Numerical approximation of the logarithmic capacity

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Abstract

The logarithmic capacity of compact sets in $\mathbb{R}^2$ plays an important role in various fields of applied mathematics. Its value can be computed analytically for a few simple sets. In this paper a new algorithm is presented that numerically approximates the logarithmic capacity for more involved sets. The algorithm requires the solution of a boundary integral equation with Dirichlet boundary data. The boundary integral equation is solved by a collocation approach or a Galerkin approach. We illustrate the effectiveness of the algorithm for a number of examples.

Key words: logarithmic capacity, numerical approximation, collocation, Galerkin

1 Introduction

The logarithmic capacity of a compact set in $\mathbb{R}^2$ is a real positive number which may be viewed as a measure for the capability of a set to support a unit amount of charge. As it is an important concept in several fields of applied mathematics, there exist different definitions of the logarithmic capacity. In potential theory the logarithmic capacity is a measure of the size of a compact set in $\mathbb{R}^2$ [15]. It is also often called the transfinite diameter, first introduced by Fekete in the 1930s [12–14]. The transfinite diameter is a key ingredient in number theory [7]. The logarithmic capacity also appears in polynomial approximation [6,16], where it is called Chebyshev constant. It can also be directly linked to the Robin constant, which plays an essential role in the field

[1] This work was carried out when the author was at Eindhoven University of Technology.

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of conformal mappings [21,22]. This constant appears in the Green's function, and by approximating this Green's function, the Robin constant, and also the logarithmic capacity, can be approximated. The logarithmic capacity is also crucial in the field of boundary integral equations, where it determines whether the integral equation for a Dirichlet problem is singular [9,10].

Despite its frequent appearance, direct evaluation of the logarithmic capacity has received little attention. An analytic expression for the logarithmic capacity is known only for a few simple sets such as ellipses and squares. For slightly more complex sets it may be estimated, but accurate approximations are rare [23,24]. In this paper we propose a new algorithm to numerically approximate the logarithmic capacity of compact sets that are bounded by a finite set of smooth Jordan curves. In the next section we introduce the logarithmic capacity from a physical point of view. A numerical algorithm to approximate the logarithmic capacity is presented in the third section. In Section 4 we analyze some properties of the method, while in Section 5 the performance of the algorithm is illustrated on a number of sets, including simple sets for which the exact value of the logarithmic capacity is known. We also compare our algorithm with the algorithms developed in [23,24].

2 Logarithmic capacity

Although the method we will propose can also be applied to sets that are not connected (see also the experiments in Section 5), for ease of presentation let Ω be a simply-connected set in $\mathbb{R}^2$. Suppose that Ω is a conducting plate and let $\rho \geq 0$ be a unit *charge distribution* on the plate. Then ρ generates a potential p_ρ, given by

$$p_\rho(\boldsymbol{x}) := \int_\Omega \log \frac{1}{\|\boldsymbol{x} - \boldsymbol{y}\|} \, d\rho(\boldsymbol{y}),$$

which is called the *logarithmic potential*. Here $\|\cdot\|$ denotes the standard Euclidean norm. The *potential energy* of the charge distribution is

$$I(\rho) := \int_\Omega p_\rho(\boldsymbol{x}) \, d\rho(\boldsymbol{x}) = \int_\Omega \int_\Omega \log \frac{1}{\|\boldsymbol{x} - \boldsymbol{y}\|} \, d\rho(\boldsymbol{y}) \, d\rho(\boldsymbol{x}).$$

The charge distributes itself such that the energy is minimized. Let V be the minimal energy,

$$V := \inf_{\rho \geq 0} I(\rho),$$

Boundary Γ	Logarithmic capacity $C(\Gamma)$
circle with radius a	a
ellipse with semi-axes a and b	$(a+b)/2$
interval of length a	$\frac{1}{4}a$
square with side l	$\frac{\Gamma(\frac{1}{4})^2}{4\pi^{3/2}}\ l \approx 0.59\ l$
equilateral triangle with side l	$\frac{\left(\Gamma(1/3)\right)^3}{4\pi^2}\ l \approx 0.18\ l$
isosceles with side l	$\frac{3^{3/4}\left(\Gamma(1/4)\right)^2}{2^{7/2}\pi^{3/2}}\ l \approx 0.48\ l$

Table 1
The logarithmic capacity of some simple boundaries.

then the *logarithmic capacity* is defined as

$$C(\Omega) := e^{-V}.$$

The charge obeys Faraday's principle and resides at the boundary of the plate. Let Γ denote the boundary of Ω. If Ω is bounded by a finite number of smooth Jordan curves, $d\rho(\boldsymbol{x})$ can be replaced by $\mu(\boldsymbol{x})d\boldsymbol{x}$ where μ is the derivative of ρ [11]. In the sequel we assume that this is the case and therefore the logarithmic potential can be written as

$$p_\mu(\boldsymbol{x}) = \int_\Gamma \log \frac{1}{\|\boldsymbol{x}-\boldsymbol{y}\|}\ \mu(\boldsymbol{y})\, d\Gamma_y,$$

and the energy can be written as

$$I(\mu) = \int_\Gamma \int_\Gamma \log \frac{1}{\|\boldsymbol{x}-\boldsymbol{y}\|}\ \mu(\boldsymbol{y})\, \mu(\boldsymbol{x})\, d\Gamma_y\, d\Gamma_x.$$

Consequently, we may consider the logarithmic capacity as a function of Γ.

