reduce the differential part. We can guarantee stability of the reduced-order model and have an a priori computable error bound. Moreover, we can easily choose the size of the reduced-order model before hand.

7.6 Limitations of the IMOR method

The limitations of the IMOR method originates from März decoupling procedure discussed in Chapter 4 since it involves matrix inversions. This lead to decoupled systems with very dense matrix coefficients which can be very difficult to reduce. Hence in the next Chapter, we develop its implicit version which we call the implicit-IMOR (IIMOR) method.

Chapter 8

Implicit Index-aware Model Order Reduction (Implicit IMOR) method

In this Chapter, we discuss the Implicit Index-aware MOR method which can be abbreviated as IIMOR method. This method uses the implicit decoupled systems derived in Chapter 6. This is due to the fact that the implicit decoupling procedure is computationally cheaper than the explicit decoupling procedure derived in Chapter 5. We consider DAEs of the form

$$\mathbf{E}\dot{\boldsymbol{x}}(t) = \mathbf{A}\boldsymbol{x}(t) + \mathbf{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{8.0.1a}$$

$$\boldsymbol{y}(t) = \mathbf{C}^{\mathrm{T}}\boldsymbol{x}(t), \tag{8.0.1b}$$

where $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{n\times \ell}$, the input vector $\boldsymbol{u}(t) \in \mathbb{R}^m$ and output vector $\boldsymbol{y}(t) \in \mathbb{R}^\ell$ of the system. $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector and $\boldsymbol{x}_0$ is the initial value. The number of state variables n is called the order of system or the state-space dimension. m and ℓ are the number of inputs and outputs, respectively. We need to reduce (8.0.1) using the IIMOR method.

8.1 Algebraic Elimination MOR method

In this Section, we discuss the reduction of the algebraic parts if the decoupled systems are derived from Chapter 6. The basic idea is the same as that presented in Chapter 7, which is to eliminate algebraic variables which do not contribute to the output solution. This can also be done using the reordering techniques.

8.1.1 Index-1 DAEs

Assume (8.0.1) is of index-1 then it can be decoupled into the form (6.1.5) given by

$$\mathbf{E}_p\xi_p' = \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u}, \tag{8.1.1a}$$
$$\mathbf{E}_q\xi_q = \mathbf{A}_q\xi_p + \mathbf{B}_q\boldsymbol{u}, \tag{8.1.1b}$$
$$\boldsymbol{y} = \mathbf{C}_p^{\mathrm{T}}\xi_p + \mathbf{C}_q^{\mathrm{T}}\xi_q, \tag{8.1.1c}$$

where $\mathbf{E}_p, \mathbf{A}_p \in \mathbb{R}^{n_p\times n_p}$, $\mathbf{B}_p \in \mathbb{R}^{n_p\times m}$, $\mathbf{E}_q \in \mathbb{R}^{n_q\times n_q}$, $\mathbf{A}_q \in \mathbb{R}^{n_q\times n_p}$, $\mathbf{B}_q \in \mathbb{R}^{n_q\times m}$ and $\mathbf{C}_p \in \mathbb{R}^{n_p\times\ell}$, $\mathbf{C}_q \in \mathbb{R}^{n_q\times\ell}$. The trivial case is when $\mathbf{C}_q = 0$, then (8.0.1) can just be reduced to an ODE of dimension $\mathbf{n}_p$. Thus, if we consider the nontrivial case, assume $\mathbf{C}_q \neq 0$. Then, the Algebraic Elimination MOR model for index-1 DAEs can derived as follows. Consider the algebraic subsystem from (8.1.1) given by

$$\mathbf{E}_q\xi_q = \mathbf{A}_q\xi_p + \mathbf{B}_q\boldsymbol{u}, \tag{8.1.2a}$$
$$\boldsymbol{y}_q = \mathbf{C}_q^{\mathrm{T}}\xi_q. \tag{8.1.2b}$$

We compute permutation matrices $\mathbf{P}_\pi, \mathbf{Q}_\pi \in \mathbb{R}^{n_q\times n_q}$ such that $\mathbf{Q}_\pi^{\mathrm{T}}\mathbf{C}_q = \begin{pmatrix}\tilde{\mathbf{C}}_{q_1}\\ 0\end{pmatrix}$ and $\mathbf{P}_\pi\mathbf{E}_q\mathbf{Q}_\pi = \begin{pmatrix}\tilde{\mathbf{E}}_{q_{11}} & \tilde{\mathbf{E}}_{q_{12}}\\ \tilde{\mathbf{E}}_{q_{21}} & \tilde{\mathbf{E}}_{q_{22}}\end{pmatrix}$, where $\tilde{\mathbf{E}}_{q_{22}}$ is a non-singular matrix. Then, the rest of the matrices can be partitioned as $\mathbf{P}_\pi\mathbf{A}_q = \begin{pmatrix}\tilde{\mathbf{A}}_{q_1}\\ \tilde{\mathbf{A}}_{q_2}\end{pmatrix}$ and $\mathbf{P}_\pi\mathbf{B}_q = \begin{pmatrix}\tilde{\mathbf{B}}_{q_1}\\ \tilde{\mathbf{B}}_{q_1}\end{pmatrix}$. If we let $\xi_q = \mathbf{Q}_\pi\tilde{\xi}_q = \begin{pmatrix}\tilde{\xi}_{q_1}\\ \tilde{\xi}_{q_2}\end{pmatrix}$, where $\tilde{\xi}_{q_1} \in \mathbb{R}^{\tau}$, $\tilde{\xi}_{q_1} \in \mathbb{R}^{n_q-\tau}$ and substituting it into (8.1.2) and left multiplying (8.1.2a) by $\mathbf{P}_\pi$, we obtain a partitioned system of (8.1.2). We can then eliminate $\tilde{\xi}_{q_2}$ since it does not contribute to the output solution. This leads to a reduced-order model of dimension

$\tau < \mathbf{n}_q$ which is given by

$$\mathbf{E}_{q_\tau}\xi_{q_\tau} = \mathbf{A}_{q_\tau}\xi_p + \mathbf{B}_{q_\tau}\boldsymbol{u}, \tag{8.1.3a}$$
$$\boldsymbol{y}_p = \mathbf{C}^{\mathrm{T}}_{q_\tau}\xi_{q_\tau} \tag{8.1.3b}$$

where $\mathbf{E}_{q_\tau} = \left[\tilde{\mathbf{E}}_{q_{11}} - \tilde{\mathbf{E}}_{q_{12}}\tilde{\mathbf{E}}^{-1}_{q_{22}}\tilde{\mathbf{E}}_{q_{21}}\right] \in \mathbb{R}^{\tau\times\tau}$, $\mathbf{A}_{q_\tau} = \left[\tilde{\mathbf{A}}_{q_1} - \tilde{\mathbf{E}}_{q_{12}}\tilde{\mathbf{E}}^{-1}_{q_{22}}\tilde{\mathbf{A}}_{q_2}\right] \in \mathbb{R}^{\tau\times n_p}$, $\mathbf{B}_{q_\tau} = \left[\tilde{\mathbf{B}}_{q_1} - \tilde{\mathbf{E}}_{q_{12}}\tilde{\mathbf{E}}^{-1}_{q_{22}}\tilde{\mathbf{B}}_{q_2}\right] \in \mathbb{R}^{\tau\times m}$ and $\mathbf{C}_{q_\tau} = \tilde{\mathbf{C}}_{q_1} \in \mathbb{R}^{\tau\times\ell}$ and the reduced dimension is given by $\tau < n_q$. Thus the DAE (8.0.1) is reduced to a reduced-order model of dimension $n_p + \tau < n$ which is given by

$$\mathbf{E}_p\xi'_p = \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u}, \tag{8.1.4a}$$
$$\mathbf{E}_{q_\tau}\xi_{q_\tau} = \mathbf{A}_{q_\tau}\xi_p + \mathbf{B}_{q_\tau}\boldsymbol{u}, \tag{8.1.4b}$$
$$\boldsymbol{y} = \mathbf{C}^{\mathrm{T}}_p\xi_p + \mathbf{C}^{\mathrm{T}}_{q_\tau}\xi_{q_\tau} \tag{8.1.4c}$$

8.1.2 Index-2 DAEs

Assume (8.0.1) is of index-2. Then, if we consider the case of the index-2 DAEs with at least one finite eigenvalue then it can be decoupled in the form (6.2.3) given by

$$\begin{aligned}
\mathbf{E}_p\xi'_p &= \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u},\\
\mathbf{E}_{q,1}\xi_{q,1} &= \mathbf{A}_{q,1}\xi_p + \mathbf{B}_{q,1}\boldsymbol{u},\\
\mathbf{E}_{q,0}\xi_{q,0} &= \mathbf{A}_{q,0}\xi_p + \mathbf{B}_{q,0}\boldsymbol{u} + \mathbf{A}_{\boldsymbol{q}_{0,1}}\left[\xi'_{q,1} - \xi_{q,1}\right],\\
\boldsymbol{y} &= \mathbf{C}^{\mathrm{T}}_p\xi_p + \mathbf{C}^{\mathrm{T}}_{q,1}\xi_{q,1} + \mathbf{C}^{\mathrm{T}}_{q,0}\xi_{q,0},
\end{aligned} \tag{8.1.5}$$

where $\mathbf{E}_p, \mathbf{A}_p \in \mathbb{R}^{n_p\times n_p}$, $\mathbf{B}_p \in \mathbb{R}^{n_p\times m}$, $\mathbf{E}_{q,1} \in \mathbb{R}^{k_1\times k_1}$, $\mathbf{A}_{q,1} \in \mathbb{R}^{k_1\times n_p}$, $\mathbf{B}_{q,1} \in \mathbb{R}^{k_1\times m}$, $\mathbf{E}_{q,0} \in \mathbb{R}^{k_2\times k_2}$, $\mathbf{A}_{q,0} \in \mathbb{R}^{k_2, n_p}$, $\mathbf{B}_{q,0} \in \mathbb{R}^{k_2\times m}$, $\mathbf{A}_{\boldsymbol{q}_{0,1}} \in \mathbb{R}^{k_2\times k_1}$ and $\mathbf{C}_p \in \mathbb{R}^{n_p\times\ell}$, $\mathbf{C}_{q,1} \in \mathbb{R}^{k_1\times\ell}$, $\mathbf{C}_{q,0} \in \mathbb{R}^{k_2\times\ell}$. From (8.1.5), we can observe that we can easily obtain a reduction for the following trivial cases; (i) If $\mathbf{C}_{q,1} = 0$ and $\mathbf{C}_{q,0} = 0$, then (8.0.1) can be reduced to an ODE of dimension n_p, (ii) If $\mathbf{A}_{\boldsymbol{q}_{0,1}} = 0$ and $\mathbf{C}_{q,1} = 0$ or $\mathbf{C}_{q,0} = 0$, then (8.0.1) can be reduced to index-1 system of dimension $n_p + k_1$ and $n_p + k_2$, respectively. After checking for the trivial cases, then we can eliminate the algebraic variables as follows. If we consider only algebraic parts of the system (8.1.5), we obtain an algebraic subsystem

given by

$$\mathbf{E}_{q,1}\xi_{q,1} = \mathbf{A}_{q,1}\xi_p + \mathbf{B}_{q,1}\boldsymbol{u}, \tag{8.1.6a}$$

$$\mathbf{E}_{q,0}\xi_{q,0} = \mathbf{A}_{q,0}\xi_p + \mathbf{B}_{q,0}\boldsymbol{u} + \mathbf{A}_{\boldsymbol{q}_{0,1}}\left[\xi'_{q,1} - \xi_{q,1}\right], \tag{8.1.6b}$$

$$\boldsymbol{y}_q = \mathbf{C}^{\mathrm{T}}_{q,1}\xi_{q,1} + \mathbf{C}^{\mathrm{T}}_{q,0}\xi_{q,0}. \tag{8.1.6c}$$

Using the same trick as for the case of index-1 system. We can first compute permutation matrices $\mathbf{V}_\pi, \mathbf{W}_\pi \in \mathbb{R}^{k_2\times k_2}$ such that $\mathbf{W}^{\mathrm{T}}_\pi\mathbf{C}_{q,0} = \begin{pmatrix}\mathbf{C}_{q_1,0}\\ \mathbf{0}\end{pmatrix}$ and $\mathbf{V}_\pi\mathbf{E}_{q,0}\mathbf{W}_\pi = \begin{pmatrix}\mathbf{E}_{q_{11},0} & \mathbf{E}_{q_{12},0}\\ \mathbf{E}_{q_{21},0} & \mathbf{E}_{q_{22},0}\end{pmatrix}$, where $\mathbf{E}_{q_{22},0}$ is a nonsingular matrix. Then,$\mathbf{V}_\pi\mathbf{B}_{q,0} = \begin{pmatrix}\mathbf{B}_{q_1,0}\\ \mathbf{B}_{q_2,0}\end{pmatrix}$, $\mathbf{V}_\pi\mathbf{A}_{q,0} = \begin{pmatrix}\mathbf{A}_{q_1}\\ \mathbf{A}_{q_2}\end{pmatrix}$. Next, we compute another set of permutation matrices $\mathbf{P}_\pi, \mathbf{Q}_\pi \in \mathbb{R}^{k_1\times k_1}$ such that $\mathbf{V}_\pi\mathbf{A}_{\boldsymbol{q}_{0,1}}\mathbf{Q}_\pi = \begin{pmatrix}\mathbf{A}_{\boldsymbol{q}_{0_{11},1}} & \mathbf{0}\\ \mathbf{A}_{\boldsymbol{q}_{0_{21},1}} & \mathbf{0}\end{pmatrix}$, $\mathbf{Q}^{\mathrm{T}}_\pi\mathbf{C}_{q,0} = \begin{pmatrix}\mathbf{C}_{q_1,1}\\ \mathbf{0}\end{pmatrix}$and $\mathbf{P}_\pi\mathbf{E}_{q,1}\mathbf{Q}_\pi = \begin{pmatrix}\mathbf{E}_{q_{11},1} & \mathbf{E}_{q_{12},1}\\ \mathbf{E}_{q_{21},1} & \mathbf{E}_{q_{22},1}\end{pmatrix}$, where $\mathbf{E}_{q_{22},1}$ is a nonsingular matrix. Then, $\mathbf{P}_\pi\mathbf{A}_{q,1} = \begin{pmatrix}\mathbf{A}_{q_1,1}\\ \mathbf{A}_{q_1,2}\end{pmatrix}$, $\mathbf{P}_\pi\mathbf{B}_{q,1} = \begin{pmatrix}\mathbf{B}_{q_1,1}\\ \mathbf{B}_{q_2,1}\end{pmatrix}$. If we let $\xi_{q,1} = \mathbf{Q}_\pi\tilde{\xi}_{q,1} = \begin{pmatrix}\tilde{\xi}_{q_1,1}\\ \tilde{\xi}_{q_2,1}\end{pmatrix}$, where $\tilde{\xi}_{q_1,1} \in \mathbb{R}^{\tau_1}$, $\tilde{\xi}_{q_2,1} \in \mathbb{R}^{k_1-\tau_1}$ and $\xi_{q,0} = \mathbf{W}_\pi\tilde{\xi}_{q,0} = \begin{pmatrix}\tilde{\xi}_{q_1,0}\\ \tilde{\xi}_{q_2,0}\end{pmatrix}$, where $\tilde{\xi}_{q_1,0} \in \mathbb{R}^{\tau_2}$, $\tilde{\xi}_{q_2,0} \in \mathbb{R}^{k_2-\tau_2}$. Then left multiply (8.1.6a) and (8.1.6b) with $\mathbf{P}_\pi$ and $\mathbf{V}_\pi$, respectively. We can obtain a partitioned system of (8.1.6). We can then eliminate the algebraic variables $\tilde{\xi}_{q_2,1}$ and $\tilde{\xi}_{q_2,0}$ which do not contribute to the output equation (8.1.6c). This leads to a reduce-order model of dimension $\tau_1 + \tau_2$ given by

$$\mathbf{E}_{q_\tau,1}\xi_{q,1} = \mathbf{A}_{q_\tau,1}\xi_p + \mathbf{B}_{q_\tau,1}\boldsymbol{u}, \tag{8.1.7a}$$

$$\mathbf{E}_{q_\tau,0}\xi_{q_\tau,0} = \mathbf{A}_{q_\tau,0}\xi_p + \mathbf{B}_{q_\tau,0}\boldsymbol{u} + \mathbf{A}_{\boldsymbol{q}_{0_\tau,1}}\left[\xi'_{q_\tau,1} - \xi_{q_\tau,1}\right], \tag{8.1.7b}$$

$$\boldsymbol{y}_{q_\tau} = \mathbf{C}^{\mathrm{T}}_{q_\tau,1}\xi_{q_\tau,1} + \mathbf{C}^{\mathrm{T}}_{q_\tau,0}\xi_{q_\tau,0}, \tag{8.1.7c}$$

where
$\mathbf{E}_{q_\tau,1} = \left[\mathbf{E}_{q_{11},1} - \mathbf{E}_{q_{12},1}\mathbf{E}^{-1}_{q_{22},1}\mathbf{E}_{q_{21},1}\right] \in \mathbb{R}^{\tau_1\times\tau_1}$, $\mathbf{B}_{q_\tau,1} = \left[\mathbf{B}_{q_1,1} - \mathbf{E}_{q_{12},1}\mathbf{E}^{-1}_{q_{22},1}\mathbf{B}_{q_2,1}\right] \in \mathbb{R}^{\tau_1\times m}$, $\mathbf{E}_{q_\tau,0} = \left[\mathbf{E}_{q_{11},0} - \mathbf{E}_{q_{12},0}\mathbf{E}^{-1}_{q_{22},0}\mathbf{E}_{q_{21},0}\right] \in \mathbb{R}^{\tau_2,\tau_2}$, $\mathbf{A}_{q_\tau,0} = \left[\mathbf{A}_{q_1} - \mathbf{E}_{q_{12},0}\mathbf{E}^{-1}_{q_{22},0}\mathbf{A}_{q_2}\right] \in \mathbb{R}^{\tau_2\times n_p}$, $\mathbf{A}_{\boldsymbol{q}_{0_\tau,1}} = \left[\mathbf{A}_{\boldsymbol{q}_{0_{11},1}} - \mathbf{E}_{q_{12},0}\mathbf{E}^{-1}_{q_{22},0}\mathbf{A}_{\boldsymbol{q}_{0_{21},1}}\right]$ and $\mathbf{C}_{q_\tau,1} = \mathbf{C}_{q_1,1} \in \mathbb{R}^{\tau_1\times\ell}$, $\mathbf{C}_{q_\tau,0} = \mathbf{C}_{q_1,0} \in \mathbb{R}^{\tau_2\times\ell}$. Hence, the DAE (8.0.1) is reduced to a reduced-order model of dimension $\mathbf{n}_p + \tau_1 + \tau_2$

given by

$$
\begin{aligned}
\mathbf{E}_p\xi_p' &= \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u}, \\
\mathbf{E}_{q_\tau,1}\xi_{q,1} &= \mathbf{A}_{q_\tau,1}\xi_p + \mathbf{B}_{q_\tau,1}\boldsymbol{u}, \\
\mathbf{E}_{q_\tau,0}\xi_{q_\tau,0} &= \mathbf{A}_{q_\tau,0}\xi_p + \mathbf{B}_{q_\tau,0}\boldsymbol{u} + \mathbf{A}_{\boldsymbol{q}_{0_\tau},1}\left[\xi_{q_\tau,1}' - \xi_{q_\tau,1}\right], \\
\boldsymbol{y} &= \mathbf{C}_p^{\mathrm{T}}\xi_p + \mathbf{C}_{q_\tau,1}^{\mathrm{T}}\xi_{q_\tau,1} + \mathbf{C}_{q_\tau,0}^{\mathrm{T}}\xi_{q_\tau,0},
\end{aligned}
$$

This same technique can used for even higher index DAEs. However, we do not gain too much reduction as compared to applying the AE method on the explicit decoupled system as illustrated in the next example. Thus the AE method is not so useful for the implicit decoupled systems. This is due to the fact that it is very difficult to eliminate all the algebraic variables which do not contribute to the output equation in the implicit form. However, this can be improved using the graph and matrix reordering algorithms.

