12 Machine-repairmen problem

So far we considered single-stage production systems. From now on we will consider multi-stage production systems, such as production lines and job shops. These systems may be modelled as networks of queues. In this chapter we start with the analysis of a very simple queueing network problem.

We consider a manufacturing system consisting of c machines. The machines now and then fail, in which case they have to be repaired by a repair man. The mean up time of a machine is $1/\lambda$, the mean repair time is $1/\mu$. While a machine is up, it is producing h parts per time unit. The situation is schematically shown in figure 1. The c jobs circulating in this network are the machines.

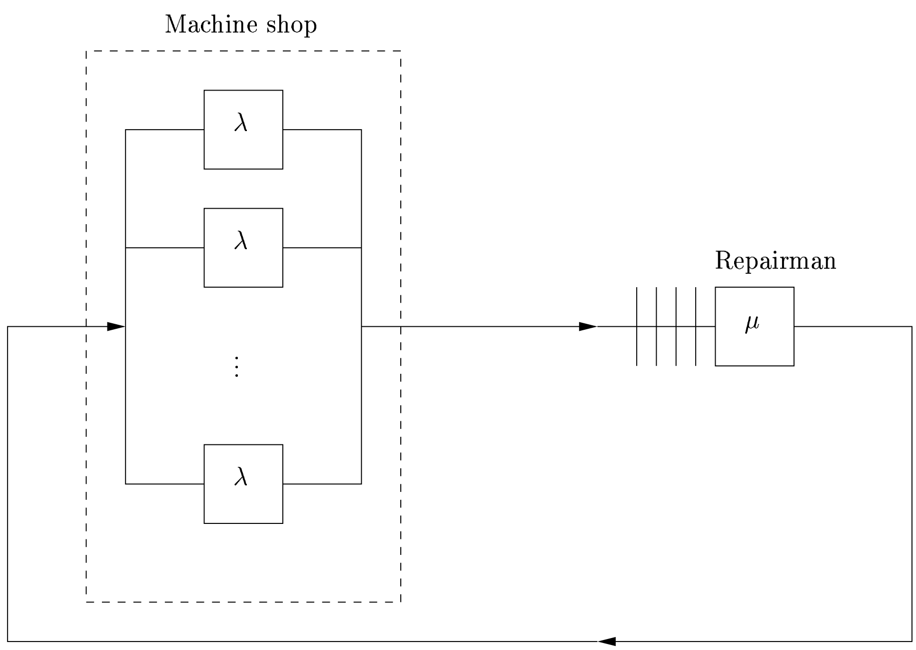

Figure 1: Machine-repairman model

There are two aspects of performance of concern: the performance of the machines and the performance of the repairman. Relevant performance characteristics are the machine efficiency η, defined as the throughput (number of parts produced per unit of time) divided by the maximal throughput (when each machine never goes down and can produce continuously), and the utilization of the repairman, ρ. If many machines are assigned to the repairman (c is large), then his utilization will be high, but the machine efficiency low; it is the other way around if a small number of machines is assigned to the repairman.

Let Λ denote the mean number of repairs per time unit, and $E(L)$ the mean number of machines that is up. Then we find (by Little's law)

$$\Lambda = \rho\mu, \qquad E(L) = \frac{\Lambda}{\lambda}.$$

Hence the throughput TH is equal to

$$TH = \frac{\rho\mu}{\lambda}h, \tag{1}$$

and thus the machine efficiency is given by

$$\eta = \frac{TH}{ch} = \frac{\rho\mu}{\lambda c}. \tag{2}$$

Clearly, to determine TH or η, we need to know the utilization ρ. To find ρ we have to take into account the variations in the up times and repair times of the machines. But before doing so, we can derive some general bounds for the throughput. Setting $\rho = 1$ in (1) yields

$$TH \leq \frac{\mu}{\lambda}h$$

and if the machines never have to wait for repair we get

$$TH \leq c\frac{\mu}{\mu+\lambda}h.$$

12.1 Exponential up times and exponential repair times

To develop further understanding of the problem we now assume that the up times and repair times are independent and exponentially distributed. Then we can describe the problem by a Markov process with states k where k is the number of machines that is up. The flow diagram is shown in figure 2