The logarithmic capacity can be computed analytically for a few simple boundaries, such as ellipses and rectangles. Table 1 lists a number of boundaries for which the logarithmic capacity is analytically known [19]. Note that $\Gamma(\cdot)$ denotes the *Gamma function*. For boundaries for which there is no analytic expression known, the following properties can be used to bound or to estimate the logarithmic capacity [3,20].

1) If Γ is the outer boundary of a closed bounded set Ω, then $C(\Gamma) = C(\Omega)$. This agrees with Faraday's principle, mentioned above.
2) If $\Omega_1 \subset \Omega_2$, then $C(\Omega_1) \leq C(\Omega_2)$. Using the logarithmic capacity of a circle (see Table 1) we therefore get that the radius of the smallest circle

in which Γ is contained is an upper bound for the logarithmic capacity of Γ.

3) If $\Gamma = \boldsymbol{x} + \alpha\Gamma_1$ for $\boldsymbol{x} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, then $C(\Gamma) = \alpha\, C(\Gamma_1)$: the logarithmic capacity behaves linearly with respect to scaling and is invariant with respect to translation.
4) For a compact set Ω,

$$C(\Omega) \geq \left(\frac{\text{area}(\Omega)}{\pi}\right)^{1/2}.$$

3 Algorithm

In this section we develop a new algorithm to numerically approximate the logarithmic capacity of a set in $\mathbb{R}^2$ that is bounded by a finite set of Jordan curves. For a convenient presentation we again restrict ourselves to simply connected sets, although the algorithm is also valid for sets that are not connected. First we introduce a boundary integral operator $\mathcal{V} : \mu \mapsto p_\mu$,

$$(\mathcal{V}\mu)(\boldsymbol{x}) := \int_\Gamma \log \frac{1}{\|\boldsymbol{x} - \boldsymbol{y}\|}\, \mu(\boldsymbol{y})\, d\Gamma_y, \quad \boldsymbol{x} \in \Gamma.$$

This operator is called the *single layer operator*. Related to this operator is the following Dirichlet problem: find a distribution μ such that

$$\mathcal{V}\mu = f, \tag{1}$$

where f represents Dirichlet data on the boundary Γ. It is well known that the classical Dirichlet problem in its differential form ($\Delta u = 0$ on Ω, u given at Γ) has a unique solution. For the boundary integral formulation of the Dirichlet problem this is not always the case. This can be observed with the following result [17,26].

Theorem 1 *Let $\mathcal{S}$ be the collection of normalized distributions, i.e.*

$$\mathcal{S} := \left\{\mu : \int_\Gamma \mu\, d\Gamma = 1\right\}.$$

For each boundary Γ there exists a unique distribution $\mu_E \in \mathcal{S}$ which minimizes the potential energy $I(\mu)$. Moreover,

$$(\mathcal{V}\mu_E)(\boldsymbol{x}) \equiv -\log C(\Gamma), \quad \textit{for all } \boldsymbol{x} \in \Gamma. \tag{2}$$

The distribution μ_E that minimizes $I(\mu)$ is called the *equilibrium distribution*. When the logarithmic capacity of Γ is equal to one, the right-hand side of (2) vanishes, and the operator has a zero eigenvalue. Hence, the homogeneous equation $\mathcal{V}\mu = 0$ has a non-trivial solution, which can be added to the solution of the non-homogeneous Dirichlet problem (1). Therefore, the boundary integral equation for the Dirichlet problem does not have a unique solution in this particular case.

This theorem also holds for a rescaled boundary $a\Gamma$, $a > 0$; there exists a unique μ_E^a satisfying

$$\begin{aligned} \mathcal{V}^a \mu_E^a &= -\log C(a\Gamma), \\ \int_{a\Gamma} \mu_E^a \, d\Gamma &= 1, \end{aligned}$$

where $\mathcal{V}^a$ is the single layer operator over $a\Gamma$. Making use of property 3) in Section 2, it follows that

$$\mathcal{V}^a \mu_E^a = -\log C(\Gamma) - \log a. \tag{3}$$

If a is such that $-\log C(\Gamma) - \log a = 0$, then $C(\Gamma)$ is related to an operator $\mathcal{V}^a$ with eigenvalue zero. Hence, to find the logarithmic capacity of Γ, we need to find an operator $\mathcal{V}^a$ that is singular. Fortunately, the family of operators $\{\mathcal{V}^a\}_{a>0}$, can be evaluated efficiently. Writing $\boldsymbol{x}' := a\boldsymbol{x}$ and $\boldsymbol{y}' := a\boldsymbol{y}$, we obtain

$$\begin{aligned} (\mathcal{V}^a \mu^a)(\boldsymbol{x}') &= \int_{a\Gamma} \log \frac{1}{\|\boldsymbol{x}' - \boldsymbol{y}'\|} \, \mu^a(\boldsymbol{y}') \, d\Gamma_{y'} \\ &= \int_{\Gamma} \log \frac{1}{a \, \|\boldsymbol{x} - \boldsymbol{y}\|} \, \mu^a(a\boldsymbol{y}) \, a \, d\Gamma_y \\ &= -a \log a \int_{\Gamma} \mu(\boldsymbol{y}) \, d\Gamma_y + a \int_{\Gamma} \log \frac{1}{\|\boldsymbol{x} - \boldsymbol{y}\|} \, \mu(\boldsymbol{y}) \, d\Gamma_y \\ &= -a \log a \int_{\Gamma} \mu(\boldsymbol{y}) \, d\Gamma_y + a(\mathcal{V}\mu)(\boldsymbol{x}). \end{aligned} \tag{4}$$

(Here we use $\mu^a(\boldsymbol{y}') = \mu^a(a\boldsymbol{y}) = \mu(\boldsymbol{y})$.) Therefore, in the process of finding the operator $\mathcal{V}^a$ that is singular, we have to evaluate $\mathcal{V}$ only once. The scaling a for which the operator $\mathcal{V}^a$ is singular is often referred to as the *critical scaling*, and we denote it by a^*. Hence, from (3) we get that the logarithmic capacity of Γ is the inverse of the critical scaling

$$C(\Gamma) = \frac{1}{a^*},$$

and is related to an eigenfunction of $\mathcal{V}^a$ with zero eigenvalue.