Example 8.1.1 In this example, we use the same system models as those used in Example 7.1.1 for comparison. These are all index-1 DAE and this time we decoupled the power system using the implicit decoupling procedure discussed in Chapter 6. If we compare Table 8.1 and 7.1, we obtain the same number of differential and algebraic equations. However, we can observe that this time the AE method leads to much larger reduced algebraic part. This is due to that fact that not all the algebraic variables which do not contribute to the output solution can easily be removed.

8.2 Implicit IMOR method for DAEs

We have seen that the using the AE method the differential part remains unreduced. Hence, we do not get good reduction for the algebraic parts. In this Section, we extend the Index-aware MOR method discussed in Chapter 7, for the case of implicitly decoupled systems which we derived in Chapter 6. In this Section, we discuss the Implicit version of the IMOR method which we call the IIMOR method. We first proposed this approach in [4], thus some of the content presented here is also in [4]. The basic idea is still the same as the IMOR method, the only difference is the starting decoupled system. Recall, for the case of IMOR method we use the explicit decoupled systems derived in Chapter 5 while for the case of IIMOR method, we have to use the implicit decoupled systems derived in Chapter 6. The advantage of using the implicit decoupling

Table 8.1: Algebraic Reduced models of power systems

Systems	# inputs/# outputs		Decoupled model		Alg. Reduced model		Reduced system	% Reduction
n	# inputs	# outputs	n_p	n_q	n_p	γ	$n_p+\gamma$	
40366	2	2	5727	34639	5727	27797	33524	17.0
40337	2	1	5723	34614	5723	27791	27791	16.9
21476	1	1	3172	18304	3172	11370	14542	32.3
21128	4	4	3078	18050	3078	11113	14191	32.8
20944	2	2	3012	17932	3012	3390	6402	69.4
20738	1	6	2940	17798	2940	0	1755	91.5
16861	4	4	2476	14385	2476	7448	9924	41.1
15066	4	4	1998	13068	1998	6140	8138	46.0
13309	8	8	1676	11633	1676	0	1676	87.4
13296	46	46	1664	11632	1664	7090	8754	34.2
13275	4	4	1693	11582	1693	7045	7045	34.2
13250	1	1	1664	11586	1664	7075	8739	34.1
13250	46	46	1664	11586	1664	7075	8739	34.1
13251	28	28	1664	11587	1664	7076	8740	34.0
13251	1	1	1664	11587	1664	0	1664	87.4
11685	1	1	1257	10428	1257	5917	7174	38.6
11305	4	4	1450	9855	1450	5320	6770	40.1
9735	4	4	1142	8593	1142	4032	5174	46.9
7135	4	4	606	6529	606	1968	2574	63.9

procedure is the computational advantage over the explicit decoupling procedure. The IIMOR method is derived as follows. Assume (8.0.1) is of index-μ, with the spectrum of its matrix pencil has at least one eigenvalue. Then, it can be decoupled into a system of the form (6.4.1) given by

$$\begin{aligned} \mathbf{E}_p\xi_p' &= \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u}, \\ -\mathcal{L}\xi_q' &= \mathbf{A}_q\xi_p - \mathcal{L}_q\xi_q + \mathbf{B}_q\boldsymbol{u}, \\ \boldsymbol{y} &= \mathbf{C}_p^{\mathrm{T}}\xi_p + \mathbf{C}_q^{\mathrm{T}}\xi_q, \end{aligned} \tag{8.2.1}$$

where $\mathcal{L}$ is a nilpotent matrix of index-μ. $\mathcal{L}_q$ is a non-singular lower triangular matrix with block diagonal matrices for $\mu > 1$. $\xi_p \in \mathbb{R}^{n_p}$, $\xi_q \in \mathbb{R}^{n_q}$, $\mathbf{A}_q \in \mathbb{R}^{n_q\times n_p}$, $\mathbf{B}_q \in \mathbb{R}^{n_q\times m}$ and $\mathbf{C}_q \in \mathbb{R}^{n_q\times \ell}$, $\mathbf{C}_p \in \mathbb{R}^{n_p\times \ell}$. Taking the Laplace transform of (8.2.1) and setting $\xi_p(0) = 0$ since it can be chosen arbitrary. Then, the output function is given by

$$\mathbf{Y}(s) := \mathbf{H}(s)\mathbf{U}(s) + \mathcal{P}(s), \tag{8.2.2}$$

where $\mathbf{H}(s)$ is decomposed as $\mathbf{H}(s) = \mathbf{H}_p(s)+\mathbf{H}_q(s)$, where $\mathbf{H}_p(s) := \mathbf{C}_p^{\mathrm{T}}(s\mathbf{E}_p-\mathbf{A}_p)^{-1}\mathbf{B}_p$ and $\mathbf{H}_q(s) := \mathbf{C}_q^{\mathrm{T}}(\mathcal{L}_q-s\mathcal{L})^{-1}\left[\mathbf{A}_q(s\mathbf{E}_p-\mathbf{A}_p)^{-1}\mathbf{B}_p+\mathbf{B}_q\right]$ are transfer functions corresponding to the differential part and algebraic parts, respectively and
$\mathcal{P}(s) := \mathbf{C}_q^{\mathrm{T}}(\mathcal{L}_q - s\mathcal{L})^{-1}\mathcal{L}\xi_q(0)$. If we let $Q(\boldsymbol{u}(0)) := \mathcal{L}\xi_q(0)$ then $\mathcal{P}(s)$ can be written

as $\mathcal{P}(s) = \mathbf{C}_q^{\mathrm{T}}(\mathcal{L}_q - s\mathcal{L})^{-1}Q(\boldsymbol{u}(0))$. Following the same steps as for the case of IMOR method. It is easy to show that if the hidden polynomial $Q(\boldsymbol{u}(0))$ has nonzero coefficient the conventional MOR methods fail or lead to reduced-order models which are very difficult to solve for the case of higher index DAEs. For the interested reader, follow the same steps as in Section 7.2.2.

The derivation of the IIMOR method goes as follows. We use the same strategy as in the IMOR method by the first splitting the decoupled system (8.2.1) into separate subsystems as

$$\begin{aligned} \mathbf{E}_p\xi_p' &= \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u}, \\ \boldsymbol{y}_p &= \mathbf{C}_p^{\mathrm{T}}\xi_p, \end{aligned} \tag{8.2.3}$$

and

$$-\mathcal{L}\xi_q' = \mathbf{A}_q\xi_p - \mathcal{L}_q\xi_q + \mathbf{B}_q\boldsymbol{u}, \tag{8.2.4a}$$

$$\boldsymbol{y}_q = \mathbf{C}_q^{\mathrm{T}}\xi_q, \tag{8.2.4b}$$

where (8.2.3) and (8.2.4) are the differential and algebraic subsystems. Then the output equation can be reconstructed using: $\boldsymbol{y} = \boldsymbol{y}_p + \boldsymbol{y}_q$. Next, we derive the reduction procedure for (8.2.3) and (8.2.4), respectively.

8.2.1 Reduction of the differential part

Consider the subsystem (8.2.3). We can use the conventional MOR methods such as the PRIMA method to reduce this subsystem as follows. Choose an expansion point $s_0 \in \mathbb{C} \setminus \sigma(\mathbf{E}_p, \mathbf{A}_p)$ and then construct an order-r Krylov subspace generated by $\mathbf{M}_p$ and $\mathbf{R}_p$ given by

$$\mathcal{V}_p := \mathcal{K}_r(\mathbf{M}_p, \mathbf{R}_p) = \mathrm{Span}\{\mathbf{R}_p, \mathbf{M}_p\mathbf{R}_p, \dots, \mathbf{M}_p^{r-1}\mathbf{R}_p\},\ r \le n_p,$$

where $\mathbf{M}_p := (s_0\mathbf{E}_p - \mathbf{A}_p)^{-1}\mathbf{E}_p$ and $\mathbf{R}_p := (s_0\mathbf{E}_p - \mathbf{A}_p)^{-1}\mathbf{B}_p$. Then, $\mathbf{V}_p \in \mathbb{R}^{n_p, r}$ denotes the orthonormal basis matrix of the above subspace, so that $\mathbf{V}_p^{\mathrm{T}}\mathbf{V}_p = \mathrm{I}$. The reduced-order subsystem is obtained by using the approximation $\xi_p = \mathbf{V}_p\hat{\xi}_p$, leading to a reduced-order subsystem

$$\begin{aligned} \hat{\mathbf{E}}_p \hat{\xi}'_p &= \hat{\mathbf{A}}_p \hat{\xi}_p + \hat{\mathbf{B}}_p \boldsymbol{u}, \\ \hat{\boldsymbol{y}}_p &= \hat{\mathbf{C}}_p^{\mathrm{T}} \hat{\xi}_p, \end{aligned} \tag{8.2.5}$$

where $\hat{\mathbf{E}}_p = \mathbf{V}_p^{\mathrm{T}} \mathbf{E}_p \mathbf{V}_p$, $\hat{\mathbf{A}}_p = \mathbf{V}_p^{\mathrm{T}} \mathbf{A}_p \mathbf{V}_p \in \mathbb{R}^{r \times r}$, $\hat{\mathbf{B}}_p = \mathbf{V}_p^{\mathrm{T}} \mathbf{B}_p \in \mathbb{R}^{r \times m}$ and $\hat{\mathbf{C}}_p = \mathbf{V}_p^{\mathrm{T}} \mathbf{C}_p \in \mathbb{R}^{r \times p}$. $\hat{\xi}_p \in \mathbb{R}^r$ is the reduced state vector and $\hat{\boldsymbol{y}}_p \in \mathbb{R}^{\ell}$ is the approximated output. Thus the dimension of the differential part is reduced to $r \leq n_p$. The transfer function of the reduced-order model (8.2.5) is given by $\hat{\mathbf{H}}_p(s) = \hat{\mathbf{C}}_p^{\mathrm{T}} (s\hat{\mathbf{E}}_p - \hat{\mathbf{A}}_p)^{-1} \hat{\mathbf{B}}_p$.

8.2.2 Reduction of the algebraic part

Here, we intend to reduce the algebraic subsystem (8.2.4). Substituting $\xi_p = \mathbf{V}_p \hat{\xi}_p$ into (8.2.4), we obtain

$$-\mathcal{L}\xi'_q = \mathbf{A}_q \mathbf{V}_p \hat{\xi}_p - \mathcal{L}_q \xi_q + \mathbf{B}_q \boldsymbol{u}, \tag{8.2.6a}$$

$$\boldsymbol{y}_q = \mathbf{C}_q^{\mathrm{T}} \xi_q. \tag{8.2.6b}$$

From (8.2.6a), we can observe that the reduction of the differential part induces a reduction on the algebraic part but the order of the algebraic part is unchanged. In order to reduce the algebraic part, we need to take the following steps. We start from (8.2.6a), which can be written as

$$\mathcal{L}_q \xi_q = \mathbf{N}_q \mathcal{L}_q \xi'_q + \mathbf{b}_q, \tag{8.2.7}$$

where $\mathbf{b}_q = \mathbf{A}_q \mathbf{V}_p \hat{\xi}_p + \mathbf{B}_q \boldsymbol{u}$ and $\mathbf{N}_q = \mathcal{L}\mathcal{L}_q^{-1}$ is also a nilpotent matrix with the same index-μ as $\mathcal{L}$. Thus, (8.2.7) can be written as

$$\mathcal{L}_q \xi_q = \sum_{k=0}^{\mu-1} \mathbf{N}_q^k \mathbf{b}_q^{(k)} = \sum_{k=0}^{\mu-1} \mathbf{N}_q^k (\mathbf{A}_q \mathbf{V}_p \hat{\xi}_p^{(k)} + \mathbf{B}_q \boldsymbol{u}^{(k)}). \tag{8.2.8}$$

We can observe that, for the algebraic variable ξ_q, we have the restriction

$$\mathcal{L}_q \xi_q \in \mathcal{W}_q = \mathcal{K}_\mu(\mathbf{N}_q, \mathbf{R}_q),$$

with $\mathbf{R}_q = \begin{pmatrix}\mathbf{B}_q & \mathbf{A}_q\mathbf{V}_p\end{pmatrix} \in \mathbb{R}^{n_q\times(r+1)m}$. Since $\mathbf{N}_q = \mathcal{L}\mathcal{L}_q^{-1}$, it follows that

$$\xi_q \in \mathcal{V}_q = \mathcal{L}_q^{-1}\mathcal{W}_q = \mathcal{K}_\mu(\mathcal{L}_q^{-1}\mathbf{N}_q, \mathcal{L}_q^{-1}\mathbf{R}_q). \tag{8.2.9}$$

We denote by $\mathbf{V}_q$ an orthonormal basis of $\mathcal{V}_q$, so that $\mathbf{V}_q^{\mathrm{T}}\mathbf{V}_q = \mathbf{I}$, and we write $\xi_q = \mathbf{V}_q\hat{\xi}_q$. Substituting $\xi_q = \mathbf{V}_q\hat{\xi}_q$ into (8.2.6) and after simplifying, we obtain a reduced-order algebraic subsystem given by

$$-\hat{\mathcal{L}}\hat{\xi}_q' = \hat{\mathbf{A}}_q\hat{\xi}_p - \hat{\mathcal{L}}_q\hat{\xi}_q + \hat{\mathbf{B}}_q\boldsymbol{u}, \tag{8.2.10a}$$
$$\hat{y}_q = \hat{C}_q^{\mathrm{T}}\hat{\xi}_q, \tag{8.2.10b}$$

with $\hat{\mathcal{L}}_q = \mathbf{V}_q^{\mathrm{T}}\mathcal{L}_q\mathbf{V}_q \in \mathbb{R}^{\tau\times\tau}$, $\hat{\mathcal{L}} = \mathbf{V}_q^{\mathrm{T}}\mathcal{L}\mathbf{V}_q\mathbb{R}^{\tau\times\tau}$, $\hat{\mathbf{A}}_q = \mathbf{V}_q^{\mathrm{T}}\mathbf{A}_q\mathbf{V}_p\mathbb{R}^{\tau\times r}$, $\hat{\mathbf{B}}_q = \mathbf{V}_q^{\mathrm{T}}\mathbf{B}_q\mathbb{R}^{\tau\times m}$ and $\hat{\mathbf{C}}_q = \mathbf{V}_q^{\mathrm{T}}\mathbf{C}_q\mathbb{R}^{\tau\times\ell}$. $\hat{\xi}_q \in \mathbb{R}^\tau$ is the reduced algebraic state vector and $\hat{\boldsymbol{y}}_q \in \mathbb{R}^\ell$ is the approximated output. Thus the dimension of the algebraic part is reduced to $\tau \le n_q$. The transfer function of this reduced-order model of the algebraic part (8.2.4) is given by $\hat{\mathbf{H}}_q(s) := \hat{\mathbf{C}}_q^{\mathrm{T}}(\hat{\mathcal{L}}_q - s\hat{\mathcal{L}})^{-1}\Big[\hat{\mathbf{A}}_q(s\hat{\mathbf{E}}_p - \hat{\mathbf{A}}_p)^{-1}\hat{\mathbf{B}}_p + \hat{\mathbf{B}}_q\Big]$. Thus, combining (8.2.5) and (8.2.10), we obtain the IIMOR reduced-order model of (8.0.1) given by

$$\begin{aligned}\hat{\mathbf{E}}_p\hat{\xi}_p' &= \hat{\mathbf{A}}_p\hat{\xi}_p + \hat{\mathbf{B}}_p\boldsymbol{u},\\ -\hat{\mathcal{L}}\hat{\xi}_q' &= \hat{\mathbf{A}}_q\hat{\xi}_q - \hat{\mathcal{L}}_q\hat{\xi}_q + \hat{\mathbf{B}}_q\boldsymbol{u},\\ \hat{\boldsymbol{y}} &= \hat{\mathbf{C}}_p^{\mathrm{T}}\hat{\xi}_p + \hat{\mathbf{C}}_q^{\mathrm{T}}\hat{\xi}_q,\end{aligned} \tag{8.2.11}$$

with total dimension $r + \tau \ll n$, where r and τ are the dimension of the reduced differential and algebraic parts, respectively. The transfer function of the IMOR reduced model is equal to the sum of the transfer function of the differential and algebraic parts given by $\mathbf{H}_r(s) = \mathbf{H}_{p_r}(s) + \mathbf{H}_{q_r}(s)$.

Remark 8.2.1 *We note that in practice the algebraic part* (8.2.4a) *has* μ *algebraic subsystems. Thus* $\mathcal{V}_q$ *can be partitioned as:* $\mathcal{V}_q = \left(\mathcal{V}_{q,\mu-1}^{\mathrm{T}}, \cdots, \mathcal{V}_{q,1}^{\mathrm{T}}, \mathcal{V}_{q,0}^{\mathrm{T}}\right)^{\mathrm{T}}$, *where the length of each partition corresponds to the block sizes of algebraic subsystems given by* k_i, $i = \mu - 1, \ldots, 1, 0$, *respectively. We can then compute the orthonormal basis matrix of each partition to build a block diagonal orthonormal matrices*
$\mathbf{V}_q = \mathrm{blkdiag}\Big[\mathbf{V}_{q,\mu-1}, \ldots, \mathbf{V}_{q,1}, \mathbf{V}_{q,0}\Big]$, *where* $\mathbf{V}_{q,i} = \mathrm{orth}(\mathcal{V}_{q,i})$. *Although this approach leads to a much larger reduced-order model especially for MIMO systems, but preserves the structure of the algebraic parts.*

For comparison with other existing MOR methods, we can rewrite the IIMOR reduced-order model in descriptor form given by

$$\begin{aligned}\hat{\mathbf{E}}\xi_r' &= \hat{\mathbf{A}}\hat{\xi} + \hat{\mathbf{B}}\boldsymbol{u}\\ \hat{\boldsymbol{y}}_r &= \hat{\mathbf{C}}^{\mathrm{T}}\hat{\xi},\end{aligned} \tag{8.2.12}$$

where $\hat{\mathbf{E}} = \begin{pmatrix}\hat{\mathbf{E}}_p & 0\\ 0 & -\hat{\mathcal{L}}\end{pmatrix}$, $\hat{\mathbf{A}} = \begin{pmatrix}\hat{\mathbf{A}}_p & 0\\ \hat{\mathbf{A}}_q & -\hat{\mathcal{L}}_q\end{pmatrix}$, $\hat{\mathbf{B}} = \begin{pmatrix}\hat{\mathbf{B}}_p\\ \hat{\mathbf{B}}_q\end{pmatrix}$, $\hat{\mathbf{C}} = \begin{pmatrix}\hat{\mathbf{C}}_p\\ \hat{\mathbf{C}}_q\end{pmatrix}$ and $\hat{\xi} = \begin{pmatrix}\hat{\xi}_p\\ \hat{\xi}_q\end{pmatrix}^{\mathrm{T}}$ for DAEs with a differential part and $\hat{\mathbf{E}} = -\hat{\mathcal{L}}$, $\hat{\mathbf{A}} = -\mathrm{I}$, $\hat{\mathbf{B}} = \hat{\mathbf{B}}$, $\hat{\mathbf{C}} = \hat{\mathbf{C}}_q$ and $\hat{\xi} = \hat{\xi}_q$ for DAEs without a differential part and the transfer function of the reduced-order model can be written as $\hat{\mathbf{H}}(s) = \hat{\mathbf{C}}(s\hat{\mathbf{E}} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}}$.

8.3 Simple examples

In order to compare IIMOR and IMOR methods, we use the same examples as in Section 7.3. Thus, we use the matrices from Examples 3.2.1 and 3.2.2. These are index-2 and -3 DAEs, respectively. In order to use the IIMOR method, we first decouple these systems using the implicit decoupling procedure derived in Chapter 6.