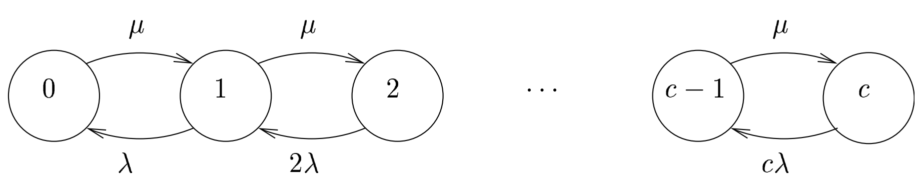

Figure 2: Flow diagram for the machine-repairman model

Let p_k denote the equilibrium probability of state k (or fraction of time in state k). Balance of flow between the set of states $\{0, 1, \ldots, k-1\}$ and $\{k, k+1, \ldots, c\}$ yields

$$p_{k-1}\mu = p_k k\lambda, \qquad k = 1, 2, \ldots, c.$$

Hence we find

$$p_k = \frac{1}{k!}\left(\frac{\mu}{\lambda}\right)^k p_0,$$

where p_0 follows from normalization, so

$$p_0^{-1} = \sum_{k=0}^{c} \frac{1}{k!}\left(\frac{\mu}{\lambda}\right)^k, \qquad k = 0, 1, \ldots, c.$$

Finally, the utilization rate of the repairman is equal to $\rho = 1 - p_c$.

We can further develop our model by including a more realistic description of the variations in the up times and repair times, e.g., by using general distributions or including dependences. Alternatively we may include new features like, e.g., multiple repairmen (pooling), non-identical machines, spare machines, etc.

12.2 Erlang up times

We now assume that the up times are Erlang distributed with r phases, each with mean $1/r\lambda$; so the variation in the up times is less than in the exponential case. The model becomes more complicated, because we have to keep track of the up phases of each machine. The state of the system can be characterized by the vector $(k_1, k_2, \ldots, k_r)$ where k_i denotes the number of machines that is in up phase i. It can be shown that the equilibrium probabilities $p(k_1, k_2, \ldots, k_r)$ are of the form

$$p(k_1, k_2, \ldots, k_r) = \frac{1}{k_1!k_2!\cdots k_r!}\left(\frac{\mu}{r\lambda}\right)^k p(0, 0, \ldots, 0),$$

where $k = k_1 + k_2 + \cdots + k_r$. This implies that p_k, the probability that k machines are up, is given by

$$p_k = \sum_{k_1+k_2+\cdots+k_r=k} p(k_1, k_2, \ldots, k_r) = \frac{1}{k!}\left(\frac{\mu}{\lambda}\right)^k p_0.$$

Hence the probabilities p_k are exactly the same as in the exponential case. This suggests (and it can be proved) that the probabilities p_k are *insensitive* to the distribution of the up times. This is, however, not true for the repair time distribution.

12.3 Pooling

Let us suppose that we have multiple repairmen, say n, assigned to the c machines ($n \leq c$). This problem can be described by a Markov process with states k where k is the number of machines that is up (i.e., the same states as in section 12.1). Its flow diagram is closely related to the one in figure 2; see figure 3.

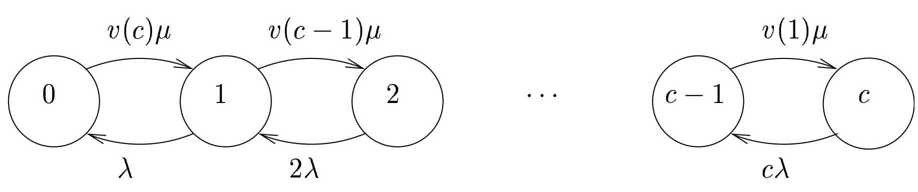

Figure 3: Flow diagram for the model with pooling, where $v(i) = \min(i, n)$

It can be readily verified that in this situation,

$$p_k = \frac{v(c)v(c-1)\cdots v(c-k+1)}{k!}\left(\frac{\mu}{\lambda}\right)^k p_0, \qquad k = 0, 1, \ldots, c.$$

The mean number of repairs per time unit is now equal to

$$\Lambda = \sum_{k=0}^{c-1} p_k v(c-k)\mu,$$

where $v(i) = \min(i, n)$, and thus we find for the throughput (cf. (1))

$$TH = \frac{\Lambda}{\lambda} h = \sum_{k=0}^{c-1} p_k v(c-k) \frac{\mu}{\lambda} h.$$

Example 12.1 Suppose we have 60 machines, 10 repairmen and $\lambda = 0.5$ per day and $\mu = 2$ per day. The production rate is $h = 1$ parts per day. In table 1 we display the throughput of the system, TH, as a function of the degree of pooling; the number of machines assigned to a pool of i repairmen is $6 \cdot i$.

Pool size	TH
1	35.3
2	37.4
5	39.1
10	39.7

Table 1: Throughput as a function of the pool size