For most boundaries it is not possible to find the critical scaling parameter a_* for which $\mathcal{V}^a$ has a zero eigenvalue analytically. Therefore we construct a numerical approximation of the operator, i.e., a matrix, and determine for which value a_* this matrix has a zero eigenvalue. To this end we discretize the boundary Γ and equation (1). Let θ be a variable that parameterizes the boundary Γ, $0 \leq \theta < 2\pi$. We choose N points x_k on the boundary Γ at $\theta_k = (k-1)d\theta$, $k = 1, \ldots, N$, where $d\theta = 2\pi/N$. (More adaptive strategies, such as adding more nodes near non-smooth parts of the boundary could be considered, but this is outside the scope of this paper.) Two consecutive points $\boldsymbol{x}_k$ and $\boldsymbol{x}_{k+1}$ are connected by a straight line element Γ_k. The union of these elements forms an approximation $\widehat{\Gamma}$ for Γ. For each element Γ_k we approximate the distribution μ by a constant value μ_k. The integral over Γ can be approximated by a sum of integrals over the elements Γ_k, which yields the following discretized form of (1),

$$\sum_{k=1}^{N} \mu_k \int_{\Gamma_k} \log \frac{1}{\|\boldsymbol{x} - \boldsymbol{y}\|} \, d\Gamma_y = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma. \tag{5}$$

In the left-hand side of this expression, the numerical approximation of the single layer operator $\mathcal{V}$ can be identified. Note that the outlined procedure is the common way to discretize and numerically solve the Dirichlet problem (1); in this paper we also use discretizations to approximate the logarithmic capacity. There are two ways to transform the equation (5) into a linear system that are commonly used in the field of boundary integral equations. For more information we refer to handbooks on numerical solutions of integral equations [4,5,8].

3.1 Collocation approach

The collocation approach is the most commonly used approach to discretize boundary integral equations such as (5). It yields a matrix whose elements are evaluated by a single evaluation of an integral. In practice, these integrals are approximated by quadrature rules. A drawback of the collocation approach is that in general the resulting matrices are asymmetric.

We choose collocation nodes $\boldsymbol{x}^p$ at the center of each element, i.e, $\boldsymbol{x}^p := (\boldsymbol{x}_p + \boldsymbol{x}_{p+1})/2$. We substitute $\boldsymbol{x} = \boldsymbol{x}^p$ in equation (5) and obtain the following approximation of (5) (note that in general $\boldsymbol{x}^p$ is not on Γ but on $\widehat{\Gamma}$)

$$\sum_{k=1}^{N} \mu_k \int_{\Gamma_k} \log \frac{1}{\|\boldsymbol{x}^p - \boldsymbol{y}\|} \, d\Gamma_y = f(\boldsymbol{x}^p) =: f_p, \quad p = 1, \ldots, N. \tag{6}$$

We introduce the matrix $\mathbf{A}$ and the vectors $\boldsymbol{\mu}$ and $\mathbf{f}$ by

$$\begin{aligned} \mathbf{A}_{pk} &:= \int_{\Gamma_k} \log \frac{1}{\|\boldsymbol{x}^p - \boldsymbol{y}\|} \, d\Gamma_y, \\ \boldsymbol{\mu} &:= [\mu_1, \ldots, \mu_N]^T, \\ \mathbf{f} &:= [f_1, \ldots, f_N]^T, \end{aligned} \tag{7}$$

and write (6) in matrix-vector notation

$$\mathbf{A}\boldsymbol{\mu} = \mathbf{f}.$$

The matrix $\mathbf{A}$ is the discrete counterpart of the single layer operator $\mathcal{V}$. Similar to the single layer operator $\mathcal{V}$, the matrix $\mathbf{A}^a$ corresponding to a rescaled boundary $a\Gamma$ can be obtained from the matrix $\mathbf{A}$. Let $\mathbf{1}$ denote the constant vector $[1, \ldots, 1]^T$ of length N, and let $\boldsymbol{l}$ denote the vector with the lengths of the elements, $\boldsymbol{l} = [|\Gamma_1|, \ldots, |\Gamma_N|]^T$, with $|\Gamma_k|$ the length of the kth element. From (4) it can be verified that, after rescaling of the boundary by a factor a, the new matrix $\mathbf{A}^a$ is given by

$$\mathbf{A}^a = a\mathbf{A} - a \log a \; \mathbf{1} \, \boldsymbol{l}^T. \tag{8}$$

Hence by rescaling the boundary, the original matrix $\mathbf{A}$ is updated by a rank-one matrix.