Example 8.3.1 In this example, we use matrices from Example 3.2.1 with both cases of the control input matrix **B**. These are index-2 DAEs with the same matrix pencil (**E**, **A**). These DAEs are solvable and they have only one finite eigenvalue, thus we expect their decoupled systems to have a differential part.
(i) For this case, we use the control input matrix **B** of Example 3.2.1(i). Using the implicit decoupling procedure this system can be decoupled in to the form (6.2.3) given by

$$\begin{aligned}
-\frac{2}{3}\xi_p' &= \xi_p + \frac{1}{2}\boldsymbol{u},\\
\begin{pmatrix}-1 & -1\\ -1 & 1\end{pmatrix}\xi_{q,1} &= \begin{pmatrix}0\\0\end{pmatrix}\xi_p + \begin{pmatrix}0\\0\end{pmatrix}\boldsymbol{u},\\
\begin{pmatrix}-\frac{20}{7} & \frac{3}{7}\\ \frac{4}{7} & -\frac{2}{7}\end{pmatrix}\xi_{q,0} &= \begin{pmatrix}\frac{11}{3}\\ -\frac{2}{3}\end{pmatrix}\xi_p + \begin{pmatrix}\frac{17}{14}\\ -\frac{1}{7}\end{pmatrix}\boldsymbol{u} + \begin{pmatrix}-1 & -\frac{13}{7}\\ 0 & \frac{4}{7}\end{pmatrix}\left[\xi_{q,1}' - \xi_{q,1}\right],\\
\boldsymbol{y} &= \frac{2}{3}\xi_p + \begin{pmatrix}1 & -1\end{pmatrix}\xi_{q,1} + \begin{pmatrix}1 & 0\end{pmatrix}\xi_{q,0}.
\end{aligned} \tag{8.3.1}$$

Before, we apply the IIMOR method on the system (8.3.1), we need to use the AE method which leads to a AE reduced-order model given by

$$
\begin{aligned}
-\frac{2}{3}\xi_p' &= \xi_p + \frac{1}{2}\boldsymbol{u},\\
\begin{pmatrix}-1 & -1\\ -1 & 1\end{pmatrix}\xi_{q,1} &= \begin{pmatrix}0\\0\end{pmatrix}\xi_p + \begin{pmatrix}0\\0\end{pmatrix}\boldsymbol{u},\\
-2\xi_{q_1,0} &= \frac{8}{3}\xi_p + \boldsymbol{u} - \begin{pmatrix}1 & 1\end{pmatrix}\left[\xi_{q,1}' - \xi_{q,1}\right],\\
\boldsymbol{y} &= \frac{2}{3}\xi_p + \begin{pmatrix}1 & -1\end{pmatrix}\xi_{q,1} + \xi_{q_1,0},
\end{aligned}
\tag{8.3.2}
$$

We can now apply the IIMOR method as follow. This done by first rewriting system (8.3.2) into the form (8.2.1) with system matrices given by

$$
\begin{aligned}
&\mathbf{E}_p = -\frac{2}{3},\ \mathbf{A}_p = 1,\ \mathbf{B}_p = \frac{1}{2},\ \mathbf{C}_p = \frac{2}{3},\\
&\mathcal{L} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -1 & -1 & 0\end{pmatrix},\ \mathcal{L}_q = \begin{pmatrix}-1 & -1 & 0\\ -1 & 1 & 0\\ -1 & -1 & -2\end{pmatrix},\ \mathbf{A}_q = \begin{pmatrix}0\\0\\\frac{8}{3}\end{pmatrix},\ \mathbf{B}_q = \begin{pmatrix}0\\0\\1\end{pmatrix},\ \mathbf{C}_q = \begin{pmatrix}1\\-1\\1\end{pmatrix}.
\end{aligned}
\tag{8.3.3}
$$

We can observe that the differential part cannot be reduced any further, thus we can just set $\mathbf{V}_p = 1$. In order to further reduce the algebraic parts, we use (8.2.9) to construct the Krylov subspace of order $\mu = 2$ given by,

$$\mathcal{K}_2(\tilde{\mathbf{N}}_q, \tilde{\mathbf{R}}_q) = \mathrm{Span}\{\tilde{\mathbf{R}}_q, \tilde{\mathbf{N}}_q\tilde{\mathbf{R}}_q\},$$

where $\mathbf{R}_q = \begin{bmatrix}\mathbf{B}_q & \mathbf{A}_q\mathbf{V}_p\end{bmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\\ 1 & \frac{8}{3}\end{pmatrix}$, $\tilde{\mathbf{N}}_q = \mathcal{L}_q^{-1}\mathbf{N}_q = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -\frac{1}{2} & 0 & 0\end{pmatrix}$ and $\tilde{\mathbf{R}}_q = \mathcal{L}_q^{-1}\mathbf{R}_q = \begin{pmatrix}0 & 0\\ 0 & 0\\ -\frac{1}{2} & -\frac{4}{3}\end{pmatrix}$. Then the orthonormal basis is given by $\mathbf{V}_q = \begin{pmatrix}0 & 0 & 1\end{pmatrix}^{\mathrm{T}}$. Thus, substituting (8.3.3) and orthonormal basis $\mathbf{V}_q$ into (8.2.11). We obtain a reduced-order model which can be written in descriptor form (8.2.12) with system matrices given by

$$
\hat{\mathbf{E}} = \begin{pmatrix}-\frac{2}{3} & 0\\ 0 & 0\end{pmatrix},\quad \hat{\mathbf{A}} = \begin{pmatrix}1 & 0\\ \frac{8}{3} & 2\end{pmatrix},\quad \hat{\mathbf{B}} = \begin{pmatrix}\frac{1}{2}\\ \frac{8}{3}\end{pmatrix},\quad \hat{\mathbf{C}} = \begin{pmatrix}\frac{2}{3}\\ 1\end{pmatrix}.
\tag{8.3.4}
$$

Thus the dimension of the DAE is reduced from 5 to 2 using the IIMOR method. It is easy to check that this reduced-order mode is accurate since its transfer function and output solution coincides with that of the original DAE. If we compare the IIMOR reduced-order model (8.3.4) and (7.3.3), we observe they have the same size.

(ii) For this case, we use the control input matrix **B** of Example 3.2.1(ii). Using the implicit decoupling procedure this system can be decoupled in to the form (6.2.3) given by

$$\begin{aligned}
-\frac{2}{3}\xi_p' &= \xi_p + \frac{1}{2}\boldsymbol{u},\\
\begin{pmatrix}-1 & -1\\ -1 & 1\end{pmatrix}\xi_{q,1} &= \begin{pmatrix}0\\0\end{pmatrix}\xi_p + \begin{pmatrix}0\\-1\end{pmatrix}\boldsymbol{u},\\
\begin{pmatrix}-\frac{20}{7} & \frac{3}{7}\\ \frac{4}{7} & -\frac{2}{7}\end{pmatrix}\xi_{q,0} &= \begin{pmatrix}\frac{11}{3}\\ -\frac{2}{3}\end{pmatrix}\xi_p + \begin{pmatrix}0\\-1\end{pmatrix}\boldsymbol{u} + \begin{pmatrix}-1 & -\frac{13}{7}\\ 0 & \frac{4}{7}\end{pmatrix}\left[\xi_{q,1}' - \xi_{q,1}\right],\\
\boldsymbol{y} &= \frac{2}{3}\xi_p + \begin{pmatrix}1 & -1\end{pmatrix}\xi_{q,1} + \begin{pmatrix}1 & 0\end{pmatrix}\xi_{q,0}.
\end{aligned} \tag{8.3.5}$$

Here, we also first apply the AE method which leads to a reduced decoupled system given by

$$\begin{aligned}
-\frac{2}{3}\xi_p' &= \xi_p + \frac{1}{2}\boldsymbol{u},\\
\begin{pmatrix}-1 & -1\\ -1 & 1\end{pmatrix}\xi_{q,1} &= \begin{pmatrix}0\\0\end{pmatrix}\xi_p + \begin{pmatrix}0\\-1\end{pmatrix}\boldsymbol{u},\\
-2\xi_{q_1,0} &= \frac{8}{3}\xi_p - \frac{3}{2}\boldsymbol{u} - \begin{pmatrix}1 & 1\end{pmatrix}\left[\xi_{q,1}' - \xi_{q,1}\right],\\
\boldsymbol{y} &= \frac{2}{3}\xi_p + \begin{pmatrix}1 & -1\end{pmatrix}\xi_{q,1} + \xi_{q_1,0},
\end{aligned} \tag{8.3.6}$$

Then, we apply the implicit IMOR method as follows. This done by first rewriting system (8.3.6) into the form (8.2.1) with system matrices given by

$$\begin{aligned}
&\mathbf{E}_p = -\frac{2}{3},\ \mathbf{A}_p = 1,\ \mathbf{B}_p = \frac{1}{2},\ \mathbf{C}_p = \frac{2}{3},\\
&\mathcal{L} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -1 & -1 & 0\end{pmatrix},\ \mathcal{L}_q = \begin{pmatrix}-1 & -1 & 0\\ -1 & 1 & 0\\ -1 & -1 & -2\end{pmatrix}, \mathbf{A}_q = \begin{pmatrix}0\\0\\ \frac{8}{3}\end{pmatrix}, \mathbf{B}_q = \begin{pmatrix}0\\-1\\-\frac{3}{2}\end{pmatrix},\ \mathbf{C}_q = \begin{pmatrix}1\\-1\\1\end{pmatrix}.
\end{aligned} \tag{8.3.7}$$

We can observe that the differential part cannot be reduced any further, thus we can also just set $\mathbf{V}_p = 1$. In order to further reduce the algebraic parts, we use (8.2.9) to construct the Krylov subspace of order $\mu = 2$ given by

$$\mathcal{K}_2(\tilde{\mathbf{N}}_q, \tilde{\mathbf{R}}_q) = \mathrm{Span}\{\tilde{\mathbf{R}}_q, \tilde{\mathbf{N}}_q\tilde{\mathbf{R}}_q\},$$

where $\mathbf{R}_q = \begin{bmatrix}\mathbf{B}_q & \mathbf{A}_q\mathbf{V}_p\end{bmatrix} = \begin{pmatrix}0 & 0\\ -1 & 0\\ -\frac{3}{2} & \frac{8}{3}\end{pmatrix}$, $\tilde{\mathbf{N}}_q = \mathcal{L}_q^{-1}\mathbf{N}_q = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -\frac{1}{2} & 0 & 0\end{pmatrix}$ and $\tilde{\mathbf{R}}_q = \mathcal{L}_q^{-1}\mathbf{R}_q = \begin{pmatrix}\frac{1}{2} & 0\\ -\frac{1}{2} & 0\\ \frac{3}{4} & -\frac{4}{3}\end{pmatrix}$. Then the orthonormal basis is given by $\mathbf{V}_q = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0\\ -\frac{1}{\sqrt{2}} & 0\\ 0 & 1\end{pmatrix}$. Substituting orthonormal basis $\mathbf{V}_q$ and

(8.3.7) into (8.2.11) , we obtain a reduced-order model with system matrices given by

$$\mathbf{E}_{p_r} = -\frac{2}{3},\ \mathbf{A}_{p_r} = 1,\ \mathbf{B}_{p_r} = \frac{1}{\sqrt{2}},\ \mathbf{C}_{p_r} = \frac{2}{3},$$
$$\mathcal{L}_r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ \mathbf{A}_{q_r} = \begin{pmatrix} 0 \\ \frac{8}{3} \end{pmatrix},\ \mathcal{L}_{q_r} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \mathbf{B}_{q_r} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{3}{2} \end{pmatrix},\ \mathbf{C}_{q_r} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}. \tag{8.3.8}$$

This reduced-order model can be written in descriptor form (8.2.12) with system matrices given by

$$\hat{\mathbf{E}} = \begin{pmatrix} -\frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad \hat{\mathbf{A}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \frac{8}{3} & 0 & 2 \end{pmatrix},\quad \hat{\mathbf{B}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{3}{2} \end{pmatrix},\quad \hat{\mathbf{C}} = \begin{pmatrix} \frac{2}{3} \\ \sqrt{2} \\ 1 \end{pmatrix}. \tag{8.3.9}$$

Thus the dimension of the DAE is reduced from 5 to 3 using the IIMOR method. It is easy to check that this reduced-order mode is accurate since its transfer function and output solution coincides with that of the original DAE. If we compare the IIMOR reduced-order model (8.3.9) and (7.3.6), we observe they have the same size.

Example 8.3.2 In this example, we used system matrices from Example 3.2.2. This is an index-3 system whose matrix pencil has at least one finite eigenvalue. Thus it can be decoupled into the form (6.3.3) given by

$$\frac{1}{2}\xi_p' = -\xi_p + \boldsymbol{u},$$
$$-\xi_{q,2} = 0\xi_p + \boldsymbol{u},$$
$$\frac{1}{2}\xi_{q,1} = 0\xi_p + \boldsymbol{u} + \frac{1}{2}\xi_{q,2} + \frac{1}{2}\xi_{q,2}',$$
$$\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{pmatrix} \xi_{q,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \xi_p + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \boldsymbol{u} + \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 0 \end{pmatrix} \xi_{q,2} + \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} \xi_{q,1} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 0 \end{pmatrix} \xi_{q,2}' + \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} \xi_{q,1}', \tag{8.3.10}$$
$$\boldsymbol{y} = \xi_p + 0\xi_{q,2} + 0\xi_{q,1} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xi_{q,0}.$$

We can observe that the DAE system of dimension is decoupled into 1 differential equation and 8 algebraic equations. Also here, we used the AE method on (8.3.10), which reduced it to an index-2 reduced order system given by

$$\frac{1}{2}\xi_p' = -\xi_p + \boldsymbol{u}, \tag{8.3.11a}$$
$$-\xi_{q,2} = 0\xi_p + \boldsymbol{u}, \tag{8.3.11b}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} \xi_{q,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \xi_p + \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \boldsymbol{u} + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \xi_{q,2} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \xi'_{q,2}, \tag{8.3.11c}$$

$$\boldsymbol{y} = \xi_p + 0\xi_{q,2} + \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \xi_{q,0}. \tag{8.3.11d}$$

We can observe the decoupled system cannot be completely reduced by the AE method, thus we can now apply the implicit IMOR method as follows. The AE reduced model can be written into the form (8.2.1) with system matrices given by

$$\mathbf{E}_p = \frac{1}{2}, \mathbf{A}_p = -1, \mathbf{B}_p = 1, \mathbf{C}_p = 1,$$
$$\mathcal{L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \mathcal{L}_q = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}, \mathbf{A}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{B}_q = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 2 \end{pmatrix}, \mathbf{C}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{8.3.12}$$

We can observe that the differential part cannot be reduced any further, thus we can just set $\mathbf{V}_p = 1$. In order to further reduce the algebraic parts, we use (8.2.9) to construct the Krylov subspace of order $\mu = 2$,

$$\mathcal{K}_2(\tilde{\mathbf{N}}_q, \tilde{\mathbf{R}}_q) = \text{Span}\{\tilde{\mathbf{R}}_q, \tilde{\mathbf{N}}_q \tilde{\mathbf{R}}_q\},$$

where $\mathbf{R}_q = \begin{bmatrix} \mathbf{B}_q & \mathbf{A}_q \mathbf{V}_p \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 0 \\ 2 & 0 \end{pmatrix}$, $\tilde{\mathbf{N}}_q = \mathcal{L}_q^{-1} \mathbf{N}_q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\tilde{\mathbf{R}}_q = \mathcal{L}_q^{-1} \mathbf{R}_q = -\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$. Then the orthonormal basis is given by $\mathbf{V}_q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$. Substituting orthonormal basis $\mathbf{V}_q$ and (8.3.12) into (8.2.11). We obtain a reduced-order model with system matrices given by

$$\mathbf{E}_{p_r} = \frac{1}{2}, \mathbf{A}_{p_r} = -1, \mathbf{B}_{p_r} = 1, \mathbf{C}_{p_r} = 1,$$
$$\mathcal{L}_r = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}, \mathcal{L}_{q_r} = \begin{pmatrix} -1 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{4} & \frac{2\sqrt{2}-1}{4} \\ -\frac{1}{2} & -\frac{2\sqrt{2}+1}{4} & -\frac{1}{4} \end{pmatrix}, \mathbf{A}_{q_r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{B}_{q_r} = \begin{pmatrix} 1 \\ \frac{\sqrt{2}-1}{2} \\ -\frac{\sqrt{2}+1}{2} \end{pmatrix}, \mathbf{C}_{q_r} = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}. \tag{8.3.13}$$

The reduced system can be written in descriptor form (8.2.12) with system matrices

given by

$$\hat{\mathbf{E}} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}, \hat{\mathbf{A}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{-2\sqrt{2}+1}{4} \\ 0 & \frac{1}{2} & \frac{2\sqrt{2}+1}{4} & \frac{1}{4} \end{pmatrix}, \quad \hat{\mathbf{B}} = \begin{pmatrix} 1 \\ 1 \\ \frac{\sqrt{2}-1}{2} \\ -\frac{\sqrt{2}+1}{2} \end{pmatrix}, \quad \hat{\mathbf{C}} = \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}. \tag{8.3.14}$$

Thus the dimension of the DAE is reduced from 9 to 4 using the IIMOR method. It is easy to check that this reduced-order mode is accurate since its transfer function and output solution coincides with that of the original DAE. If we compare the IIMOR reduced-order model (8.3.14) and (7.3.9), we observe that the IIMOR reduced model is a much larger reduced model.

8.4 Extension of IIMOR method to truncation methods

In this Section, we discuss how the IIMOR method can be extended to the truncation methods especially the balanced truncation method. This approach is the same as that discussed in Section 7.4 for the case of the IMOR method. However, for this case we need to solve the generalized Lyapunov equations in order to compute the system controllability and observability Gramians. This is done as follows. Consider a stable DAE (8.0.1) which is decoupled into a stable implicit decoupled system in the form (8.2.1) given by

$$\begin{aligned} \mathbf{E}_p\xi_p' &= \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u} \\ -\mathcal{L}\xi_q' &= \mathbf{A}_q\xi_p - \mathcal{L}_q\xi_q + \mathbf{B}_q\boldsymbol{u}, \\ \boldsymbol{y} &= \mathbf{C}_p^{\mathrm{T}}\xi_p + \mathbf{C}_q^{\mathrm{T}}\xi_q, \end{aligned} \tag{8.4.1}$$

where $\mathcal{L}$ is a nilpotent matrix of index μ. $\mathcal{L}_q$ is a nonsingular lower triangular matrix with block diagonal matrices for $\mu > 1$. $\xi_p \in \mathbb{R}^{n_p}$, $\xi_q \in \mathbb{R}^{n_q}$, $\mathbf{A}_q \in \mathbb{R}^{n_q \times n_p}$, $\mathbf{B}_q \in \mathbb{R}^{n_q \times m}$ and $\mathbf{C}_q \in \mathbb{R}^{n_q \times \ell}$, $\mathbf{C}_p \in \mathbb{R}^{n_p \times \ell}$. Then (8.4.1) can be strictly separated obtaining,

$$\begin{aligned} \mathbf{E}_p\xi_p' &= \mathbf{A}_p\xi_p + \mathbf{B}_p\boldsymbol{u}, \\ \boldsymbol{y}_p &= \mathbf{C}_p^{\mathrm{T}}\xi_p, \end{aligned} \tag{8.4.2}$$

and

$$-\mathcal{L}\xi_q' = \mathbf{A}_q\xi_p - \mathcal{L}_q\xi_q + \mathbf{B}_q\boldsymbol{u}, \tag{8.4.3a}$$
$$\boldsymbol{y}_q = \mathbf{C}_q^{\mathrm{T}}\xi_q, \tag{8.4.3b}$$

where (8.4.2) and (8.4.3) are the differential and algebraic subsystems. Then the output equation can be reconstructed using: $\boldsymbol{y} = \boldsymbol{y}_p + \boldsymbol{y}_q$. We have already discussed that if the DAE system is stable then the differential part must also be stable. This implies that $\sigma(\mathbf{E}_p, \mathbf{A}_p) \subset \mathbb{C}^-$.

