Nonlinear Diffusion in an Inhomogeneous Aquifer

M. Guedda
Université de Picardie Jules Verne
Faculté de Mathématiques et d'Informatique
33, rue Saint-Leu, 80039 Amiens Cedex 01
France

D. Hilhorst
Université Paris Sud
Laboratoire d'Analyse Numérique d'Orsay
Bâtiment 425, 91405 Orsay Cedex
France

M. A. Peletier
Delft University of Technology
P.O. Box 5031, 2600 GA Delft
The Netherlands

1 Introduction

The evolution of the interface between salt and fresh water in underground aquifers can be modelled by the equation

$$\rho(x)u_t - \operatorname{div}(u(1-u)\nabla u) = 0 \quad \text{for } (x,t) \in \boldsymbol{R}^N \times \boldsymbol{R}^+, \tag{1.1}$$

where the space dimension N is equal to either one or two. The unknown u represents the height of the interface, scaled as to vary between 0 and 1. The function ρ is a combination of several coefficients: the porosity and permeability of the aquifer, the viscosity of the water and the density difference between the salt and fresh water. In Section 2 we give a derivation of the model that leads to this equation.

From the mathematical point of view the case of a constant function ρ is well studied. We are interested in the influence of variation of ρ on the qualitative behaviour of solutions of (1.1). More specifically, when ρ tends to zero for large values of $|x|$, the large-time behaviour of the solutions and the interfaces can change fundamentally. We mention two cases of interest:

- If $\rho \in L^1(\boldsymbol{R}^N)$, then solutions with finite initial mass do not decrease to zero as t tends to infinity, but converge to a constant value;

- For appropriate functions ρ, the interfaces between the sets $\{u = 0\}$, $\{0 < u < 1\}$, and $\{u = 1\}$ disappear in finite time.

In Section 3 we investigate these phenomena and give characterisations for the cases in which they occur.

In the sequel we shall only sketch the proofs; the interested reader can find the details in [4].

2 Derivation of the model

Consider a horizontal aquifer of even thickness h, confined between two impermeable layers, which will be represented by the domain $\Omega = \{(x, y, z) \in \boldsymbol{R}^3 : 0 < z < h\}$. The aquifer is fully saturated with water with a varying salt content, giving rise to spatial variations in the mass density δ of the water. The water is supposed to flow according to Darcy's law,

$$\frac{\mu}{\kappa}\boldsymbol{q} + \nabla p + \delta g \boldsymbol{e}_z = 0, \tag{2.1}$$

subject to the incompressibility condition

$$\operatorname{div} \boldsymbol{q} = 0. \tag{2.2}$$

Here $\boldsymbol{q}$ denotes the specific discharge and p the pressure; g is the acceleration of gravity, the parameter μ denotes the dynamic viscosity, and both are assumed constant. The remaining coefficient κ is termed permeability, and we assume that it depends on the spatial variables x and y, but not on z and t (time). The unit vector $\boldsymbol{e}_z$ is supposed to point upwards. Typically we are interested in the situation where the porosity ε varies in space, and especially where ε approaches zero for large $|x|$ and $|y|$. In such circumstances it is reasonable to expect κ to vary in space, too.

Let Ω contain a layer of salt water with constant density δ^s, underneath a layer of fresh water with density δ^f, with a sharp interface separating the two. We introduce the function $u : \boldsymbol{R}^2 \to [0, h]$, denoting the height of the interface, such that $\Omega \cap \{z < u(x, y)\}$ is filled with salt water and $\Omega \cap \{z > u(x, y)\}$ with fresh water. At every time t, the equations (2.1) and (2.2) determine the discharge field $\boldsymbol{q}$, and from the evolution of the discharge one can deduce the evolution of the interface. Our goal is to derive an equation for u that describes the evolution of the interface but which does not explicitly refer to the discharge $\boldsymbol{q}$. We set for brevity $\gamma = (\delta^s - \delta^f)g$.