3.2 Galerkin approach

Another well-known approach to discretize boundary integral equations is the Galerkin approach. The matrices that occur in this approach are symmetric. The computation of a matrix element, however, requires the evaluation of a double integral. Again, in practice, these integrals are approximated with quadrature rules. Define the shape function ϕ_i by

$$\phi_i(\boldsymbol{x}) = \begin{cases} 1 \text{ at } \Gamma_i, \\ 0 \text{ elsewhere}, \end{cases}$$

for $i = 1, \ldots, N$. We multiply (5) by ϕ_i and integrate over $\widehat{\Gamma}$, to get

$$\sum_{j=1}^{N} \sum_{k=1}^{N} \mu_k \int_{\Gamma_j} \phi_i(\boldsymbol{x}) \int_{\Gamma_k} \log \frac{1}{\|\boldsymbol{x} - \boldsymbol{y}\|} \, d\Gamma_y \, d\Gamma_x = \sum_{j=1}^{N} \int_{\Gamma_j} f(\boldsymbol{x}) \, d\Gamma_x,$$

for $i = 1, \ldots, N$. Assume that $f \approx f_i := f(\boldsymbol{x}^i)$ at Γ_i. As ϕ_i is zero at Γ_j for $j \neq i$, we obtain

$$\sum_{k=1}^{N} \mu_k \int_{\Gamma_i} \int_{\Gamma_k} \log \frac{1}{\|\boldsymbol{x} - \boldsymbol{y}\|} \, d\Gamma_y \, d\Gamma_x = f_i \, |\Gamma_i|, \quad i = 1, \ldots, N. \tag{9}$$

We introduce the matrix $\mathbf{B}$ and the vector $\mathbf{g}$ by

$$\begin{aligned} \mathbf{B}_{ik} &:= \int_{\Gamma_i} \int_{\Gamma_k} \log \frac{1}{\|\boldsymbol{x} - \boldsymbol{y}\|} \, d\Gamma_y \, d\Gamma_x, \\ \mathbf{g} &:= [f_1 \, |\Gamma_1|, \ldots, f_N \, |\Gamma_N|]^T, \end{aligned} \tag{10}$$

and write (9) in matrix-vector form

$$\mathbf{B}\boldsymbol{\mu} = \mathbf{g}.$$

The new matrix $\mathbf{B}^a$, corresponding to a rescaled boundary $a\Gamma$, is given by (cf. (4), (8))

$$\mathbf{B}^a = a^2 \mathbf{B} - a^2 \log a \; \boldsymbol{l} \, \boldsymbol{l}^T.$$

Again, by rescaling the boundary, a rank-one matrix is added to the original matrix $\mathbf{B}$.

To determine the value a_* for which $\mathbf{A}^a$ or $\mathbf{B}^a$ have a zero eigenvalue, for both the collocation and Galerkin approach we realize the following. Given a matrix $\mathbf{D}$ (representing $\mathbf{A}$ or $\mathbf{B}$) and two vectors $\mathbf{u}$ and $\mathbf{v}$, we define the following rank-one modification $\mathbf{D}^{\gamma,a}$ of $\mathbf{D}$,

$$\mathbf{D}^{\gamma,a} := \gamma \mathbf{D} - \gamma \log a \; \mathbf{u}\mathbf{v}^T, \quad \gamma \in \mathbb{R}.$$

(For the collocation approach: $\gamma = a$, $\mathbf{u} = \mathbf{1}$, $\mathbf{v} = \boldsymbol{l}$; for the Galerkin approach: $\gamma = a^2$, $\mathbf{u} = \mathbf{v} = \boldsymbol{l}$.) Assume for the moment that $\mathbf{D}$ is invertible. The determinant of the matrix $\mathbf{D}^{\gamma,a}$ is computed with

$$\begin{aligned} \det(\mathbf{D}^{\gamma,a}) &= \det\left(\gamma \mathbf{D} - \gamma \log a \; \mathbf{u}\mathbf{v}^T\right) \\ &= \gamma^N \det(\mathbf{D}) \det\left(\mathbf{I} - \log a \; \mathbf{D}^{-1}\mathbf{u}\mathbf{v}^T\right) \\ &= \gamma^N \det(\mathbf{D}) \left(1 - \log a \; \mathbf{v}^T\mathbf{D}^{-1}\mathbf{u}\right). \end{aligned}$$

If additionally $\mathbf{v}^T\mathbf{D}^{-1}\mathbf{u} \neq 0$, this result implies that $\mathbf{D}^{\gamma,a}$ has a zero eigenvalue when

$$a = \exp\left(\frac{1}{\mathbf{v}^T\mathbf{D}^{-1}\mathbf{u}}\right).$$

If $\mathbf{D}$ is singular, then this matrix, which corresponds to the unscaled boundary ($a = 1$), already has a zero eigenvalue. Applying this result to the collocation approach, we find that the matrix $\mathbf{A}^a$ has a zero eigenvalue if

$$a = a_1 := \exp\left(\frac{1}{\boldsymbol{l}^T\mathbf{A}^{-1}\mathbf{1}}\right), \tag{11}$$

when $\mathbf{A}$ is invertible and $\boldsymbol{l}^T\mathbf{A}^{-1}\mathbf{1} \neq 0$, and $a = 1$ when $\mathbf{A}$ is not invertible. For the Galerkin approach this means that the matrix $\mathbf{B}^a$ has a zero eigenvalue if

$$a = a_2 := \exp\left(\frac{1}{\boldsymbol{l}^T\mathbf{B}^{-1}\boldsymbol{l}}\right), \tag{12}$$

when $\mathbf{B}$ is invertible and $\boldsymbol{l}^T\mathbf{B}^{-1}\boldsymbol{l} \neq 0$, and $a = 1$ when $\mathbf{B}$ is not invertible.

In the following algorithm we summarize the two methods to approximate the logarithmic capacity.

Algorithm: A boundary integral method for approximating the logarithmic capacity.