8.4.1 Reduction of the differential part

Here, we follow the same steps as in Section 7.4.1. After applying the balancing transformation $\xi_p = \mathbf{T}_p\tilde{\xi}_p$ on (8.4.2). We seek a balanced system given by

$$\begin{aligned}\tilde{\mathbf{E}}_p\tilde{\xi}_p' &= \tilde{\mathbf{A}}_p\tilde{\xi}_p + \tilde{\mathbf{B}}_p\boldsymbol{u},\\ \tilde{\boldsymbol{y}}_p &= \tilde{\mathbf{C}}_p^{\mathrm{T}}\tilde{\xi}_p,\end{aligned} \tag{8.4.4}$$

where $\tilde{\mathbf{E}}_p = \mathbf{T}_p^{-1}\mathbf{E}_p\mathbf{T}_p$, $\tilde{\mathbf{A}}_p = \mathbf{T}_p^{-1}\mathbf{A}_p\mathbf{T}_p$, $\tilde{\mathbf{B}}_p = \mathbf{T}_p^{-1}\mathbf{B}_p$ and $\tilde{\mathbf{C}}_p = \mathbf{T}_p^{\mathrm{T}}\mathbf{C}_p$. Using the coefficient matrices of (8.4.2), we need to compute the balancing transformation nonsingular matrix $\mathbf{T}_p$. This is done by solving the two generalized Lyapunov equations

$$\mathbf{E}_p\mathcal{P}_p\mathbf{A}_p^{\mathrm{T}} + \mathbf{A}_p\mathcal{P}_p\mathbf{E}_p^{\mathrm{T}} = -\mathbf{B}_p\mathbf{B}_p^{\mathrm{T}}, \quad \mathbf{E}_p^{\mathrm{T}}\mathcal{Q}_p\mathbf{A}_p + \mathbf{A}_p^{\mathrm{T}}\mathcal{Q}_p\mathbf{E}_p = -\mathbf{C}_p\mathbf{C}_p^{\mathrm{T}}. \tag{8.4.5}$$

for the controllability Gramian $\mathcal{P}_p$ and observability Gramian $\mathcal{Q}_p$ instead of (7.4.5). We then follow procedure (7.4.6) – (7.4.8) to derive the balancing transformation $\mathbf{T}_p \in \mathbb{R}^{n_p\times n_p}$ and its inverse given by

$$\mathbf{T}_p = \mathbf{U}_p\mathbf{K}_p\Sigma_p^{-1/2} \quad \text{and} \quad \mathbf{T}_p^{-1} = \Sigma_p^{1/2}\mathbf{K}_p^{\mathrm{T}}\mathbf{U}_p^{-1}. \tag{8.4.6}$$

It can also easily be shown that $\mathbf{T}_p$ indeed balances the system (8.4.2) that is, $\mathbf{T}_p^{-1}\mathcal{P}_p\mathbf{T}_p^{-\mathrm{T}} = \Sigma_p$ and $\mathbf{T}_p^{\mathrm{T}}\mathcal{Q}_p\mathbf{T}_p = \Sigma$. We then truncate the balanced system (8.4.4) in order to obtain a reduced-order model of (8.4.2). Since a transformation was defined which transforms the system according to the Hankel singular values, now very easily a truncation can be defined using the partition of the diagonal matrix Σ_p given by

$$\Sigma_p = \begin{pmatrix} \Sigma_{p_1} & \mathbf{0} \\ \mathbf{0} & \Sigma_{p_2} \end{pmatrix},$$

where Σ_{p_1} contains the largest Hankel singular values. Thus $\tilde{\mathbf{E}}_p$, $\tilde{\mathbf{A}}_p$, $\tilde{\mathbf{B}}_p$ and $\tilde{\mathbf{C}}_p$ can be partitioned in conformance with Σ_p:

$$\tilde{\mathbf{E}}_p = \begin{pmatrix} \tilde{\mathbf{E}}_{p_{11}} & \tilde{\mathbf{E}}_{p_{12}} \\ \tilde{\mathbf{E}}_{p_{21}} & \tilde{\mathbf{E}}_{p_{22}} \end{pmatrix}, \quad \tilde{\mathbf{A}}_p = \begin{pmatrix} \tilde{\mathbf{A}}_{p_{11}} & \tilde{\mathbf{A}}_{p_{12}} \\ \tilde{\mathbf{A}}_{p_{21}} & \tilde{\mathbf{A}}_{p_{22}} \end{pmatrix}, \quad \tilde{\mathbf{B}}_p = \begin{pmatrix} \tilde{\mathbf{B}}_{p_1} \\ \tilde{\mathbf{B}}_{p_2} \end{pmatrix}, \quad \tilde{\mathbf{C}}_p = \begin{pmatrix} \tilde{\mathbf{C}}_{p_1} \\ \tilde{\mathbf{C}}_{p_2} \end{pmatrix} \tag{8.4.7}$$

and transformed variables can also be partitioned as $\tilde{\xi}_p = (\tilde{\xi}_{p_1}^{\mathrm{T}}, \tilde{\xi}_{p_2}^{\mathrm{T}})^{\mathrm{T}}$. Thus the reduced-order model of the differential subsystem (8.4.2) is given by

$$\begin{aligned} \mathbf{E}_{p_r}\xi'_{p_r} &= \mathbf{A}_{p_r}\xi_{p_r} + \mathbf{B}_{p_r}\boldsymbol{u}, \\ \boldsymbol{y}_{p_r} &= \mathbf{C}_{p_r}^{\mathrm{T}}\xi_{p_r}, \end{aligned} \tag{8.4.8}$$

where $\mathbf{E}_{p_r} = \tilde{\mathbf{E}}_{p_{11}}$, $\mathbf{A}_{p_r} = \tilde{\mathbf{A}}_{p_{11}} \in \mathbb{R}^{r\times r}$, $\mathbf{B}_{p_r} = \tilde{\mathbf{B}}_{p_1} \in \mathbb{R}^{r\times m}$, $\mathbf{C}_{p_r} = \tilde{\mathbf{C}}_{p_1} \in \mathbb{R}^{r\times \ell}$ and $\xi_{p_r} = \tilde{\xi}_{p_1} \in \mathbb{R}^r$, $\boldsymbol{y}_{p_r} = \tilde{\boldsymbol{y}}_p \in \mathbb{R}^{\ell\times r}$. The reduced-order model (8.4.8) is also stable with Hankel singular values given by diagonal elements of $\Sigma_{p_1} = \mathrm{diag}(\sigma_1, \cdots, \sigma_r)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, where $r \ll n_p$ is the order of the reduced system (8.4.8). It is also possible to choose r via computable error bound (7.4.12).

8.4.2 Reduction of the algebraic part

We have seen that in the previous Section, we have just reduced the differential subsystem (8.4.2) but the algebraic subsystem (8.4.3) is left unreduced. However, if we make a transformation $\xi_p = \mathbf{T}_p\tilde{\xi}_p$, where $\mathbf{T}_p$ is the balancing transformation of the differential part (8.4.2), it induces a balancing also in the algebraic part. Thus (8.4.3a) can be written as

$$-\mathcal{L}\tilde{\xi}'_q = \mathbf{A}_q\mathbf{T}_p\tilde{\xi}_p - \mathcal{L}_q\tilde{\xi}_q + \mathbf{B}_q\boldsymbol{u}, \tag{8.4.9}$$

where $\tilde{\xi}_q$ is the approximation of ξ_q induced by balancing of the differential subsystem (8.4.2). Thus, we can also partition $\tilde{\mathbf{A}}_q = \mathbf{A}_q\mathbf{T}_p$ as $\tilde{\mathbf{A}}_q = \begin{pmatrix} \tilde{\mathbf{A}}_{q_{11}} & \tilde{\mathbf{A}}_{q_{12}} \end{pmatrix}$ corresponding to the partition in (8.4.7). Thus, (8.4.9) can be reduced to

$$-\mathcal{L}\tilde{\xi}_q' = \tilde{\mathbf{A}}_{q_{11}}\tilde{\xi}_{p_1} - \mathcal{L}_q\tilde{\xi}_q + \mathbf{B}_q\boldsymbol{u}. \tag{8.4.10}$$

From (8.4.10), without loss of generality and using (8.2.9), we have

$$\xi_q \in \mathcal{T}_q = \mathcal{K}_\mu(\mathcal{L}_q^{-1}\mathbf{N}_q, \mathcal{L}_q^{-1}\mathbf{R}_q), \tag{8.4.11}$$

with $\mathbf{R}_q = \begin{pmatrix}\mathbf{B}_q & \tilde{\mathbf{A}}_{q_{11}}\end{pmatrix} \in \mathbb{R}^{n_q\times(r+1)m}$ and $\mathbf{N}_q = \mathcal{L}\mathcal{L}_q^{-1}$. We denote by $\mathbf{T}_q$ an orthonormal basis of $\mathcal{T}_q$, so that $\mathbf{T}_q^{\mathrm{T}}\mathbf{T}_q = \mathrm{I}$, and we write $\tilde{\xi}_q = \mathbf{T}_q\xi_{q_r}$. Thus, substituting $\tilde{\xi}_q = \mathbf{T}_q\xi_{q_r}$ into (8.4.11) and (8.4.3b), we obtain the reduced-order model of the algebraic part (8.4.3) given by

$$-\mathcal{L}_r\xi_{q_r}' = \mathbf{A}_{q_r}\xi_{p_r} - \mathcal{L}_{q_r}\xi_{q_r} + \mathbf{B}_{q_r}\boldsymbol{u}, \tag{8.4.12a}$$
$$\boldsymbol{y}_{q_r} = \mathbf{C}_{q_r}^{\mathrm{T}}\xi_{q_r}. \tag{8.4.12b}$$

where $\mathcal{L}_{q_r} = \mathbf{T}_q^{\mathrm{T}}\mathcal{L}_q\mathbf{T}_q$, $\mathcal{L}_r = \mathbf{T}_q^{\mathrm{T}}\mathcal{L}\mathbf{T}_q \in \mathbb{R}^{\tau\times\tau}$, $\mathbf{A}_{q_r} = \mathbf{T}_q^{\mathrm{T}}\tilde{\mathbf{A}}_{q_{11}} \in \mathbb{R}^{\tau\times r}$, $\mathbf{B}_{q_r} = \mathbf{T}_q^{\mathrm{T}}\mathbf{B}_q \in \mathbb{R}^{\tau\times m}$ and $\mathbf{C}_{q_r} = \mathbf{T}_q^{\mathrm{T}}\mathbf{C}_q \in \mathbb{R}^{\tau\times\ell}$. We note that the dimension of the reduced algebraic system is given by $\tau = \dim(\mathcal{T}_q) < n_q$. Thus, the IMOR reduced-order model based on the balanced truncation method of a DAE is given by

$$\begin{aligned}\mathbf{E}_{p_r}\xi_{p_r}' &= \mathbf{A}_{p_r}\xi_{p_r} + \mathbf{B}_{p_r}\boldsymbol{u},\\ -\mathcal{L}_r\xi_{q_r}' &= \mathbf{A}_{q_r}\xi_{p_r} - \mathcal{L}_{q_r}\xi_{q_r} + \mathbf{B}_{q_r}\boldsymbol{u},\\ \boldsymbol{y}_r &= \mathbf{C}_{p_r}^{\mathrm{T}}\xi_{p_r} + \mathbf{C}_{q_r}^{\mathrm{T}}\xi_{q_r},\end{aligned} \tag{8.4.13}$$

with total dimension $r + \tau \ll n$, where r and τ are the dimension of the reduced differential and algebraic parts, respectively. We note that this reduced-order model will also be always stable and have a computable error bound for the differential part. This is illustrated in the next example.

Example 8.4.1 For comparison, we use system matrices (7.4.19) from Example 7.4.1. This DAE can be decoupled into the form (8.4.1) with system matrices

$$\mathbf{E}_p = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{A}_p = \begin{pmatrix} -0.2 & 0.1 & 0 \\ 0.1 & -0.1 & -1.0 \\ 0 & 1.0 & 0.0 \end{pmatrix}, \mathbf{B}_p = \begin{pmatrix} 0.1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C}_p = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathcal{L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{L}_q = \begin{pmatrix} 0 & 0.1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0.1 & 0 & 1 & 0 \end{pmatrix}, \mathbf{A}_q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.1 & 0 & 0 \end{pmatrix}, \mathbf{B}_q = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

We can observe that the DAE is decoupled into $n_p = 3$ differential equations and $n_q = 4$ algebraic equations, and the decoupled system is still stable since $\sigma(\mathbf{E}_p, \mathbf{A}_p) = \sigma_f(\mathbf{E}, \mathbf{A})$. Next, we use the balanced truncation method to reduce the dimension of the system. This can be done as follows. We substitute matrices $\mathbf{E}_p$, $\mathbf{A}_p$, $\mathbf{B}_p$ and $\mathbf{C}_p$ into (8.4.5) to solve for the Gramians given by

$$\mathcal{P}_p = \begin{pmatrix} 0.0169 & 0.0086 & 0.0025 \\ 0.0086 & 0.0167 & 0.0000 \\ 0.0025 & 0.0000 & 0.0250 \end{pmatrix} \quad \text{and} \quad \mathcal{Q}_p = \begin{pmatrix} 2.5 & 5.0 & 0.25 \\ 5.0 & 10.0 & 0.5 \\ 0.25 & 0.5 & 15.025 \end{pmatrix}.$$

We can then use these Gramians to construct the balancing transformation and its inverse using (8.4.6) obtaining

$$\mathbf{T}_p = \begin{pmatrix} 0.6486 & 1.2972 & 4.7167 \\ 2.0457 & 4.0913 & -1.5068 \\ 0.0007 & -0.0005 & -0.0001 \end{pmatrix} \quad \text{and} \quad \mathbf{T}_p^{-1} = \begin{pmatrix} 0.054218 & 0.12336 & 1095.7 \\ 0.04379 & 0.16026 & -547.85 \\ 0.19251 & -0.061038 & 0 \end{pmatrix}. \qquad (8.4.14)$$

Substituting (8.4.14) into (8.4.4), we obtain a balanced system which can then be partitioned in conformance with Σ_p to obtain a reduced-order algebraic subsystem of the form (8.4.8) with coefficient matrices given by

$$\mathbf{E}_{p_r} = \begin{pmatrix} 0.99324 & 0.023928 \\ -0.021333 & 1.0755 \end{pmatrix}, \mathbf{A}_{p_r} = \begin{pmatrix} -0.046022 & 0.82467 \\ -0.86258 & -0.024527 \end{pmatrix},$$
$$\mathbf{B}_{p_r} = \begin{pmatrix} 0.064859 \\ 0.20457 \end{pmatrix}, \mathbf{C}_{p_r} = \begin{pmatrix} 0.19251 \\ -0.061038 \end{pmatrix}. \qquad (8.4.15)$$

We observe that the reduced-order subsystem is of dimension $r = 2$ and the computable error bound is given by $\|\mathbf{H}_p - \mathbf{H}_{p_r}\|_2 \leq 11.6 \times 10^{-9} \|\boldsymbol{u}\|_2$. We can then compute the orthonormal basis $\mathbf{T}_q$ using (8.4.11) which reduces the algebraic subsystem which is given by

$$\mathbf{T}_q = \begin{pmatrix} -9.9504 \cdot 10^{-1} & 7.3880 \cdot 10^{-4} & -9.9519 \cdot 10^{-2} \\ 0 & 0 & 0 \\ 9.9521 \cdot 10^{-2} & 7.0866 \cdot 10^{-3} & -9.9501 \cdot 10^{-1} \\ 2.9859 \cdot 10^{-5} & 9.9997 \cdot 10^{-1} & 7.1250 \cdot 10^{-3} \end{pmatrix}. \tag{8.4.16}$$

Substituting (8.4.16) into (8.4.12), we obtain reduced-order algebraic subsystem which is given by

$$\mathcal{L}_r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{L}_{q_r} = \begin{pmatrix} -2.9711 \cdot 10^{-5} & -0.99501 & -0.0071196 \\ 1.7923 \cdot 10^{-5} & 0.0078991 & -1.0049 \\ -2.844 \cdot 10^{-6} & -0.099465 & -0.0078694 \end{pmatrix}, \mathbf{A}_{q_r} = \begin{pmatrix} -0.19156 & 0.060735 \\ 0.0055639 & 0.012291 \\ -0.01912 & 0.0061623 \end{pmatrix},$$

$$\mathbf{B}_{q_r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{C}_{q_r} = \begin{pmatrix} 2.9859 \cdot 10^{-5} \\ 0.99997 \\ 0.007125 \end{pmatrix}. \tag{8.4.17}$$

Thus the algebraic system is reduced to dimension $\tau = 3$. Substituting (8.4.15) and (8.4.17) into (8.4.13) we obtained a IMOR reduced-order model based on balanced truncation which we can write in the descriptor form with system matrices given by

$$\mathbf{E}_r = \begin{pmatrix} 0.99324 & 0.023928 & 0 & 0 & 0 \\ -0.021333 & 1.0755 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{B}_r = \begin{pmatrix} 0.064859 \\ 0.20457 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C}_r = \begin{pmatrix} 0.19251 \\ -0.061038 \\ 2.9859 \times 10^{-5} \\ 0.99997 \\ 0.007125 \end{pmatrix},$$

$$\mathbf{A}_r = \begin{pmatrix} -0.046022 & 0.82467 & 0 & 0 & 0 \\ -0.86258 & -0.024527 & 0 & 0 & 0 \\ 0 & 0 & -2.9711 \times 10^{-5} & -0.99501 & -0.0071196 \\ 0 & 0 & 1.7923 \times 10^{-5} & 0.0078991 & -1.0049 \\ 0 & 0 & -2.844 \times 10^{-6} & -0.099465 & -0.0078694 \end{pmatrix}. \tag{8.4.18}$$

Hence the DAE (7.4.19) is reduced to a reduced-order model (8.4.18) of dimension $r + \tau = 5$. We note the solution and transfer function coincides with those illustrated in Figure 7.1 and 7.2.

8.5 Properties of the IIMOR method

The IIMOR method is an extension of the IMOR method, thus it will also inherits all the properties of the IMOR method. The main difference between the IMOR and IIMOR methods is that IMOR method leads to explicit decoupled reduced-order models while IIMOR method leads to implicit decoupled reduced-order models. We note that these two methods coincide if and only if $\mathbf{E}_p = \mathrm{I}$ and $\mathcal{L}_q = \mathrm{I}$. However, the IIMOR method is computationally cheaper than the IMOR method. The IIMOR and IMOR methods always preserves the index of the DAEs. Following the same procedure as for the case of IMOR method in Section 7.5. The properties of the IIMOR method are discussed as follows. From (8.2.2), the transfer function of the DAE (8.0.1) can be decomposed as

$$\mathbf{H}(s) = \begin{pmatrix}\mathbf{C}_p^{\mathrm{T}} & \mathbf{C}_q^{\mathrm{T}}\end{pmatrix} \begin{pmatrix} s\mathbf{E}_p - \mathbf{A}_p & 0 \\ -\mathbf{A}_q & \mathcal{L}_q - s\mathcal{L} \end{pmatrix}^{-1} \begin{pmatrix}\mathbf{B}_p \\ \mathbf{B}_q\end{pmatrix} = \mathbf{H}_p(s) + \mathbf{H}_q(s), \tag{8.5.1}$$

where $\mathbf{H}_p(s) := \mathbf{C}_p^{\mathrm{T}}(s\mathbf{E}_p - \mathbf{A}_p)^{-1}\mathbf{B}_p$ and $\mathbf{H}_q(s) := \mathbf{C}_q^{\mathrm{T}}(\mathcal{L}_q - s\mathcal{L})^{-1}\left[\mathbf{A}_q(s\mathbf{E}_p - \mathbf{A}_p)^{-1}\mathbf{B}_p + \mathbf{B}_q\right]$ are transfer functions corresponding to the differential part and algebraic parts, respectively. We can use (8.5.1) to show that the IMOR methods also preserves the properties of the MOR. These properties are discussed as follows.