A local analysis of the situation near the interface, as in [3], [1], or [5], shows that conservation of mass and continuity of the pressure p require certain conditions on $\boldsymbol{q}$ and the inclination angle of the interface. If we denote the discharge at the interface on the salt side and on the fresh side by $\boldsymbol{q}^s$ and $\boldsymbol{q}^f$, these conditions read

$$(\boldsymbol{q}^s - \boldsymbol{q}^f)\cdot\boldsymbol{\nu} = 0, \quad (\boldsymbol{q}^s - \boldsymbol{q}^f)\cdot\boldsymbol{\tau} = -\frac{\kappa\gamma}{\mu}\boldsymbol{\tau}\cdot\boldsymbol{e}_z, \quad (\boldsymbol{q}^s - \boldsymbol{q}^f)\cdot\boldsymbol{\sigma} = 0, \quad (2.3)$$

where $\boldsymbol{\nu}$ is any vector normal to the interface, $\boldsymbol{\tau}$ is the unit vector tangential to the interface in the direction of steepest ascent, and $\boldsymbol{\sigma} = \boldsymbol{\nu}\times\boldsymbol{\tau}$. If u is differentiable, and if we denote the components of $\operatorname{grad} u$ by u_x and u_y, then $\boldsymbol{\nu}$, $\boldsymbol{\tau}$, and $\boldsymbol{\sigma}$ are parallel to $(-u_x, -u_y, 1)^T$, $(u_x, u_y, u_x^2 + u_y^2)^T$, $(-u_y, u_x, 0)^T$, respectively.

We introduce a two-dimensional discharge $\boldsymbol{Q} = (Q_x, Q_y)^T$ with

$$Q_x(x,y) = \int_0^h \boldsymbol{q}(x,y,z)\cdot\boldsymbol{e}_x\,dz \quad \text{and} \quad Q_y(x,y) = \int_0^h \boldsymbol{q}(x,y,z)\cdot\boldsymbol{e}_y\,dz.$$

The incompressibility condition $\operatorname{div}\boldsymbol{Q} = 0$ (in which the divergence is taken in $\boldsymbol{R}^2$) is automatically satisfied, and by integrating Darcy's law over the height of the aquifer it follows that

$$\frac{\mu}{\kappa}\boldsymbol{Q} + \nabla P = 0 \quad \text{in} \quad \boldsymbol{R}^2$$

where $P(x,y) = \int_0^h p(x,y,z)\,dz$ and where the gradient is taken in $\boldsymbol{R}^2$. We combine these two to find

$$\operatorname{div}(\kappa(x,y)\nabla P(x,y)) = 0 \quad \text{in} \quad \boldsymbol{R}^2.$$

With appropriate conditions on the behaviour of P at infinity, this implies $P \equiv 0$ and therefore $\boldsymbol{Q} \equiv 0$ in $\boldsymbol{R}^2$.

It is common in the literature on hydrology to further simplify the problem by adopting the *Dupuit assumption:* within each flow region, fresh and salt, the x- and y-components of $\boldsymbol{q}$ are assumed to be independent of z, i.e. constant on vertical lines (see Fig. 1). If we denote these constant compo-