Input: A set of nodes describing a boundary Γ.
Output: Approximations C of the logarithmic capacity.

1: **(Collocation)** Compute matrix $\mathbf{A}$ with (7).
2: Construct the vector $\mathbf{1}$ with ones of length N, $\mathbf{1} = [1, \ldots, 1]^T$.
3: Construct the vector $\boldsymbol{l}$ containing the lengths of the boundary elements, $\boldsymbol{l} = [|\Gamma_1|, \ldots, |\Gamma_N|]^T$.
4: Compute $C = \exp\left(-\frac{1}{\boldsymbol{l}^T(\mathbf{A}\backslash\mathbf{1})}\right)$.
5: **(Galerkin)** Compute matrix $\mathbf{B}$ with (10).
6: Compute $C = \exp\left(-\frac{1}{\boldsymbol{l}^T(\mathbf{B}\backslash\boldsymbol{l})}\right)$.

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4 Analysis

As $\mathbf{D}$ (representing either $\mathbf{A}$ or $\mathbf{B}$) is an approximation obtained by discretization, we now look at the sensitivity of the algorithm as a function of $\mathbf{D}$. Suppose $\mathbf{D}$ is nonsingular and $\boldsymbol{v}^T\mathbf{D}^{-1}\boldsymbol{u} \neq 0$. Let $\widehat{\mathbf{D}}$ be a perturbation of a given $\mathbf{D}$. If $\|\widehat{\mathbf{D}} - \mathbf{D}\|$ is small enough, then also $\widehat{\mathbf{D}}$ is nonsingular and $\boldsymbol{v}^T\widehat{\mathbf{D}}^{-1}\boldsymbol{u} \neq 0$. Furthermore, denote the approximations to the logarithmic capacity correspond-

ing to $\mathbf{D}$ and $\widehat{\mathbf{D}}$ by $C := \exp(-(\boldsymbol{v}^T\mathbf{D}^{-1}\boldsymbol{u})^{-1})$ and $\widehat{C} := \exp(-(\boldsymbol{v}^T\widehat{\mathbf{D}}^{-1}\boldsymbol{u})^{-1})$, respectively.

Now define the following condition number:

$$\kappa_{\mathbf{D}} := \limsup_{\varepsilon\to 0}\left\{\left|\frac{\widehat{C}-C}{\varepsilon C}\right| : \|\widehat{\mathbf{D}}-\mathbf{D}\| \le \varepsilon\right\}.$$

By bounding this condition number, the next theorem describes the sensitivity of the approximation of the logarithmic capacity as function of the matrix $\widehat{\mathbf{D}}$.

Theorem 1 *If* $\mathbf{D}$ *is nonsingular and* $\boldsymbol{v}^T\mathbf{D}^{-1}\boldsymbol{u} \neq 0$*, then*

$$\kappa_{\mathbf{D}} \le \frac{\|\mathbf{D}^{-1}\|^2\,\|\boldsymbol{u}\|\|\boldsymbol{v}\|}{(\boldsymbol{v}^T\mathbf{D}^{-1}\boldsymbol{u})^2}.$$

Proof: Write $\widehat{\mathbf{D}} = \mathbf{D} + \mathbf{E}$ with $\|\mathbf{E}\| = \varepsilon$. We have

$$\widehat{\mathbf{D}}^{-1} = (\mathbf{D}+\mathbf{E})^{-1} = \mathbf{D}^{-1}(\mathbf{I}+\mathbf{E}\mathbf{D}^{-1})^{-1} = \mathbf{D}^{-1}(\mathbf{I}-\mathbf{E}\mathbf{D}^{-1}+\text{h.o.t.}),$$

so disregarding higher order terms we have

$$\|\widehat{\mathbf{D}}^{-1}-\mathbf{D}^{-1}\| \le \|\mathbf{D}^{-1}\|^2\,\varepsilon.$$

Therefore, again up to higher order terms,

$$|\boldsymbol{v}^T\widehat{\mathbf{D}}^{-1}\boldsymbol{u} - \boldsymbol{v}^T\mathbf{D}^{-1}\boldsymbol{u}| \le \|\mathbf{D}^{-1}\|^2\,\|\boldsymbol{u}\|\|\boldsymbol{v}\|\,\varepsilon,$$

$$|(\boldsymbol{v}^T\widehat{\mathbf{D}}^{-1}\boldsymbol{u})^{-1} - (\boldsymbol{v}^T\mathbf{D}^{-1}\boldsymbol{u})^{-1}| \le \|\mathbf{D}^{-1}\|^2\,\|\boldsymbol{u}\|\|\boldsymbol{v}\|\,(\boldsymbol{v}^T\mathbf{D}^{-1}\boldsymbol{u})^{-2}\varepsilon,$$

and finally

$$\begin{aligned}|\exp(-(\boldsymbol{v}^T\widehat{\mathbf{D}}^{-1}\boldsymbol{u})^{-1}) - \exp(-(\boldsymbol{v}^T\mathbf{D}^{-1}\boldsymbol{u})^{-1})| \\ \le \|\mathbf{D}^{-1}\|^2\,\|\boldsymbol{u}\|\|\boldsymbol{v}\|\,(\boldsymbol{v}^T\mathbf{D}^{-1}\boldsymbol{u})^{-2}\,|\exp(-(\boldsymbol{v}^T\mathbf{D}^{-1}\boldsymbol{u})^{-1})|\,\varepsilon\end{aligned}$$

which proves the theorem.