(i) Moment matching property.
After the reduction of the differential part of the decoupled system using the block Arnoldi process, it preserves the first r moments of the differential component $\mathbf{H}_p(s)$ of the decomposed transfer function (8.5.1). This leads to the following Theorem.

Theorem 8.5.1 *IIMOR methods preserves the moment matching property if and only if the conventional MOR method applied on the differential part preserves the moment matching property.*

Proof 8.5.1 *The proof can be summarized following the same procedure as in [49]. If we choose the expansion point as $s_0 = 0$ and assume $\mathbf{A}_p$ is nonsingular. Then the transfer function $\mathbf{H}_p(s)$ of the differential part can be written as $\mathbf{H}_p(s) = \sum_{k=0}^{\infty}\mathbf{h}_p^{(k)}s^k$ where $\mathbf{h}_p^{(k)} = (-1)^k\mathbf{C}_p^{\mathrm{T}}\mathbf{M}_p^k\mathbf{R}_p$ are the (block) moments of $\mathbf{H}_p(s)$, $\mathbf{M}_p = -\mathbf{A}_p^{-1}\mathbf{E}_p$ and $\mathbf{R}_p = -\mathbf{A}_p^{-1}\mathbf{B}_p$. Likewise, the transfer function of the PRIMA reduced-order*

differential part can be written as $\tilde{\mathbf{H}}_p(s) = \sum_{k=0}^{\infty} \tilde{\mathbf{h}}_p^{(k)} s^k$ *where* $\tilde{\mathbf{h}}_p^{(k)} = (-1)^k \tilde{\mathbf{C}}_p^{\mathrm{T}} \tilde{\mathbf{M}}_p{}^k \tilde{\mathbf{R}}_p$, *are the moments,* $\tilde{\mathbf{M}}_p = -\tilde{\mathbf{A}}_p^{-1}\tilde{\mathbf{E}}_p$ *and* $\tilde{\mathbf{R}}_p = -\tilde{\mathbf{A}}_p^{-1}\tilde{\mathbf{B}}_p$. *Then,*
$\tilde{\mathbf{C}}_p = \mathbf{V}_p^{\mathrm{T}}\mathbf{C}_p$, $\tilde{\mathbf{A}}_p = \mathbf{V}_p^{\mathrm{T}}\mathbf{A}_p\mathbf{V}_p$, $\tilde{\mathbf{B}}_p = \mathbf{V}_p^{\mathrm{T}}\mathbf{B}_p$. *We can observe that* $\tilde{\mathbf{h}}_p^{(k)}$ *can be written as*

$$\tilde{\mathbf{h}}_p^{(k)} = -\mathbf{C}_p^{\mathrm{T}}\mathbf{V}_p\Big[(\mathbf{V}_p^{\mathrm{T}}\mathbf{A}_p\mathbf{V}_p)^{-1}(\mathbf{V}_p^{\mathrm{T}}\mathbf{E}_p\mathbf{V}_p)\Big]^k(\mathbf{V}_p^{\mathrm{T}}\mathbf{A}_p\mathbf{V}_p)^{-1}\mathbf{V}_p^{\mathrm{T}}\mathbf{B}_p,$$

By construction $\mathbf{V}_p\mathbf{V}_p^{\mathrm{T}}$ *is a projector onto* $\mathcal{K}_r(\mathbf{M}_p, \mathbf{R}_p)$. *Thus it holds*

$$\mathbf{V}_p\mathbf{V}_p^{\mathrm{T}}\mathbf{M}_p^k\mathbf{R}_p = \mathbf{M}_p^k\mathbf{R}_p, \; k = 0, 1, \cdots, r-1.$$

This in turn implies $\mathbf{V}_p^{\mathrm{T}}\mathbf{M}_p^{\mathrm{T}}\mathbf{R}_p = \tilde{\mathbf{M}}_p^k\tilde{\mathbf{R}}_p$, *hence* $\tilde{\mathbf{h}}_p^{(k)} = \mathbf{h}_p^{(k)}$, $k = 0, 1, \cdots, r-1$. *Next, we can show that the induced reduction on the algebraic part of the DAE also preserves the first r moments of the algebraic component of the transfer function,* $\mathbf{H}_q(s)$, *which can be written as* $\mathbf{H}_q(s) = \sum_{j=0}^{\mu-1} \mathbf{h}_q^{(j)} s^j$, *where*
$\mathbf{h}_q^{(j)} = \mathbf{C}_q\mathcal{L}_q^{-1}\sum_{j=0}^{\mu-1}\mathbf{N}_q^j\Big[\mathbf{A}_q\mathbf{R}_p + \mathbf{B}_q\Big]$, $j = 0, \cdots, \mu-1$, $\mathbf{N}_q = \mathcal{L}\mathcal{L}_q^{-1}$ *are the moments.*
Also, by construction $\mathbf{V}_q\mathbf{V}_q^{\mathrm{T}}$ *is a projector onto* $\mathcal{K}_\mu(\mathcal{L}_q^{-1}\mathbf{N}_q, \mathcal{L}_q^{-1}\mathbf{R}_q)$, *where*
$\mathbf{R}_q = \Big[\mathbf{B}_q \quad \mathbf{A}_q\mathcal{K}_r(\mathbf{M}_p, \mathbf{R}_p)\Big]$. *Thus it holds* $\mathbf{V}_q\mathbf{V}_q^{\mathrm{T}}(\mathbf{A}_q\mathbf{M}_p^k\mathbf{R}_p + \mathbf{B}_q) = \mathbf{A}_q\mathbf{M}_p^k\mathbf{R}_p + \mathbf{B}_q$. *Then, using the identity* $\mathbf{V}_p^{\mathrm{T}}\mathbf{M}_p^{\mathrm{T}}\mathbf{R}_p = \tilde{\mathbf{M}}_p^k\tilde{\mathbf{R}}_p$. *It is possible to show that*
$\tilde{\mathbf{h}}_q^{(k)} = \mathbf{h}_q^{(k)}, \quad k = 0, 1, \cdots, r-1$, *see [4].*

(ii) Passivity preservation property.
Using Theorem 7.5.2, we can also discuss the passivity preservation of IIMOR methods as follows.

$$\begin{aligned}
\mathbf{H}(s) &= \mathbf{C}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}, \\
&= \mathbf{H}_p(s) + \mathbf{H}_q(s), \\
&= \mathbf{C}_p^T\mathbf{R}_p(s) + \mathbf{C}_q^T(\mathcal{L}_q - s\mathcal{L})^{-1}\left[\mathbf{A}_q\mathbf{R}_p(s) + \mathbf{B}_q\right], \\
&= \mathbf{C}_p^T\mathbf{R}_p(s) + \mathbf{C}_q^T\mathcal{L}_q^{-1}\sum_{j=0}^{\mu-1}\mathbf{N}_q^j\mathbf{N}(s)s^j, \text{ since } \mathbf{N}(s) = \mathbf{A}_q\mathbf{R}_p(s) + \mathbf{B}_q,
\end{aligned}$$

$$= \underbrace{\mathbf{C}_p^T \mathbf{R}_p(s) + \mathbf{M}_0(s)}_{\mathbf{H}_{pr}(s)} + \underbrace{\sum_{j=1}^{\mu-1} s^j \mathbf{M}_j(s)}_{\mathbf{H}_{impr}(s)},$$

where $\mathbf{R}_p(s) = (s\mathbf{E}_p - \mathbf{A}_p)^{-1}\mathbf{B}_p$, $\mathbf{N}_q = \mathcal{L}\mathcal{L}_q^{-1}$, $\mathbf{M}_0(s) = \mathbf{C}_q^T \mathcal{L}_q^{-1}\mathbf{N}(s)$ and $\mathbf{M}_j(s) = \mathbf{C}_q^T \mathcal{L}_q^{-1}\mathbf{N}_q^j \mathbf{N}(s)$. $\mathbf{H}_{pr}(s)$ is the proper part (bounded as $s \to \infty$) and $\mathbf{H}_{impr}(s)$ the improper part (unbounded as $s \to \infty$) of $\mathbf{H}(s)$. Thus, the transfer function $\mathbf{H}_p(s) = \mathbf{C}_p^T \mathbf{R}_p(s)$ of the differential part is a strictly proper part of $\mathbf{H}(s)$. Based on Theorem 7.5.2, $\mathbf{H}(s)$ is positive real if and only if $\mathbf{H}_{pr}(s)$ and $\mathbf{M}_j(s)$ are positive real. As we mentioned in Section 7.5, a key to testing the passivity of DAEs is to first decouple DAEs into their proper and improper parts [66]. s Hence also the matrices coefficients of the implicit decoupled systems derived in Chapter 6, can be used to test the passivity of the DAEs using the passivity test for DAEs proposed in [66]. Hence the matrices coefficients of the decoupled systems derived in Chapter 6, can be used to test the passivity of the DAEs using the passivity test for DAEs proposed in [66]. Following the proof for passivity preserving in [49], it can also be proved that if the conventional MOR method applied on the differential part is passivity preserving then the differential part of the IIMOR reduced-order model is also passive, i.e, $\tilde{\mathbf{H}}_{pr}(s)$ is positive real. However, in order to ensure that the IIMOR methods are passivity preserving one need to also prove that $\tilde{\mathbf{M}}_j(s)$ is also positive real which is still an open question.

(iii) Approximation error

The approximation error of the IIMOR methods can also be defined in the same way as the IMOR methods from Section 7.5. Thus, using (8.2.2) the approximation error of the IIMOR methods can be computed using

$$\|\mathbf{Y}(s) - \tilde{\mathbf{Y}}(s)\| \leq \|\mathbf{H}(s) - \tilde{\mathbf{H}}(s)\|\,\|\mathbf{U}(s)\| + \|\mathcal{P}(s) - \tilde{\mathcal{P}}(s)\|, \tag{8.5.2}$$

where $\|\mathbf{H}(s) - \tilde{\mathbf{H}}(s)\| \leq \|\mathbf{H}_p(s) - \tilde{\mathbf{H}}_p(s)\| + \|\mathbf{H}_q(s) - \tilde{\mathbf{H}}_q(s)\|$. If we let $Q(\boldsymbol{u}(0)) := \mathcal{L}\xi_q(0)$ to be the hidden polynomial that depends on the input data and its derivatives at $t = 0$ and $(\mathcal{L}_q - s\mathcal{L})^{-1} = \mathcal{L}_q \sum_{j=0}^{\mu-1} \mathbf{N}_q^j$. Then, $\|\mathcal{P}(s) - \tilde{\mathcal{P}}(s)\| \leq \|\tilde{\mathbf{C}}_q \tilde{\mathcal{L}}_q \sum_{j=0}^{\mu-1} \tilde{\mathbf{N}}_q^j - \mathbf{C}_q \mathcal{L}_q \sum_{j=0}^{\mu-1} \mathbf{N}_q^j\|\,\|Q(\boldsymbol{u}(0)) - \tilde{Q}(\boldsymbol{u}(0))\|$. Hence also the output-transfer function of the IIMOR reduced-order model has a small approximation error if and only if

(a) $\|\mathbf{H} - \tilde{\mathbf{H}}\|$ is small

(b) and $\|\mathcal{P}(s) - \tilde{\mathcal{P}}(s)\|$ is also very small in a suitable norm $\|.\|$.

Thus, IIMOR reduced-order models can be validate more efficiently using the above tools.

(iv) Stability

In Section 5.4, we already discussed that for the case of DAEs with a differential part the decoupled system inherits the stability properties of DAEs since $\sigma(\mathbf{E}_p^{-1}\mathbf{A}_p) = \sigma_f(\mathbf{E}, \mathbf{A})$. Hence stability preservation of the IIMOR method also depends on the MOR method used to reduce the differential part.

Chapter 9

Large scale problems

In this Chapter all experiments were done using Matlab2012b on a laptop of 6.00GB of RAM with 64 bit operating system. In the next example, we illustrate the limitation of the conventional MOR methods on higher index DAEs using a large scale example.

Example 9.0.1 This benchmark originates from [25]. Consider an RLC circuit in Figure

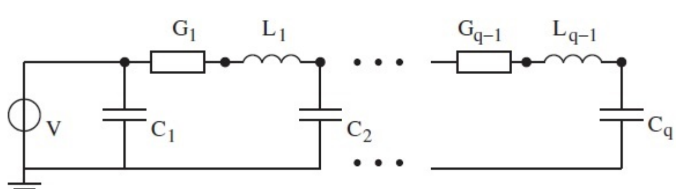

Figure 9.1: RLC circuit

9.1 which can modeled using the modified nodal analysis leading to a DAE of the form

(2.4.1) given by [5]

$$\underbrace{\begin{pmatrix} \boldsymbol{A}_C \boldsymbol{C} \boldsymbol{A}_C^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{L} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}}_{\mathbf{E}} \frac{d\boldsymbol{x}}{dt} = \underbrace{\begin{pmatrix} -\boldsymbol{A}_R \boldsymbol{\mathcal{G}} \boldsymbol{A}_R^{\mathrm{T}} & -\boldsymbol{A}_L & -\boldsymbol{A}_V \\ \boldsymbol{A}_L^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{A}_V^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}}_{\mathbf{B}} \boldsymbol{x} + \underbrace{\begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ -\mathrm{I} \end{pmatrix}}_{\mathbf{B}} \boldsymbol{v}(t). \tag{9.0.1}$$

We can use $\mathbf{C} = \mathbf{B}$ as the control output matrix and the input function $\boldsymbol{u}(t) = \boldsymbol{v}(t)$. This is a SISO system. We can also observe that $n_C = q$, $n_L = q - 1$ and $n_G = q - 1$ are the number of capacitor, inductors and resistors, respectively in the RLC circuit. It can be checked that this RLC circuit leads to an index-2 DAE of the form (9.0.1). For our case, we use $C_i = 0.1$, $i = 1, \cdots, q$, $L_i = 0.5$, $i = 1, \cdots, q-1$ and $G = 1/i$, $i = 1, \cdots, q-1$ as capacitance, inductance and conductances values, respectively. Using the same constant $q = 500$ as in [25] leads to an index-2 DAE of order $n = 1499$. Using conventional MOR method (PRIMA method) and $s_0 = 0$ as the expansion point, we obtained a reduced-order model of dimension 210. We observed that the conventional MOR reduced-order model is an ODE. For comparison, we reduced this DAE using our newly developed IMOR method. This is done as follows. Using the explicit and implicit decoupling methods, we were able to decouple the DAE system into 998 and 501 differential and algebraic equations, respectively. We then used the AE method on the algebraic parts of both algebraic parts and we were able to reduce them to only 2 algebraic equations. Thus, we were able to reduce both decoupled systems from dimension 1499 to 1000 exactly. Using the same expansion point with the PRIMA method, we were able to reduce the differential part of the explicit decoupled system from 998 to 208. Thus, the DAE system is reduced to a IMOR reduced-order model of total dimension 210. We note the IMOR reduced-order model is also an index-2 DAE, thus it preserves the index of the original model. We then compared the transfer functions and the phase angles of the original and reduced-order models. We observed the transfer function and the phase angle coincides as shown in Figure 9.2 with small approximation error as shown in Figure 9.3. However, the IMOR reduced-order model seems to be more accurate than the conventional MOR model. We numerically solved the IMOR and conventional MOR reduced-order models using as $\boldsymbol{u}(t) = 10\cos(t)$, $t \in (0, \pi)$ as the input function. We observe that the IMOR reduced-order model leads to accurate solutions which coincides with the solution of the original model but the conventional MOR reduced-order model leads to wrong solutions as shown in Figure 9.4, if one used higher order implicit integration techniques. Figure

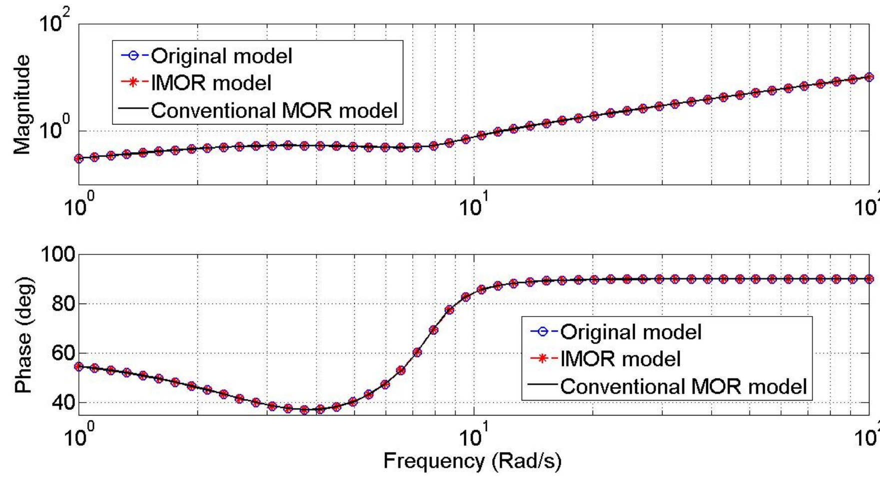

Figure 9.2: Comparison of the transfer function and phase angle.

9.5, shows the approximation error in the output solution. We can observe the solutions of the conventional MOR model has large error near the initial condition. This is not surprising to us, since in Section 7.2.2, we discussed that conventional MOR methods fail if $\mathcal{L}\mathbf{B}_q \neq \mathbf{0}$. However, if one uses the lower order implicit integration techniques such as the backward Euler method this problem is not visible.

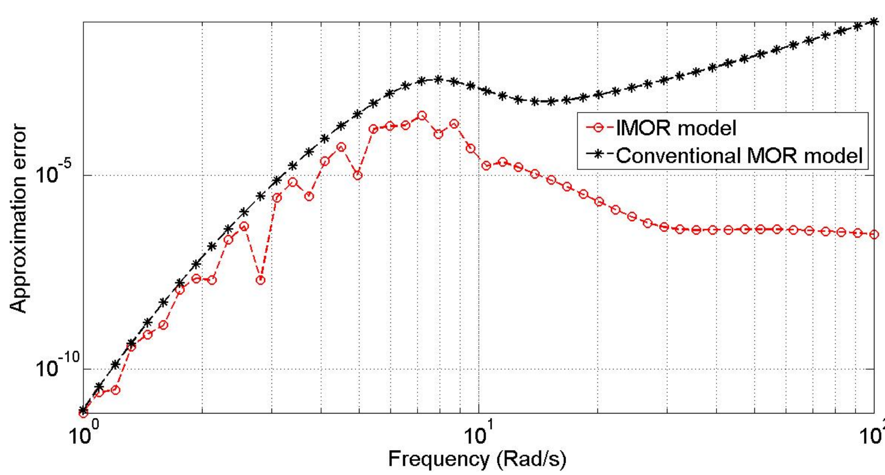

Figure 9.3: Comparison of the approximation error.

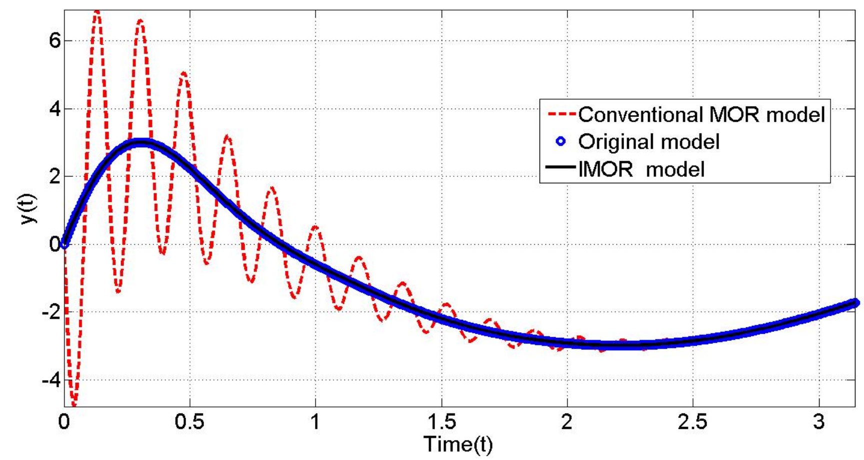

Figure 9.4: Comparison of the output solutions, $\boldsymbol{u}(t) = 10\cos(t)$.