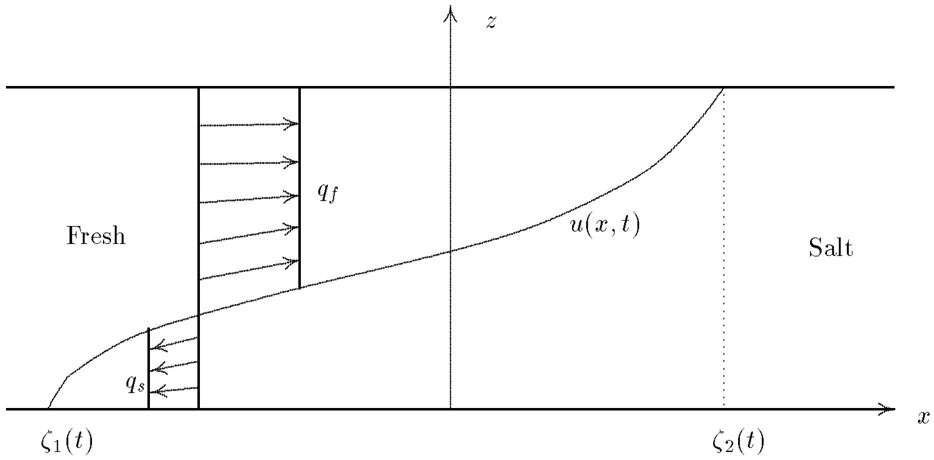

Figure 1: The Dupuit assumption.

nents by q_x^s and q_y^s for the salt water region and q_x^f and q_y^f for the fresh water region we can write

$$uq_x^s + (1-u)q_x^f = Q_x \;=\; 0 \tag{2.4}$$
$$uq_y^s + (1-u)q_y^f = Q_y \;=\; 0. \tag{2.5}$$

Solving (2.3), (2.4), and (2.5) for q_x^s and q_y^s we find

$$\begin{pmatrix} q_x^s \\ q_y^s \end{pmatrix} = -\frac{\kappa\gamma}{\mu}(1-u)\frac{\nabla u}{1+|\nabla u|^2}. \tag{2.6}$$

The motion of the interface is governed by the equation ([1], eq. (9.5.60))

$$\varepsilon\frac{\partial u}{\partial t} + \operatorname{div}\boldsymbol{Q}^s = 0 \tag{2.7}$$

where ε denotes the (space-dependent) porosity and $\boldsymbol{Q}^s = (Q_x^s, Q_y^s)^T$ is the total discharge of salt water:

$$Q_x^s(x,y) = \int_0^{u(x,y)} q(x,y,z)\cdot\boldsymbol{e}_x\,dz \quad \text{and} \quad Q_y^s(x,y) = \int_0^{u(x,y)} q(x,y,z)\cdot\boldsymbol{e}_y\,dz.$$

With the Dupuit assumption $Q_x^s = uq_x^s$ and $Q_y^s = uq_y^s$, and with equation (2.6) we can write equation (2.7) in the form

$$\varepsilon(x,y)\frac{\mu}{\gamma}\frac{\partial u}{\partial t} - \operatorname{div}\left(\kappa(x,y)\,u(1-u)\frac{\nabla u}{1+|\nabla u|^2}\right) = 0. \tag{2.8}$$

Since we shall mainly be interested in solutions u with relatively small gradients, we replace the quotient $\nabla u/(1+|\nabla u|^2)$ by ∇u. Furthermore, we shall mostly consider either one-dimensional or two-dimensional axially symmetric solutions, and then suppose that all data are either one-dimensional or axially symmetric. In the two-dimensional case with axial symmetry, equation (2.8) reduces to

$$\varepsilon(r)\frac{\mu}{\gamma}u_t - \frac{1}{r}(r\kappa(r)\,u(1-u)u_r)_r = 0 \tag{2.9}$$

where $r^2 = x^2 + y^2$ and subscripts denote differentiation. If we introduce a new space variable $\tilde{r}$, defined by

$$\log\tilde{r} \stackrel{def}{=} \int_1^r \frac{ds}{s\kappa(s)}$$

then (2.9) takes the form

$$\rho(\tilde{r})u_t - \frac{1}{\tilde{r}}(\tilde{r}u(1-u)u_{\tilde{r}})_{\tilde{r}} = 0 \tag{2.10}$$

in which $\tilde{r}^2\rho(\tilde{r}) = (\mu/\gamma)\,r^2\varepsilon(r)\kappa(r)$. We shall henceforth drop the tilde from $\tilde{r}$. In one space dimension, the equation becomes

$$\rho(x)u_t - (u(1-u)u_x)_x = 0. \tag{2.11}$$

The derivation presented above is an extension of the derivation given in [5] to the case of non-constant coefficients ε and κ.