Next, we note that $\boldsymbol{z} := \mathbf{D}^{-1}\boldsymbol{u}$ is the approximate solution of the Dirichlet problem in boundary integral form (1). This means that the estimation of the logarithmic capacity may be viewed as a cheap post-processing step after solving the boundary integral equation, and that the computational costs of our algorithm are virtually the same as the effort to solve the boundary integral equation.

Let $\boldsymbol{z}_*$ denote the true solution interpolated at the nodes. Under some assumptions, various estimates exist of the form

$$\|\boldsymbol{z} - \boldsymbol{z}_*\| \leq \gamma h^p \tag{13}$$

for certain γ and p and mesh size h [1,2,18,25]. These observations suggest to look at the sensitivity of the algorithm as function of $\boldsymbol{z}$. To this aim, define

$$\kappa_{\boldsymbol{z}} := \limsup_{\varepsilon \to 0} \left\{ \left| \frac{\widehat{C} - C}{\varepsilon C} \right| : \|\boldsymbol{z} - \boldsymbol{z}_*\| \leq \varepsilon \right\}.$$

The next theorem bounds this condition number to describe the sensitivity of the approximation of the logarithmic capacity as a function of the approximate solution of the Dirichlet problem (1).

Theorem 2 *Let* $\mathbf{D}$ *be nonsingular and* $\boldsymbol{v}^T \mathbf{D}^{-1} \boldsymbol{u} \neq 0$. *We have*

$$\kappa_{\boldsymbol{z}} \leq \frac{\|\boldsymbol{v}\|}{(\boldsymbol{v}^T \boldsymbol{z})^2} = \frac{\|\boldsymbol{v}\|}{(\boldsymbol{v}^T \mathbf{D}^{-1} \boldsymbol{u})^2}.$$

Proof: First, up to higher order terms in $\varepsilon = \|\boldsymbol{e}\|$, with $\boldsymbol{e} := \boldsymbol{z} - \boldsymbol{z}_*$:

$$(\boldsymbol{v}^T(\boldsymbol{z} + \boldsymbol{e}))^{-1} = (\boldsymbol{v}^T \boldsymbol{z})^{-1} \left(1 - \frac{\boldsymbol{v}^T \boldsymbol{e}}{\boldsymbol{v}^T \boldsymbol{z}} \right),$$

so

$$\frac{\exp(-(\boldsymbol{v}^T(\boldsymbol{z} + \boldsymbol{e}))^{-1}) - \exp(-(\boldsymbol{v}^T \boldsymbol{z})^{-1})}{\exp(-(\boldsymbol{v}^T \boldsymbol{z})^{-1})} = \exp\left(\frac{\boldsymbol{v}^T \boldsymbol{e}}{(\boldsymbol{v}^T \boldsymbol{z})^2} \right) - 1.$$

Considering only the linear term in ε we get

$$\frac{|\boldsymbol{v}^T \boldsymbol{e}|}{(\boldsymbol{v}^T \boldsymbol{z})^2} \leq \|\boldsymbol{v}\| \, (\boldsymbol{v}^T \boldsymbol{z})^{-2} \varepsilon,$$

which proves the result.

Let C_* denote the exact logarithmic capacity. Theorem 2 combined with (13) implies a first-order estimate for the computed capacity C:

$$\left| \frac{C - C_*}{C} \right| \lesssim \|\boldsymbol{v}\| \, (\boldsymbol{v}^T \boldsymbol{z})^{-2} \, \gamma h^p.$$

5 Numerical experiments

We illustrate the algorithms to numerically determine the logarithmic capacity for a number of simple boundaries for which we know the exact values of the

logarithmic capacity. All test are performed in Matlab on a Pentium 4 1.66 GHz with 504 GB RAM. We compare the performance of our algorithms with the performance of the algorithm developed by Rostand in 1997 [24], which we will refer to as the Rostand algorithm. This algorithm is based on a relation between the logarithmic capacity and the Green's function together with a uniform harmonic approximation theorem. In the algorithm the boundary is also approximated with N linear elements, which allows us to get a fair comparison. The Rostand algorithm uses a second parameter N_1, which we set to 5 in all our examples to reproduce the results of [24]. We would like to stress that fact that we did not directly copy the values from [24] but recomputed them using current software and hardware.

The first boundary that we consider is an ellipse whose semi-axes have lengths 6 and 4. For this ellipse the logarithmic capacity can be computed analytically, see Table 1, and for the particular ellipse in this example the logarithmic capacity is equal to 5. As described in the previous section, we approximate the boundary of the ellipse with N linear elements and compute the matrices $\mathbf{A}$ and $\mathbf{B}$. The integrals that need to be computed to obtain the matrix elements are approximated with a Gauss quadrature rule. For the examples in this section we use three Gauss points. We have noticed that the accuracy of the logarithmic capacity does not decrease when more Gauss points are used. Hence, the error in the logarithmic capacity depends mainly on the discretization error (13). The critical scaling parameters a_1 and a_2 are found with (11) and (12) and the inverse of these scaling parameters are approximations to the logarithmic capacity. We repeat this process for several values of N. In Table 2 we show the error between the approximation and the exact value of the logarithmic capacity, as well as the cpu time.

We clearly see that the Rostand algorithm outperforms both the Galerkin and collocation algorithm. The Rostand algorithm obtains a high accuracy with a small number of boundary elements. The errors for the Galerkin and collocation algorithms also get smaller, although with a much slower rate. The data in the table suggests that the convergence rate for both the Galerkin and the collocation method is quadratic, that is, the value of p in (13) is equal to 2.