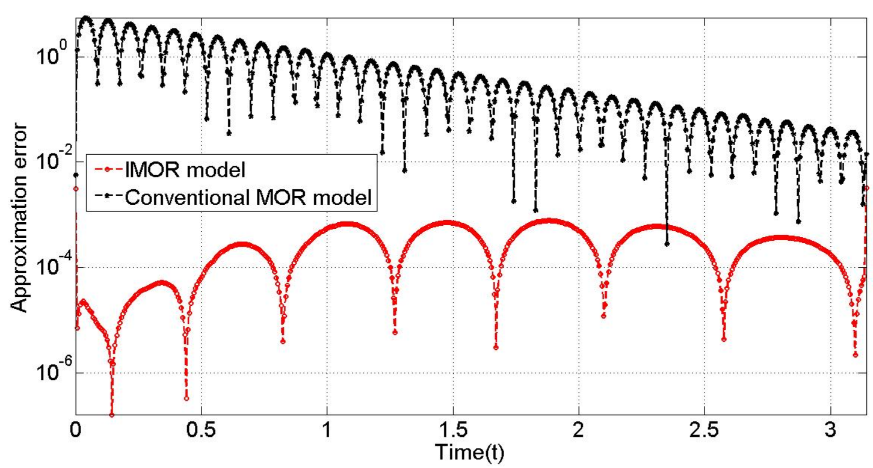

Figure 9.5: Comparison of the approximation error.

The above example shows how unreliable the conventional MOR methods can be. Having a good approximation of the transfer function does not guarantee accuracy of the output solution. Hence, the most reliable way is to use split MOR methods such as the IMOR and IIMOR methods to reduce DAEs.

Next, we apply the two newly developed IMOR and IIMOR methods for DAEs on large scale problems from real-life applications. These applications include problems from computational fluid dynamics (CFD), multibody systems and electrical networks. However these methods can be applied to any application that leads to a linear constant DAE.

CFD problems

These are applications from computational fluid dynamics (CFD).

Example 9.0.2 In this example, we used system matrices from supersonic inlet flow example discussed in Section 2.4.2. Consider the Euler equations modeling the unsteady flow through a supersonic diffuser as described in [35]. Linearization around a steady-state solution and spatial discretization using a finite volume method leads to a semi-explicit descriptor system of the form (2.4.3) of dimension $n = 11730$ and the CFD model had 3078 grid points. This is an index-1 DAE with $m = 2$ inputs and $\ell = 1$ output. According to [35], the reduced-order model must capture the dynamics of the output: the average Mach number at diffuser throat in response to two inputs: the incoming flow disturbance and the bleed actuation as shown in Figure 2.2. According to [35], they are two transfer functions of interest in this problem. Thus, the problem can be viewed as 2 single input single output (SIS0) subsystems and the frequencies of practical interest lie in the range $\frac{f}{f_0} = 0$ to $\frac{f}{f_0} = 2$, where $f_0 = \frac{a_0}{h}$, a_0 is the freestream speed of sound and h is the height of the diffuser. We decoupled this subsystems system into $n_p = 11323$ differential equations and $n_q = 407$ algebraic equations using both implicit decoupling and the explicit decoupling methods for index-1 DAEs. Figure 9.6 and 9.7 show the sparsity of the matrix pencil of the implicit and explicit decoupled system in descriptor form. We observe that the implicit decoupling procedure leads to a sparser matrix $\tilde{\mathbf{A}}$ than the matrix $\hat{\mathbf{A}}$ of the explicit decoupling procedure. Next, we compared the IMOR and IIMOR methods on these two subsystems. We used the PRIMA method to reduce the differential part of both decoupled systems using $s_0 = 0$ as the expansion point. We were able to reduce the differential and algebraic parts of both subsystems to 15 differential and 16 algebraic equations, respectively. Thus both DAE subsystems were reduced from dimension 11730 to 31. In Figure 9.8, we compare the magnitude of the transfer function and its approximation error from bleed actuation to average throat Mach

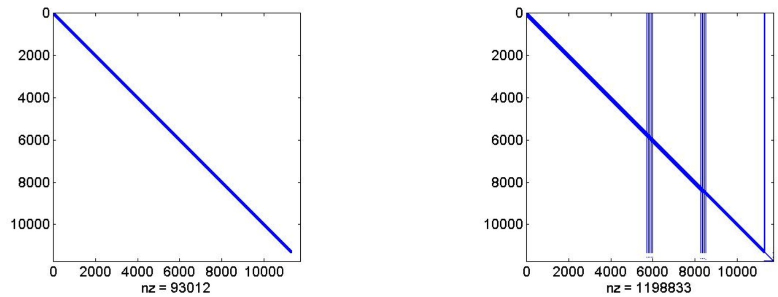

Figure 9.6: Sparsity of matrix pencil $(\tilde{\mathbf{E}}, \tilde{\mathbf{A}})$

number for supersonic diffuser. We observed that all reduced-order models are accurate in the desired frequencies but the IIMOR is more accurate than the IMOR method. In Figure 9.10, we compare the magnitude of the transfer function and its approximation error from the incoming flow disturbance to average throat Mach number for supersonic diffuser. We also observed that all the reduced-order models are accurate in the desired low frequencies. Hence, the IIMOR method is more accurate than the IMOR method for this problems.

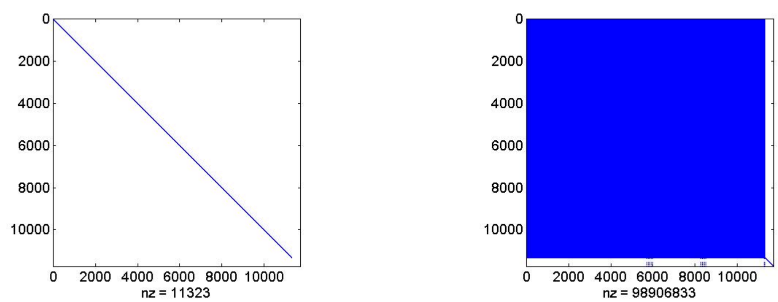

Figure 9.7: Sparsity of matrix pencil $(\hat{\mathbf{E}}, \hat{\mathbf{A}})$

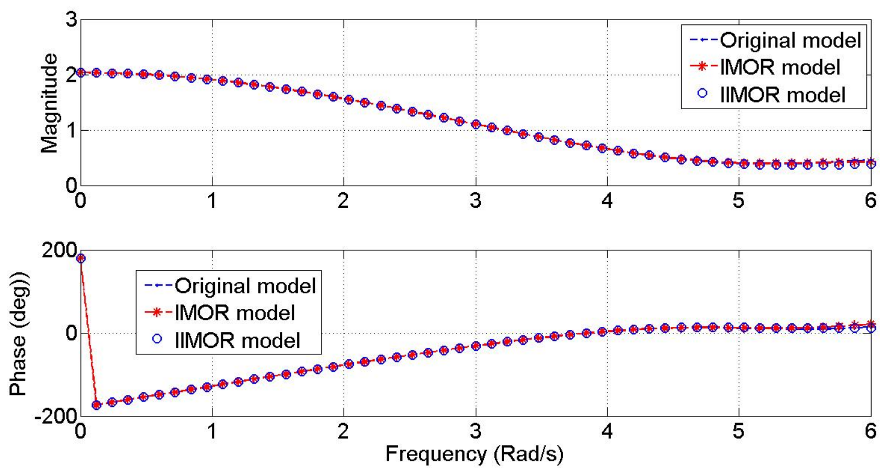

Figure 9.8: Transfer function from bleed actuation to average throat Mach number for supersonic diffuser.

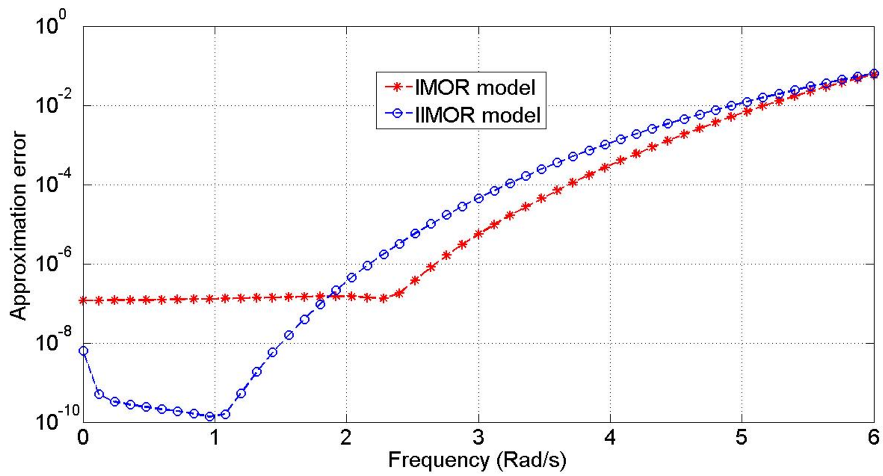

Figure 9.9: Approximation error of the Transfer function from bleed actuation to average throat Mach number for supersonic diffuser.

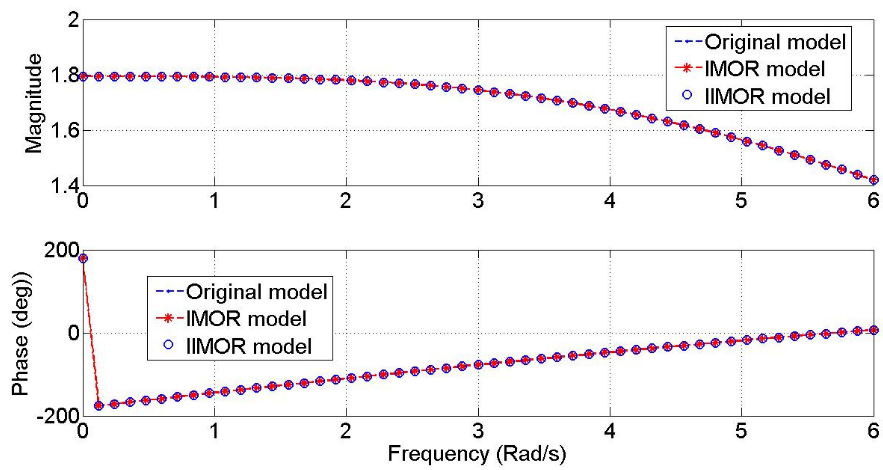

Figure 9.10: Transfer function from incoming flow disturbance to average throat Mach number for supersonic diffuser.

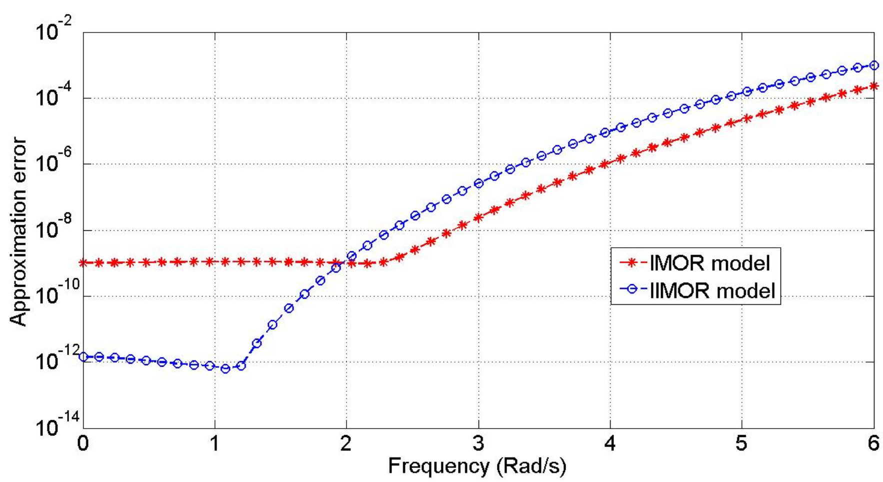

Figure 9.11: Approximation error of the Transfer function from incoming flow disturbance to average throat Mach number for supersonic diffuser.

Example 9.0.3 In this example, we apply the IMOR and IIMOR method on the semidiscretized Stokes problem which we earlier discussed in Section 2.4.2. This is an index-2 DAE with system matrices of the form (2.4.5). We note that these results presented here are also presented in [4]. We performed a spatial discretization of the Stokes equation

(2.4.4) on a square domain $\Omega = [0, 1] \times [0, 1]$ by the finite volume method on a uniform staggered grid. In order to compare the computational cost of the implicit and explicit decoupling methods, we carried out experiments on different grid sizes as shown in Table 9.1. From Table 9.1, we can observe that as the mesh becomes finer the larger the size of the problem. Hence solving the problem becomes computationally more expensive. We can also observe that both methods were able to decouple the problem but the implicit method is computationally cheaper than the explicit method as expected since it does not involve matrix inversion of matrix $\mathbf{E}_2$. We then reduced the decoupled Stokes problems

Table 9.1: Comparison of the computational cost

Grid	Order	Decoupled model			Computational cost	
	n	n_p	k_1	k_2	Implicit method	Explicit method
64×64	12159	3969	4095	4095	5521.2	-
60×60	10679	3481	3599	3599	3667.6	30653.3
56×56	9295	3025	3135	3135	5937.8	8604.0
52×52	8007	2601	2703	2703	1574.9	5569.7

using the IMOR and IIMOR methods. We applied the IMOR and IIMOR method, to the explicit and implicit decoupled problems, respectively as shown in Table 9.2. We used the PRIMA method and $s_0 = 0$ as expansion point to reduce the differential part of both decoupled systems. The differential and algebraic equations are reduced to order r and τ, respectively and $r + \tau$ is the order of the reduced-order DAE as shown in Table 9.2. We observe that the IMOR method takes less time than the IIMOR method this is due to the inversion of lower triangular matrix $\mathcal{L}_q$ but this is small compared to the time it takes to generate the explicit decoupled system. We used the system matrices from

Table 9.2: Comparison of the IMOR methods

Grid	Order	Decoupled model		IIMOR model			IMOR model		
	n	n_p	n_q	r	τ	Time(s)	r	τ	Time(s)
64×64	12159	3969	8190	11	12	63.3	-	-	-
60×60	10679	3481	7198	11	12	48.8	11	12	13.1
56×56	9295	3025	6270	32	33	28.8	32	33	11.7
52×52	8007	5406	3599	22	23	19.3	22	23	6.3

grid 52×52 to compare the transfer function and the phase angle of the IMOR model and IIMOR model with that of the original as shown in Figure 9.12. We can observe that the transfer function and phase angle of the IMOR, IIMOR and original models coincide. However, the IMOR model is more accurate than the IIMOR model as shown by the approximation error plot in Figure 9.13. We finally compared the solutions of the

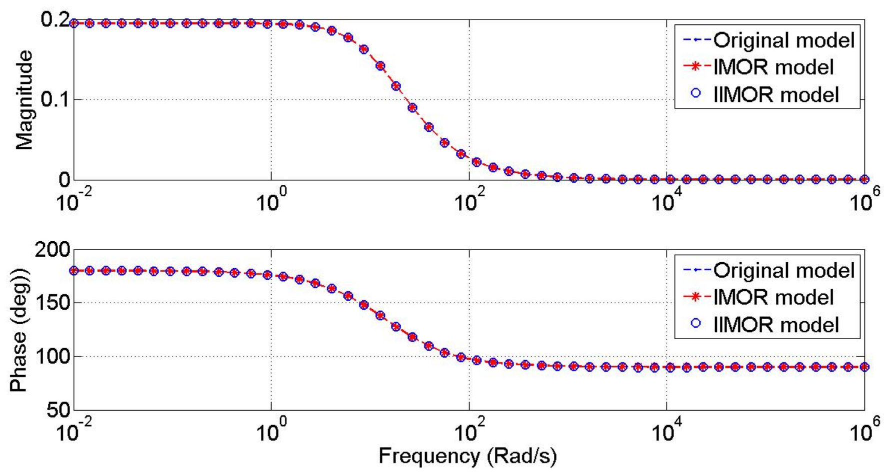

Figure 9.12: Comparison of the transfer function and phase angle.

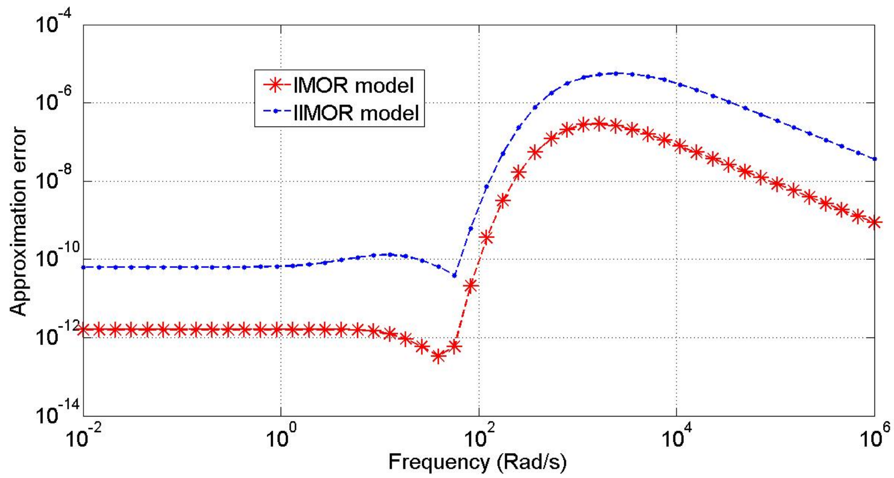

Figure 9.13: Comparison of the approximation error.

reduced-order models with that of the original model. From Figure 9.14, we observe that the solutions of the reduced-order models coincides with that of the original model with a small approximation error as shown in Figure 9.15. Both reduced-order models took 10 seconds while the original model took 148 seconds. Thus the decoupling techniques also makes solving much cheaper.

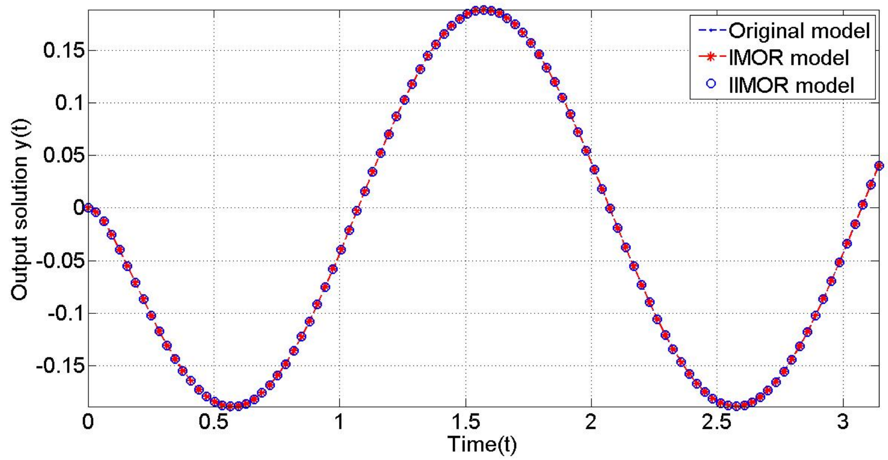

Figure 9.14: Comparison output solution $y(t)$, $u(t) = \sin(\pi t)$.

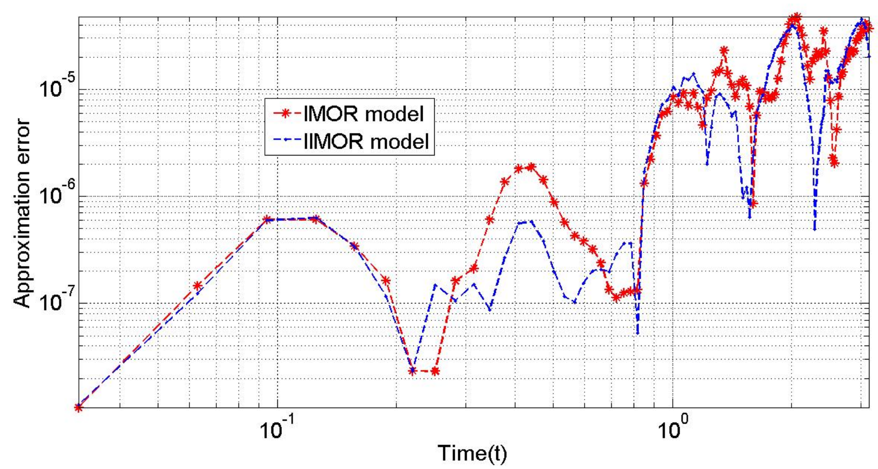

Figure 9.15: Output solution.

