3 Markov chains and Markov processes

Important classes of stochastic processes are Markov chains and Markov processes. A Markov chain is a discrete-time process for which the future behaviour, given the past and the present, only depends on the present and not on the past. A Markov process is the continuous-time version of a Markov chain. Many queueing models are in fact Markov processes. This chapter gives a short introduction to Markov chains and Markov processes focussing on those characteristics that are needed for the modelling and analysis of queueing problems.

3.1 Markov chains

A Markov chain, studied at the discrete time points $0, 1, 2, \ldots$, is characterized by a set of states S and the transition probabilities p_{ij} between the states. Here, p_{ij} is the probability that the Markov chain is at the next time point in state j, given that it is at the present time point at state i. The matrix P with elements p_{ij} is called the *transition probability matrix* of the Markov chain. Note that the definition of the p_{ij} implies that the row sums of P are equal to 1. Under the conditions that

- all states of the Markov chain *communicate* with each other (i.e., it is possible to go from each state, possibly in more than one step, to every other state),
- the Markov chain is not periodic (a periodic Markov chain is a chain in which, e.g., you can only return to a state in an even number of steps),
- the Markov chain does not drift away to infinity,

the probability $p_i(n)$ that the system is in state i at time point n converges to a limit π_i as n tends to infinity. These limiting probabilities, or equilibrium probabilities, can be computed from a set of so-called balance equations. The balance equations balance the probability of leaving and entering a state in equilibrium. This leads to the equations

$$\pi_i \sum_{j \neq i} p_{ij} = \sum_{j \neq i} \pi_j p_{ji}, \qquad i \in S$$

or

$$\pi_i = \sum_{j \in S} \pi_j p_{ji}, \qquad i \in S.$$

In vector-matrix notation this becomes, with π the row vector with elements π_i,

$$\pi = \pi P. \tag{1}$$

Together with the normalization equation

$$\sum_{i \in S} \pi_i = 1,$$

the solution of the set of equations (1) is unique.

3.2 Markov processes

In a Markov process we also have a discrete set of states S. However, the transition behaviour is different from that in a Markov chain. In each state there are a number of possible events that can cause a transition. The event that causes a transition from state i to j, where $j \neq i$, takes place after an exponential amount of time, say with parameter q_{ij}. As a result, in this model transitions take place at random points in time. According to the properties of exponential random variables (cf. section 1.2.3) we have:

- In state i a transition takes place after an exponential amount of time with parameter $\sum_{j \neq i} q_{ij}$.
- The system makes a transition to state j with probability

$$p_{ij} := q_{ij} / \sum_{k \neq i} q_{ik}.$$

Define

$$q_{ii} := -\sum_{j \neq i} q_{ij}, \qquad i \in S.$$

The matrix Q with elements q_{ij} is called the *generator* of the Markov process. Note that the definition of the q_{ii} implies that the row sums of Q are 0. Under the conditions that

- all states of the Markov process *communicate* with each other,
- the Markov process does not drift away to infinity,

the probability $p_i(t)$ that the system is in state i at time t converges to a limit p_i as t tends to infinity. Note that, different from the case of a discrete time Markov chain, we do not have to worry about periodicity. The randomness of the time the system spends in each state guarantees that the probability $p_i(t)$ converges to the limit p_i. The limiting probabilities, or equilibrium probabilities, can again be computed from the balance equations. The balance equations now balance the *flow* out of a state and the flow into that state. The flow is the mean number of transitions per time unit. If the system is in state i, then events that cause the system to make a transition to state j occur with a frequency or *rate* q_{ij}. So the mean number of transitions per time unit from i to j is equal to $p_i q_{ij}$. This leads to the balance equations

$$p_i \sum_{j \neq i} q_{ij} = \sum_{j \neq i} p_j q_{ji}, \qquad i \in S$$

or

$$0 = \sum_{j \in S} p_j q_{ji}.$$

In vector-matrix notation this becomes, with p the row vector with elements p_i,

$$0 = pQ. \tag{2}$$

Together with the normalization equation

$$\sum_{i \in S} p_i = 1,$$

the solution of the set of equations (2) is unique.

3.3 The embedded Markov chain

An interesting way of analyzing a Markov process is through the *embedded* Markov chain. If we consider the Markov process only at the moments upon which the state of the system changes, and we number these instances 0, 1, 2, etc., then we get a Markov chain. This Markov chain has the transition probabilities p_{ij} given by $p_{ij} = q_{ij} / \sum_{k \neq i} q_{ik}$ for $j \neq i$ and $p_{ii} = 0$. The equilibrium probabilities π_i of this embedded Markov chain satisfy

$$\pi_i = \sum_{j \in S} \pi_j p_{ji} \ .$$

Then the equilibrium probabilities of the Markov process can be computed by multiplying the equilibrium probabilities of the embedded chain by the mean times spent in the various states. This leads to,

$$p_i = C \pi_i / \sum_{j \neq i} q_{ij}$$

where the constant C is determined by the normalization condition. One easily verifies that these probabilities indeed satisfy $0 = pQ$.