The second boundary for which we approximate the logarithmic capacity is an equilateral triangle with side 3/2. For this boundary the logarithmic capacity can be computed analytically, and is given by (see Table 1)

$$C(\Gamma) = \frac{3\left(\Gamma\left(\frac{1}{3}\right)\right)^3}{8\pi^2} \approx 0.7305.$$

In Table 3 we show the accuracy and the cpu time for approximating the logarithmic capacity for the triangle. (Note that Matlab timings are known

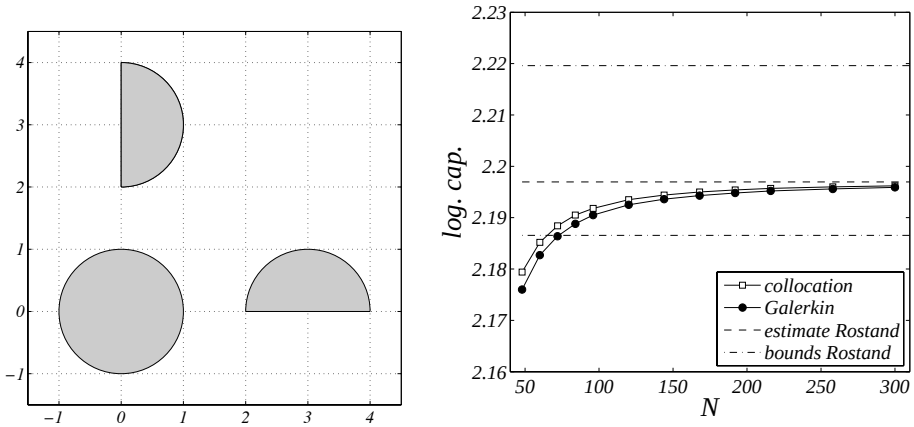
		Rostand [24]		Galerkin		collocation	
N	h	abs. error	time	abs. error	time	abs. error	time
10	3.173	$3.1 \cdot 10^{-2}$	0.001	$1.6 \cdot 10^{-1}$	0.11	$1.9 \cdot 10^{-1}$	0.02
20	1.587	$8.1 \cdot 10^{-5}$	0.001	$4.0 \cdot 10^{-2}$	0.11	$5.0 \cdot 10^{-2}$	0.02
30	1.058	$9.5 \cdot 10^{-8}$	0.001	$1.8 \cdot 10^{-2}$	0.17	$2.2 \cdot 10^{-2}$	0.03
40	0.793	$7.3 \cdot 10^{-11}$	0.002	$1.0 \cdot 10^{-2}$	0.24	$1.3 \cdot 10^{-2}$	0.06
100	0.317	$1.8 \cdot 10^{-15}$	0.003	$1.6 \cdot 10^{-3}$	0.91	$2.0 \cdot 10^{-3}$	0.33
500	0.063	$8.9 \cdot 10^{-16}$	0.018	$5.9 \cdot 10^{-5}$	16.7	$8.2 \cdot 10^{-5}$	8.3
1000	0.032	$2.7 \cdot 10^{-15}$	0.037	$9.2 \cdot 10^{-6}$	65.2	$2.1 \cdot 10^{-5}$	33.6

Table 2
Error and computation time for approximating the logarithmic capacity of an ellipse with aspect ratio 3/2.

		Rostand [24]		Galerkin		collocation	
N	h	abs. error	time	abs. error	time	abs. error	time
15	0.346	$2.8 \cdot 10^{-2}$	0.001	$7.7 \cdot 10^{-3}$	0.38	$1.4 \cdot 10^{-2}$	0.05
30	0.173	$3.8 \cdot 10^{-2}$	0.002	$3.4 \cdot 10^{-3}$	0.16	$6.1 \cdot 10^{-3}$	0.03
60	0.087	$3.9 \cdot 10^{-2}$	0.002	$1.5 \cdot 10^{-3}$	0.32	$2.7 \cdot 10^{-3}$	0.12
120	0.043	$4.0 \cdot 10^{-2}$	0.023	$6.5 \cdot 10^{-4}$	1.0	$1.2 \cdot 10^{-3}$	0.48
600	0.009	$4.0 \cdot 10^{-2}$	0.031	$9.4 \cdot 10^{-5}$	23.9	$1.7 \cdot 10^{-4}$	12.0
900	0.006	$4.0 \cdot 10^{-2}$	0.047	$5.7 \cdot 10^{-5}$	53.5	$1.0 \cdot 10^{-4}$	27.3

Table 3
Error and computation time for approximating the logarithmic capacity of a triangle.

to be potentially inaccurate, in particular for small running times; cf. the cpu times for $N = 15$ which are partly higher than those for $N = 30$.) We observe that while the errors for the Galerkin and collocation algorithms reduce, this is not the case for the Rostand algorithm. In [24] a number of modifications to the standard Rostand algorithm is introduced to obtain convergence for the triangle. Although convergence is reached, it still requires many boundary elements ($N = 600$) to get comparable convergence to our Galerkin and BEM algorithm for low numbers of boundary elements. It turns out that the Rostand algorithm shows performs very well for smooth boundaries, such as the ellipse, but performs poorer on boundaries that have corners, such as the triangle.



(a) Domain (b) Logarithmic capacity

Fig. 1. *The set consists of a circle and two semi-circles. The logarithmic capacity is approximated with the collocation and Galerkin approach. The estimate of Ransford and Rostand and the corresponding error bounds are represented by horizontal lines.*

Our algorithm has comparable accuracies for both smooth and non-smooth boundaries. Still, as the boundary of the triangle is not as smooth as the boundary of the ellipse, the convergence order is not quadratic anymore. The data in the table suggests that the convergence order is approximately 1.2.