Multibody problems

Example 9.0.4 In this example, we consider a constrained damped mass-spring system as described in Section 2.4.3. This is a DAE of index-3 with its matrix pencil has at least one finite eigenvalue. Thus, we expect its decoupled system to have a differential part. We used the same constant $g = 6000$ as used in [45] to generate the same system matrices for comparison. This generates a DAE of order $n = 12001$ with 1 input and 3 outputs in the form (2.4.6). We used both the explicit and implicit decoupling methods for index-3 DAEs derived in Chapter 5 and 6, respectively. Both methods, were able to decouple the DAE into 1198 differential and 3 algebraic equations. The explicit and implicit methods took 60 and 54 seconds, respectively to decouple the system. Thus the implicit decoupling method is computationally cheaper than the explicit decoupling method. Then, We used the IMOR and IIMOR methods to decouple the respective decoupled systems. For both methods, we used the PRIMA method to reduce the differential part using $s_0 = 10^{-4}$ as the expansion point. The IMOR and IIMOR methods reduced their respective decoupled systems to 10 differential and 1 algebraic equations. The IMOR and IIMOR methods took 17 and 15 seconds, respectively. Thus the original DAE is reduced to a reduced-order model of order 11. In Figure 9.16, we compare the

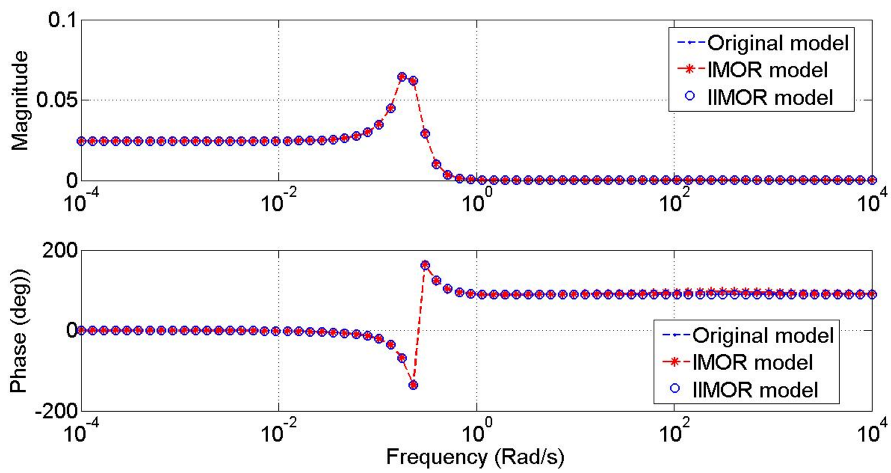

Figure 9.16: Magnitude and phase plots of $\mathbf{H}_{3,1}(i\omega)$

magnitude and phase plots of the (3, 1) components of the frequency responses for the reduced-order models and the original model. Figure 9.17 compares the approximation error of the IIMOR and IMOR reduced-order models. We see that the reduced-order models approximate the original system very well at low frequencies. However the IMOR reduced-order model is more accurate than the IIMOR reduced-order model. In Figure 9.18, we compare the output solutions $\boldsymbol{y}_1(t)$ and $\boldsymbol{y}_2(t)$ of the reduced-order models with that of the original model using $\boldsymbol{u}(t) = 10\sin(\pi t)$ as the input function. We observe that the solutions coincide with that of the original model. However the solutions of the IMOR model are more accurate than the IIMOR model as illustrated in the approximation error curve in Figure 9.19.

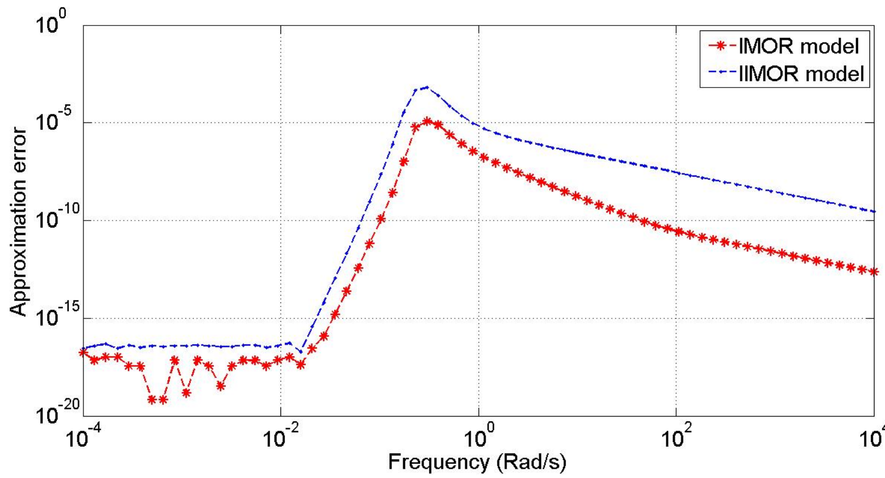

Figure 9.17: Approximation error

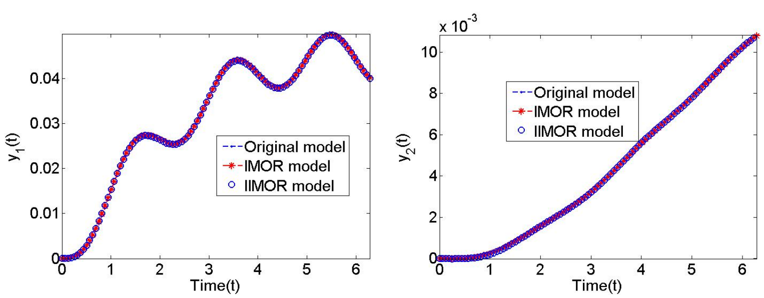

Figure 9.18: Comparison of the output solutions, $\boldsymbol{u}(t) = 10\sin(\pi t)$.

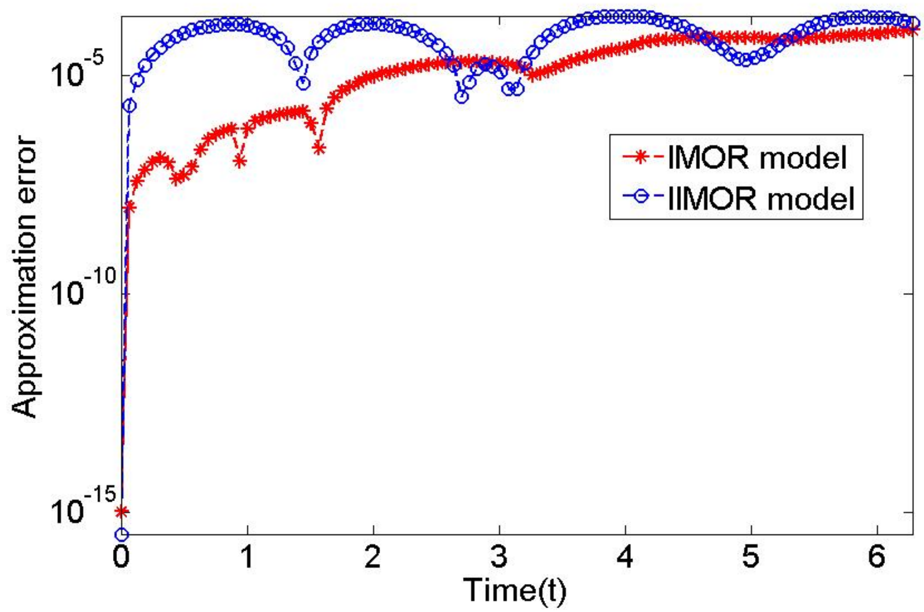

Figure 9.19: Approximation error of $\boldsymbol{y}_1(t)$

Electrical network problems

In this Section, we consider an application from the circuit community.

Example 9.0.5 In this example, we used a MNA model of dimension $n = 10913$. This DAE model originates from [11]. This is a MIMO index-2 DAE with $m = 9$ inputs and $\ell = 9$ outputs. The spectrum of its matrix pencil $(\mathbf{E}, \mathbf{A})$ has at least one finite eigenvalue. Thus its explicit and implicit decoupled systems takes the form (5.3.15) and (6.2.3), respectively. We used both the explicit and the implicit decoupling methods in order to split this DAE into differential and algebraic parts. We observed that both methods lead to $n_p = 10790$ differential equations, $k_1 = 26$ 1st algebraic equations and $k_0 = 97$ 2nd algebraic equations. This means that the DAE can be decoupled into 10790 differential and 123 algebraic equations. Thus the total dimension of the system is equal to the dimension of the DAE as expected, i.e $n = n_p + k_1 + k_0 = 10913$ as expected. The implicit decoupling procedure is computationally cheaper than its counter part because it does not involves computing the inverse of $\mathbf{E}_2$ which is very expensive. For this example implicit and explicit decoupling methods took 306 and 1965 seconds, respectively to decouple the DAE.

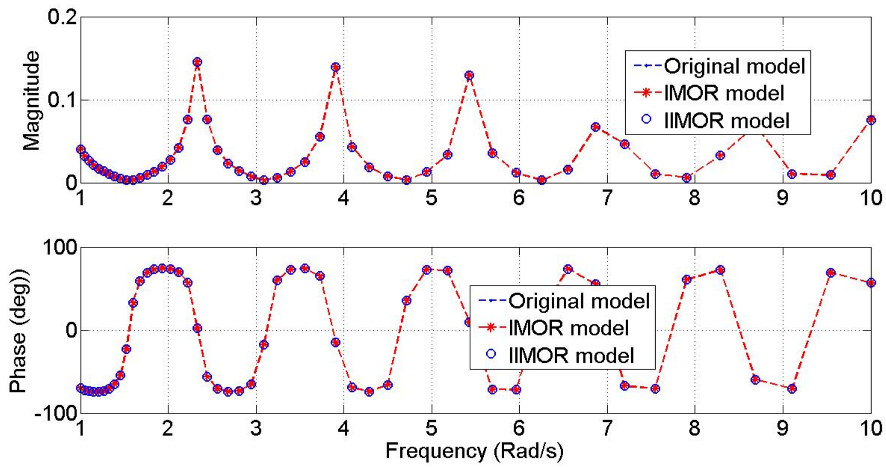

Figure 9.20: Magnitude and phase of the transfer functions.

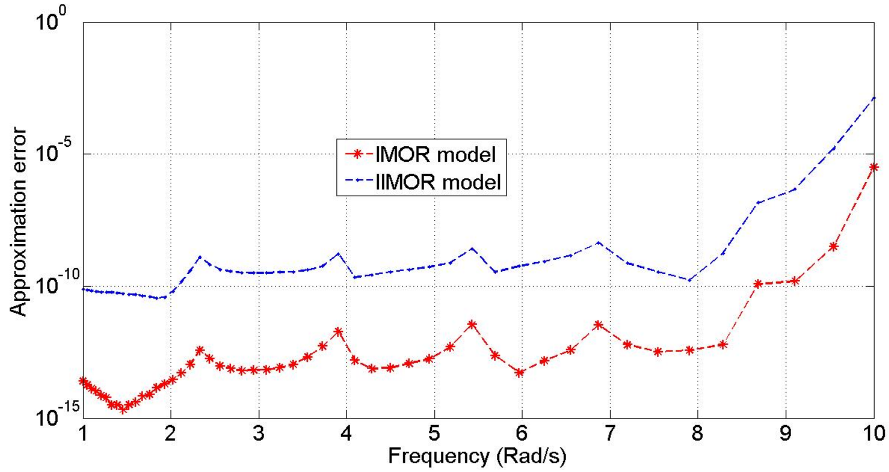

Figure 9.21: Approximation error

We used both the IMOR and IIMOR methods to reduce the DAE for comparison. In both methods we used the PRIMA method to reduced the differential part. The IMOR method lead to reduced-order model with 900 differential and 35 algebraic equations, while the IIMOR method lead to a reduced-order model with 900 differential and 99 algebraic equations. We can observe that the IIMOR reduced-order model is much larger. The IIMOR and IMOR method took 443 and 9662 seconds, respectively. Thus, the

IIMOR method is computationally cheaper than the IMOR method. In Figure 9.20, we can observe that the magnitude and phase of the transfer functions coincide with that of the original model with a small approximation error as shown in Figure 9.21. However, the IMOR reduced-order model is more accurate than the IIMOR model.

Chapter 10

Conclusions and Recommendations

In this thesis, two new model order reduction techniques for linear constant coefficient DAEs have been proposed. These methods are: the Index-aware MOR (IMOR) and Implicit IMOR (IIMOR) methods. They are both robust and lead to simple reduced-order models. However the Implicit IMOR method is computationally cheaper than the IMOR method, since the former does not involve matrix inversions. However, experiments show that the IMOR method leads to more accurate reduced-order model than the IIMOR method. Both methods have an attractive property that they preserve the index of the original DAE. Another interesting feature of our methods is the reduction of the algebraic variables. These methods were tested on both small and large scale problems from different applications which lead to accurate reduced-order models. We have also discussed that conventional MOR methods such as PRIMA method, may lead to reduced-order models which are: difficult to solve numerically, lead to wrong solutions or even unsolvable, while reduced-order models obtained by our methods do not present numerical difficulties when applied to higher index DAEs. It was noted that they are some special case where conventional MOR methods can lead to accurate reduced-order model even for higher index DAEs. This happens when initial condition does not depend on the derivatives if the input function $\boldsymbol{u}$ at time $t = 0$. The implicit and explicit decoupling procedure used in the IMOR and IIMOR methods are also new, they can be used to efficiently solve DAEs numerically using the conventional ODEs integration methods [5]. These decoupling procedures relies on the construction of projectors onto the nullspace of singular sparse matrices, which used to be its main drawback. For some applications with special structures they can be constructed explicitly and for general sparse DAEs one can use the LU based routine which is a very fast way of constructing projectors onto the nullspace of singular matrices [66]. This same routine can be used to construct bases of these projectors. However, one has to be aware that the numerical computation of these bases for the decoupling may involve serious difficulties because of the accuracy sensitive rank decisions. But it is expected to be profitable if the bases functions can be computed in a robust way, for example some applications such as the electrical network problems which are modeled using the incidence matrices. Thus, we recommend one to use the incidence matrices to construct these bases instead of using singular matrices which may be ill-conditioned for the case of circuit problems.

Recommendations for future work

Proper orthogonal decomposition (POD) model order reduction method is commonly used method to reduce nonlinear ODEs but there have been recent attempts to extend it to nonlinear DAEs, see [53]. However, this extension heavily relies on the idea of the balanced truncation MOR for the descriptor systems [45] which uses the Kronecker forms of the DAE and we have already discussed that these forms are numerically infeasible which limits their practical use. Fortunately, our implicit and explicit decoupling procedures are based on the matrix and projector chain [42] to decoupled DAEs, which are numerically feasible, and can be extended to nonlinear DAEs, linear DAEs with time varying coefficients or parametric DAEs, see [24,34]. The decoupling strategy of nonlinear DAEs involves a mixture of tractability index and strangeness index concepts, which can be used to split the nonlinear DAEs into a differential and algebraic parts [24]. Then the traditional proper orthogonal decomposition MOR method can be used to reduce the differential part and as a result the algebraic part can also be reduced. Although this is in general computationally expensive and highly sensitive with respect to perturbations, one may exploit it in a robust manner for model order reduction if the DAE to reduce has a time and state independent structure, i.e., if one can find bases functions that are both time and state independent as is the case for circuit parts without controlled sources [2]. This may be an interesting strategy to exploit in the future.

In Section 7.1 and 8.1, we proposed the Algebraic Elimination (AE) method which reduces the algebraic part of the decoupled system exactly by eliminating algebraic variables which do not contribute to the output solution. However, for the case of implicit decoupled systems as presented in Section 8.1, we do not get a good reduction of the algebraic part since it very difficult to find these algebraic variables which do not contribute to the output solution by just using the traditional permutation algorithms. We suggest if one uses the graph and matrix reordering algorithm such as the Vertex cut algorithms [31] to find the connected graphs in the matrices. This approach may lead to a better reduction of the algebraic part. This strategy can be used as a foundation for the development of MOR methods for algebraic systems since it is also an undeveloped area.

Bibliography

[1] G. Alí, N. Banagaaya, W.H.A. Schilders, and C. Tischendorf. Index-aware model order reduction for differential-algebraic equations. *Mathematical and Computer Modeling of Dynamical Systems: Methods, Tools and Applications in Engineering and Related Sciences*, 20(4):345–373, 2013.

[2] G. Alí, N. Banagaaya, W.H.A. Schilders, and C. Tischendorf. Index-aware model order reduction for index-2 differential-algebraic equations with constant coefficients. *SIAM J. Sci. Comput.*, 35(3):A1487–A1510, 2013.

[3] A.C. Antoulas. *Approximation of Large-scale Dynamical Systems*. SIAM, Philadelphia, 2005.

[4] N. Banagaaya, G. Alí, and W.H.A. Schilders. Implicit-IMOR method for index-1 and index-2 linear constant DAEs. *Numerical Algebra, Control and Optimization (NACO)*, 2014, In press.

[5] N. Banagaaya and W.H.A. Schilders. Simulation of electromagnetic descriptor models using projectors. *Journal of Mathematics in industry*, 3(1):1–18, 2013.

[6] N. Banagaaya and W.H.A. Schilders. Index-aware model order reduction for higher index DAEs. In *Progress in Differential-Algebraic Equations,Schöps, S., Bartel, A., Günther, M., ter Maten, E.J.W., Müller, P.C. (Eds)*. Springer, Berlin, 2014.

[7] N. Banagaaya, W.H.A. Schilders, G. Alí, and C. Tischendorf. Index-aware model order reduction: LTI DAEs in electric networks. *COMPEL: The interntional Journal for Computation and Mathematics in Electrical and Electronic Engineering*, 33(4):1123 – 1144, 2014.

[8] S. Baumanns. *Coupled Electromagnetic Field/Circuit Simulation: Modeling and Numerical Analysis*. PhD thesis, Universität zu Köln, Germany, 2012.

[9] P. Benner, M. Hinze, and E.J.W. ter Maten (Eds). *Model Order Reduction for Circuit Simulation*. Springer, Dordrecht Heidelberg London New York, 2011.

[10] S.L. Campbell. *Singular Systems of Differential Equations*. I. Pitman, San Fransisco, 1980.

[11] Y. Chahlaoui and P. Van Dooren. Benchmark examples for model reduction of linear time invariant dynamical systems, SLICOT Working Note 2002-2, February 2002.

[12] L. Dai. *Singular Control Systems. Lecture Notes in Control and Information Sciences*. Springer, New York Berlin Heidelberg, 1989.

[13] U. Davoudi. *Reduced-Order Modeling of Power Electronics Components and Systems*. PhD thesis, University of Illinois at Urbana-Champaign, Urbana, Illinois, USA, 2010.

[14] R.C. Degeneff, S.J. Gutierrez, D.W. Salaon, D.W. Burow, and R.J. Nevins. Kron reduction method applied to the time stepping finite element analysis of induction machines. *IEEE Transactions on Energy Conversion*, 10(4):669–674, 1995.

[15] F. Dorfler and F. Bullo. Kron reduction of graphs with applications of electrical networks. *Circuits and Systems , IEEE Transactions on*, 60(1):150 – 163, 2013.

[16] G. Duan. *Analysis and Design of Descriptor Linear Systems*. Springer, New York Dordrecht Heidelberg London, 2010.

[17] F.D. Freitas, N. Martins, S.L. Varrichio, J. Rommes, and F.C. Véliz. Reduced-order transfer matrices from network descriptor models of electric power grids. *IEEE transactions on power systems*, 26(4):1905–1919, 2011.

[18] F.D. Freitas, J. Rommes, and N. Martins. Gramian-based reduction method applied to large sparse power system descriptor models. *IEEE Transactions on Power systems*, 23(3):1258–1270, 2008.

[19] R.W. Freund. SPRIM: Structure-preserving reduced-order interconnect macromodelling. In *Proceedings of the 2004 IEEE/ACM International conference on*

Computer-aided design, pages 80–87. IEEE Computer Society, Washington, DC, USA, 2004.

[20] M. Gerdin. *Parameter estimation in Linear Descriptor Systems*. PhD thesis, Linköping University, Sweden, 2004.

[21] W.B. Gragg and A. Lindquist. On the partial realization problem. *Linear Algebra Appl.*, 50:277–319, 1983.

[22] E. Griepentrog and R. März. Basic properties of some differential-algebraic equations. *Zeitschrift für Analysis und ihre Anwendungen*, 8(1):25–40, 1989.

[23] E. J. Grimme. *Krylov projection methods for model reduction*. PhD thesis, University of Illinois, Urbana, USA, 1997.

[24] S. Grundel, L. Jansen, N. Hornung, P. Benner, T. Clees, and C. Tischendorf. Model order reduction of differential algebraic equations arising from the simulation of gas transport networks. Technical Report MPIMD/13-09, MPI-Magdeburg, 2013.

[25] S. Gugercin, T. Stykel, and S. Wyatt. Model Reduction of Descriptor Systems by Interpolatory Projection Methods. *SIAM J. Sci. Comput.*, 35(5):B1010–B1033, 2013.

[26] E. Hairer and G. Wanner. *Solving Ordinary Differential Equations II. Stiff and Differential-Algebraic Problems*. Springer, Berlin Heidelberg Newyork Barcelona Budapest Hong Kong London Milan Paris Santa Clara Singapore Tokyo, 2002.

[27] B. Hassen. Linear differential algebraic equations, Seminarbericht. Technical Report 92-1, Humboldt-Univ. Berlin, Fachbereich Mathematik, 1992.

[28] C.W. Ho, A.E. Ruehli, and P.A. Brennan. The modified nodal approach to network analysis. *IEEE Trans. Circuits and Systems, CAS*, 22(6):504–509, 1975.

[29] R. Ionutiu. *Model Order Reduction for Multi-terminals Systems with Applications to Circuit Simulation*. PhD thesis, Eindhoven University of Technology, Eindhoven, The Netherlands, 2011.

[30] G.A. Baker Jr and P.R. Graves-Morris. *Padé Approximants. Second edition. Encyclopedia of Mathematics and its applications, 59*. Cambridge University Press, Cambridge, 1996.

[31] P. Kitanov, O. Marcotte, W.H.A. Schilders, and S.M. Shontz. A vertex cut algorithm for model order reduction of parasitic resistive networks. *COMPEL: The International Journal for Computation and Mathematics in Electrical and Electronic Engineering*, 31(6):1850–1871, 2012.

[32] G. Kron. *Tensor Analysis of Networks*. General Electric Series, New York, 1939.

[33] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations. Analysis and Numerical Solution*. EMS, Zürich, 2006.

[34] R. Lamour, R. März, and C. Tischendorf. Projector based treatment of linear constant coefficient DAEs. Technical Report 11-15, Humboldt-University, Dep. of Mathematics, 2011.

[35] G. Lassaux and K. Wlillcox. Model reduction of an actively controlled supersonic diffuser. In *Dimension Reduction of Large Scale Systems, P. Benner, V. Mehrmann and D.C. Sorensen (Eds), Lecture notes in Computational Science and Engineering*, pages 357–361. Springer Berlin Heidelberg, 2005.

[36] W.Q. Liu and V. Sreeram. Model reduction of singular systems. In *Proceedings of the 39th IEEE Conference on Decision and Control (Sydney, Australia, 2000)*, pages 2373–2378. IEEE, 2000.

[37] Y. Liu and B.D.O. Anderson. Singular perturbation approximation of balanced systems. *Internat. J. Control*, 50:1379–1405, 1989.

[38] A. Lutowska. *Model Order Reduction for Coupled Systems using Low-rank Approximations*. PhD thesis, Eindhoven University of Technology, Eindhoven, The Netherlands, 2012.

[39] R. März. Numerical methods for differential algebraic equations. *Cambridge University press, Acta Numerica*, 21(5):141–198, 1992.

[40] R. März. On quasilinear index-2 differential-algebraic equations, Seminarbericht. Technical Report 92-1, Humboldt-Univ. Berlin, Fachbereich Mathematik, 1992.

[41] R. März. Progress in handling differential algebraic equations. *Advances in Computational Mathematics*, 1:279–292, 1994.

[42] R. März. Canonical projectors for linear differential algebaric equations. *Computers Math. Applications*, 31(4/5):121–135, 1996.

[43] W.J. McCalla. *Fundamentals of Computer Aided Circuit Simulation*. Acad. Publ. Group, Dordrecht, Kluwer., 1988.

[44] V. Mehrmann, R. Nabben, and E. Virnik. Generalisation of the Perron-Frobenius theory to matrix pencils. *Linear Algebra and its Applications*, 428(1):20–38, 2008.

[45] V. Mehrmann and T. Stykel. Balanced truncation model reduction for large scale systems in descriptor form. In *Dimension Reduction of Large Scale Systems,P. Benner, V. Mehrmann and D.C. Sorensen (Eds), Lecture notes in Computational Science and Engineering*, volume 45, pages 83–115. Springer Berlin Heidelberg, 2005.

[46] V. Mehrmann and T. Stykel. Descriptor systems: A general mathematical framework for modeling, simulation and control. *Automatisierungstechnik*, 54(8):405–415, 2009.

[47] C.D. Meyer. *Matrix Analysis and Applied Linear Algebra*. SIAM, Philadephia, 2000.

[48] J. Mohammadpour and K. M. Grigoriadis (Eds). *Efficient Modeling and Control of Large-Scale Systems*. Springer, Dordrecht Heidelberg London New york, 2010.

[49] A. Odabasioglu, M. Celik, and L.T. Pileggi. PRIMA: passive reduced-order interconnect macromodeling algorithm. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 17(8):645–654, 1998.

[50] S. Reich. On a geometric interpretation of DAEs. *Circuit Systems Signal Process*, 9(4):367–382, 1990.

[51] W.C. Rheinboldt. Differential-algebraic equations as differential equations on manifolds. *Math. Comp.*, 43(168):473–482, 1984.

[52] R. Riaza. *Differential-Algebraic Systems: Analytical Aspects and Circuit Applications*. World Scientific Publishing Co. Pte. Ltd, Singapore, 2008.

[53] R.C. Romijn, S. Weiland, and W. Marquardt. Proper orthogonal decomposition for model reduction of linear differential-algebraic equation systems. In *Proceedings*

of the 18th IFAC World Congress, 2011, pages 4582–4587. International Federation of Automatic Control, 2011.

[54] J. Rommes and N. Martins. Efficient computation of multivariable transfer function dominant poles using subspace acceleration. *IEEE transactions on power systems*, 21(4):1471–1483, 2006.

[55] J. Rommes and N. Martins. Computing large-scale system eigenvalues most sensitive to parameter changes, with applications to power system small-signal stability. *IEEE transactions on power systems*, 23(2):1905–1919, 2008.

[56] J. Rommes, N. Martins, and F.D. Freitas. Computing rightmost eigenvalues for small-signal stability assessment of large-scale power systems. *IEEE transactions on power systems*, 25(2):1905–1919, 2010.

[57] J. Rommes, N. Martins, and P.C. Pellanda. Computation of transfer function dominant zeros with applications to oscillation damping control of large power systems. *IEEE transactions on power systems*, 22(4):1657–1664, 2007.

[58] W.H.A. Schilders, H. Van der Vorst, and J. Rommes (Eds). *Model Oorder Reduction: Theory, Research Aspects and Applications*. Springer, Berlin Heidelberg, 2008.

[59] W.H.A. Schilders and J. Rommes. Efficient methods for large resistor networks. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 29(1):28–39, 2010.

[60] J. Schon, M. Gerdin, T. Glad, and F. Gustafsson. A Modeling and Filtering Framework for Linear Differential-Algebraic Equations. In *Decision and Control, 2003. Proceedings. 42nd IEEE Conference on*, pages 892–897. IEEE, 2003.

[61] S. Schulz. Four lectures on differential algebraic equations. Technical Report 497, Humboldt Universität zu Berlin, Germany, 2003.

[62] T. Stykel. *Anaylsis and Numerical solution of Generalized Lyapunov Equations*. PhD thesis, Technical University Berlin, Berlin, Germany, 2002.

[63] T. Stykel. Balancing-related model reduction of circuit equations using topological structure. In *Model Reduction for Circuit Simulation, P. Benner, M. Hinze and J.*

ter Maten (Eds), Lecture Notes in Electrical Engineering, volume 74, pages 53–80. Springer Berlin Heidelberg, 2011.

[64] C. Tischendorf. *Coupled Systems of Differential Algebraic and Partial Differential Equations in Circuit and Device Simulation*. PhD thesis, Humboldt-Universität zu Berlin, Berlin,Germany, 2003.

[65] M.V. Ugryumova. *Applications of Model Order Reduction for IC Modeling*. PhD thesis, Eindhoven University of Technology, Eindhoven, The Netherlands, 2011.

[66] Z. Zhang and N. Wong. An efficient projector-based passivity test for descriptor systems. *IEEE Trans. On Computer Aided Design of Integrated Circuits And Systems*, 29(8):1203–1214, 2010.

Index

Summary

Index-aware Model Order Reduction Methods for DAEs

Large scale DAEs arise in a variety of applications such as modeling of constrained multibody systems, electrical networks, aerospace engineering, chemical processes, computational fluid dynamics (CFD), gas transport networks. Characteristic of such systems is that they lead to state space descriptions of high dimension in which the coefficient of the first order derivative is a singular matrix. In practice, applications lead to DAEs with very large dimension compared to the number of inputs and the desired outputs. Despite the ever increasing computational power, simulation of these systems in real time for such large scale is very difficult because of the storage requirements and expensive computations. This is an attractive feature to apply model order reduction. However, if the initial condition is inconsistent or when the smoothness of the input does not correspond to the index of the DAE, currently available MOR techniques may lead to inaccurate reduced-order models. These reduced-order models may lead to wrong solutions that do not adequately represent the hidden truly fast modes or are very difficult to solve numerically.

The aim of this PhD project is to investigate model order reduction techniques for DAEs. The ultimate goal of the project is to deliver fundamental mathematical knowledge and efficient numerical tools for the next generation of MOR techniques for differential algebraic equations. This thesis addresses the mathematical aspects of the reduction of differential algebraic equations including the limitations of the conventional MOR methods. We have developed two new dedicated reduction methods for DAEs, using the underlying structure of DAEs, with the aim of obtaining robust reduction methods that can be applied to linear constant coefficient DAEs with arbitrary index. Our two new MOR

methods for DAEs are: the Index-aware MOR (IMOR) method and its implicit version the Implicit-IMOR (IIMOR) method. The explicit and implicit decoupling procedure used in these two methods are also new and can be used to solve DAEs more efficiently and effectively using conventional ODE integration methods.

This thesis begins with a brief overview of MOR methods for DAEs in Chapter 1 and why there was need to develop new robust MOR methods for DAEs. We also briefly explain the underlying mathematical frame work of the IMOR and IIMOR methods.

Chapter 2 introduces the theory of the DAEs and also discusses why DAEs are very difficult both to solve and to reduce. In this thesis, we restrict ourselves to linear time invariant DAEs or linear constant coefficient DAEs but the same applies to other types of DAEs. In this Chapter, we also discuss the assumptions under which DAEs can be solved as well as their mathematical properties such as stability. This is done by first transforming the DAE into a Kronecker form in order to reveal their underlying structure. We use the underlying structure of DAEs to discuss the index of DAEs and how the index of DAEs affects the choice of their initial conditions. We further used this form to discuss how the index of DAEs affects the conventional MOR methods especially for higher index DAEs. We then discuss the reason why it is a best practice to first split DAEs into differential and algebraic parts before applying model order reduction. We finally describe some of the real-life applications that lead to DAEs.

In Chapter 3, we discuss model order reduction methods in general and also illustrate numerically the limitations of conventional MOR methods using small examples. In this Chapter, we also give an overview of the existing MOR methods for DAEs and their limitations. We observed that the most successful methods for DAEs are: the balanced truncation method and Interpolatory projection methods for DAEs. Both methods lead to accurate reduced-order models for DAEs, however they both use Kronecker canonical forms to construct spectral projectors used in decoupling which are well known to be numerically infeasible. This limits their application to DAEs with special structures and cannot be used on general DAEs. These methods can not be extended to linear DAEs with variable coefficients since they use spectral projectors to decouple DAEs. We finally discuss the MOR methods for algebraic systems specifically reduction methods for resistor networks.

In Chapter 4, we discuss the decoupling of DAEs using the matrix and projector chain based on the definition of tractability index proposed by März. We used these matrix and projector chains to decouple DAEs into differential and algebraic parts using the März decoupling procedure. However, we found out that we cannot apply model order reduction on these decoupled systems since the März decoupling procedure leads to a much larger decoupled system of dimension $n(\mu + 1)$, where μ is the index of a DAE of dimension n, and it does not preserve stability of DAEs. We also discuss a fast way of constructing these matrix and projector chains using an LU decomposition based routine.

In Chapter 5, we modify the März decoupling procedure using projector bases. Using this approach, we were able to remove the redundancy in the decoupled systems. The modified decoupled system preserves both the dimension and the stability of DAEs. We call this decoupling procedure: explicit decoupling procedure since it leads to explicit differential and algebraic parts. However this modified decoupling procedure relies on the foundation of the März decoupling procedure which involves matrix inversions. Hence the explicit decoupling procedure cannot be applied to large scale DAEs. This motivated us, in Chapter 6, to develop another decoupling procedure which does not involve matrix inversions. This procedure is an implicit version of the decoupled procedure in Chapter 5. Experiments, also show that the implicit decoupling is computationally cheaper than the explicit decoupling procedure as expected.

In Chapter 7, we developed one of our new MOR methods for DAEs which we call the Index-aware MOR (IMOR) method. This is done by reducing the differential and algebraic parts, separately of the explicit decoupled systems derived in Chapter 5. One can use any conventional MOR method to reduce the differential part, while we developed new methods to reduce the algebraic part. For illustration, we used the PRIMA method and the balanced truncation method to reduce the differential part. The IMOR method leads to simple reduced-order models which preserve the index of the original DAE and also make it easy to solve. We also discussed the properties of the IMOR method and observed that it depends on the conventional MOR method used to reduce the differential part. However the IMOR method is impractical to be used to reduce large-scale DAEs since it uses the computationally expensive explicit decoupling procedure in Chapter 5. In Chapter 8, we developed the implicit version of the IMOR method which is based on the implicit decoupling procedure derived in Chapter 6, called the Implicit-IMOR (IIMOR) method. It has the same properties as the IMOR method, but it is computationally cheaper than the IMOR method. In Chapter 9, we applied both the IMOR and IIMOR methods on large-scale real-life applications. Experiments show that both methods are very accurate and robust, and lead to simple reduced-order models which are accurate and easy to solve. However, the IMOR is more accurate. Thus, one needs to trade off between accuracy and complexity. In the final Chapter, we discussed the conclusion and the future recommendations of the IMOR and IIMOR methods.

Acknowledgements

I would like to thank people who have helped me both directly and indirectly with my PhD project. Because it has been a journey full of obstacles and frustrations, without them this project would not have been a success. First of all, I would like to thank my 1st promotor Prof.Dr. Wil. H. A. Schilders for believing me and appointing me to this interesting project. He introduced me to his network of collaborators and friends who have shared with me their professional experience and the materials needed to complete this thesis. He also always allowed me to go to international conferences and workshops which gave me an international scientific exposure. I would like to thank my 2nd promotor Prof.Dr. Caren Tischendorf (Humboldt University of Berlin, Germany) for the collaboration, contribution and sharing her great experience in the field of differential algebraic equations. Her comments have helped me to produce sounding results in this thesis. My special thanks and sincere gratitude go to Prof.Dr. Guiseppe Alí (University of Calabria, Italy) for his great ideas, contributions, suggestions and time throughout this project. I would like to also thank Prof.Dr. Stykel Tatjana (University of Augsburg, Germany) and Dr. Joost Rommes (NXP Semiconductors, Netherlands) for accepting to share with me their benchmarks and test cases of large scale real-life applications of DAEs used throughout this thesis. I would like to thank Dr. Odile Marcotte (University of Montreal, Canada) and Dr. S.M. Schontz (Mississippi State University, USA) for their contribution on the reduction of the algebraic systems and sharing with their experience of graph theory algorithms. I would like to also thank my co-promotor Dr. Michiel Hochstenbach for reading and commenting on this thesis. I would like to thank CASA members (Centre for Analysis, Scientific computing and Applications) and staff for the conducive environment. The social events such as the CASA outings and the CASA cooking events which I (partly) organized made my PhD project and my stay in Netherlands a memorable experience. Special Thanks goes to Stan Korzilius whom I shared an office for the most times of my PhD project. I am also thankful for the help I received from Enna van Dijk who not only took care of all the administrative tasks but with whom I also had great time in organizing the CASA cooking event in our group. I would like to thank heartfelt gratitude to my friends: Robert Setekera, Cao Xuilei, Sangye Lungten and Simplice Nemadjieu whom I cannot thank enough for always being their for me in both happy and stressful times, thank you very much for all the jokes we shared together. I would like to thank my family: my father (James Katwe), my sisters (Hellen Nakawunde, Lydia Namutebi, Sophia Nnabigavu, Gloria Nambi-dyona and Rhoda) and brother (Joshua Mwesigwa) for the moral support. I am grateful to Netherlands Organisation for Scientific Research (NWO) for (partly) financing this research programme *Model Order Reduction for Differential Algebraic Systems* with project number 613.001.003. I dedicate this PhD thesis to my grandmother Jaja Tamali Zalwago.

Nicodemus Banagaaya

Eindhoven, September 2014

Curriculum Vitae

Nicodemus Banagaaya was born on February 6, 1985 in Kampala, Uganda. After finishing high school in 2003 at Mengo Senior School in Kampala, Uganda, he then pursued the Bachelor of Science with Education in Physics and Mathematics at Makerere University (MUK), Uganda in 2004–2007 on the Uganda government scholarship. In 2007–2008, he studied a Postgraduate diploma in mathematical sciences at the African institute for mathematical sciences (AIMS), Muizenberg, South Africa on the AIMS scholarship. In 2008–2010, he studied a double masters degree in Industrial and Applied mathematics at Johannes Kepler University (JKU) in 2008–2009 and Eindhoven University of Technology (TU/e) in 2009–2010 on the Erasmus Mundus scholarship from the European Commission. He spent the last 6 months of his double masters working as an intern at MAGWEL BV, Leuven, Belgium where he was working on the EV-formulation and implementation in the MAGWEL software. During the internship he also wrote his masters thesis: "The EV-formulation and an investigation of eigenvalues for electromagnetic problems", under the supervision of Prof.Dr. W.H.A. Schilders (TU/e) and Dr. Wim Schoenmaker (CTO, MAGWEL BV). He graduated in September 2010 with a double masters degree in Industrial and Applied mathematics within the Department of Mathematics and Computer Science at the Eindhoven University of Technology (TU/e).

From 2010 to 2014, he was a PhD candidate at the Eindhoven University of Technology, within the CASA group. During his PhD he developed two MOR methods for DAEs which are the IMOR and IIMOR methods as presented in this dissertation. He spent a total of two weeks in Calabria, Italy, working on the topic of his thesis with Prof.Dr. Guiseppe Alí.