Similar trends are observed when approximating the logarithmic capacity for the unit square. Also in this case an analytical expression for the logarithmic capacity is known, and we find

$$C(\Gamma) = \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{4\pi^{3/2}} \approx 0.5902.$$

Table 4 shows the accuracy and cpu time for the approximation of the logarithmic capacity for the square. Again the error for the Rostand algorithm does not get smaller; the Galerkin and collocation algorithms display a better performance. The data in the table suggest a convergence order of 1.3.

For the next example we turn back to the ellipse. We keep the number of boundary elements fixed, $N = 48$, and vary the aspect ratio of the ellipse. For each ellipse we approximate the logarithmic capacity. In Table 5 we give the error and the cpu time for these approximations. We see that the Rostand algorithm performs very well for ellipses with aspect ratios near one. For slender ellipses however, the error becomes much larger. The Galerkin and collocation algorithm have the same accuracy for all ellipses. They even seem to provide more accurate results when the aspect ratio gets smaller.

So far we have only looked at simply connected sets. We now show that our algorithms can also be used to approximate the logarithmic capacity for sets

		Rostand [24]		Galerkin		collocation	
N	h	abs. error	time	abs. error	time	abs. error	time
20	0.400	$1.0 \cdot 10^{-2}$	0.001	$4.9 \cdot 10^{-3}$	0.15	$9.7 \cdot 10^{-3}$	0.03
40	0.200	$1.9 \cdot 10^{-2}$	0.002	$2.0 \cdot 10^{-3}$	0.19	$3.9 \cdot 10^{-3}$	0.06
60	0.133	$2.0 \cdot 10^{-2}$	0.002	$1.2 \cdot 10^{-3}$	0.35	$2.3 \cdot 10^{-3}$	0.12
80	0.100	$2.1 \cdot 10^{-2}$	0.002	$7.9 \cdot 10^{-4}$	0.53	$1.5 \cdot 10^{-3}$	0.22
100	0.080	$2.1 \cdot 10^{-2}$	0.003	$5.8 \cdot 10^{-4}$	0.79	$1.2 \cdot 10^{-3}$	0.33
500	0.016	$2.1 \cdot 10^{-2}$	0.018	$6.6 \cdot 10^{-5}$	16.6	$1.3 \cdot 10^{-4}$	8.3
1000	0.008	$2.1 \cdot 10^{-2}$	0.069	$2.6 \cdot 10^{-5}$	66.1	$5.3 \cdot 10^{-5}$	33.8

Table 4
Error and computation time for approximating the logarithmic capacity of a square.

	Rostand [24]		Galerkin		collocation	
aspect ratio	abs. error	time	abs. error	time	abs. error	time
0.50	$3.5 \cdot 10^{-9}$	0.002	$1.1 \cdot 10^{-3}$	0.29	$1.3 \cdot 10^{-3}$	0.091
0.40	$7.5 \cdot 10^{-7}$	0.002	$9.8 \cdot 10^{-4}$	0.23	$1.2 \cdot 10^{-3}$	0.080
0.30	$6.4 \cdot 10^{-5}$	0.002	$9.1 \cdot 10^{-4}$	0.22	$1.1 \cdot 10^{-3}$	0.078
0.20	$2.0 \cdot 10^{-3}$	0.001	$8.4 \cdot 10^{-4}$	0.22	$1.1 \cdot 10^{-3}$	0.078
0.10	$8.2 \cdot 10^{-2}$	0.001	$7.7 \cdot 10^{-4}$	0.22	$9.7 \cdot 10^{-4}$	0.078
0.05	$2.8 \cdot 10^{-1}$	0.001	$7.1 \cdot 10^{-4}$	0.22	$9.1 \cdot 10^{-4}$	0.078

Table 5
Error and computation time for approximating the logarithmic capacity of an ellipse with varying aspect ratio. The number of boundary elements is $N = 48$.

that are not connected, for instance the union of several non-intersecting subsets. In Figure 1(a) we show a compact set that consists of the union of a circle and two semi-circles. This example has also been presented by Ransford and Rostand [23] and we will refer to their algorithm as the Ransford algorithm. This algorithm determines an upper and a lower bound for the logarithmic capacity and an extrapolated approximation. We discretize the boundaries of the three subsets and use the Galerkin and collocation approach to approximate the logarithmic capacity. In Figure 1(b) we plot the approximations as a function of the number of boundary elements N. We observe that the approximations of the collocation approach and the approximations of the Galerkin approach a value close to the value that was obtained with the Rostand algorithm. The dashed horizontal lines represent the upper and lower bounds for the logarithmic capacity as determined in the Ransford algorithm.

6 Concluding remarks

We developed two algorithms to numerically approximate the logarithmic capacity of compact sets in $\mathbb{R}^2$ that are bounded by a finite set of Jordan curves. Our algorithms have approximately the same quality for sets with smooth and non-smooth boundaries, which implies that they are generally better for the latter category than the algorithm in [24]. The algorithms also work on sets that are the union of several non-overlapping subsets. Although the idea behind our algorithms is conceptually relatively simple, the results are quite encouraging. The cpu time needed to approximate the logarithmic capacity is low, and even a small number of boundary elements already produces a satisfactory accuracy.

7 Acknowledgements

We thank Dr. Timo Betcke of the University of Manchester and Prof. Bob Mattheij of the Eindhoven University of Technology for helpful comments. The authors are also very grateful to the referees for their excellent remarks.

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