3 Statement of results

We will be concerned with the problem

$$(\mathrm{P})\begin{cases} \rho(x)u_t = \operatorname{div}(u(1-u)\nabla u) & \text{in } Q = \boldsymbol{R}^N \times \boldsymbol{R}^+ \\ u(x,0) = u_0(x) & \text{for } x \in \boldsymbol{R}^N \end{cases} \tag{1.1}$$

The definition of a weak solution to this problem is similar to the one in [2] and can be found in [4]. For constant functions ρ this problem is well studied. When $\rho > 0$ is allowed to approach zero for large values of $|x|$, however, the qualitative behaviour of solutions can change fundamentally. The first result that we prove in this direction is the following.

Theorem A (Large-time behaviour) *Let N be equal to either one or two, $\rho \in C(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$, $\rho > 0$ on $\boldsymbol{R}^N$, and let $u_0 \in C(\boldsymbol{R}^N)$, $0 \le u_0 \le 1$, and $\rho u_0 \in L^1(\boldsymbol{R}^N)$. Let u be the solution of Problem (P) with initial data u_0. Then*

$$u(t) \to \bar{u} \stackrel{def}{=} \frac{\displaystyle\int_{\boldsymbol{R}^N} \rho(x) u_0(x)\, dx}{\displaystyle\int_{\boldsymbol{R}^N} \rho(x)\, dx} \quad as \quad t \to \infty,$$

as $t \to \infty$, uniformly on compact subsets of $\boldsymbol{R}^N$.

This result clearly generalizes the case of constant ρ, in which a solution with finite initial mass decays to zero as t tends to infinity.

A result which now is classical states that when ρ is constant, solutions of Problem (P) exhibit so-called interfaces, curves in the x,t-plane separating the regions $\{u = 0\}$, $\{0 < u < 1\}$, and $\{u = 1\}$, provided that the initial condition has an appropriate form. When ρ is supposed positive but no longer constant, the same holds for small time; however, it is possible that the interfaces disappear in finite time. This was first pointed out for a different equation by Kamin and Kersner [6]. We wish to investigate the behaviour of solutions of Problem (P) when ρ tends to zero at infinity.

For reasons of clarity we restrict ourselves to the interface between the regions $\{u = 0\}$ and $\{u > 0\}$. It follows from the symmetry present in (1.1) that all arguments can be transposed to the interface between the regions $\{u = 1\}$ and $\{u < 1\}$. Let the support of a function f ($\operatorname{supp} f$) be defined as the closure of the set $\{x : f(x) > 0\}$. A solution u of Problem (P) for $N = 1$ is said to exhibit finite time blow-up if its support is bounded from above initially and there exists a time T such that $\operatorname{supp} u(t)$ is unbounded from above for all time $t > T$. For the formulation of Theorem B we shall need an auxilliary density function σ defined by

$$\sigma(x) = \min\{\rho(\xi) : 0 \le \xi \le x\},$$

the reason being that the function σ is monotonic while ρ need not be.

Theorem B (Blow-up in one dimension) *Let $\rho \in C(\boldsymbol{R}) \cap L^\infty(\boldsymbol{R})$, $\rho > 0$ on $\boldsymbol{R}$, and let $u_0 \in C(\boldsymbol{R})$, $0 \le u_0 \le 1$. Let u be a solution of Problem (P) for $N = 1$, with initial data u_0, such that the support of u is bounded from above at time $t = 0$. Then the following implications hold:*

(i) $\displaystyle\int_0^\infty x\rho(x)\,dx < \infty \implies$ *finite time blow-up;*

(ii) $\displaystyle\int_0^\infty x\sigma(x)\,dx = \infty \implies$ *no finite time blow-up.*

It follows immediately that

Corollary C *If $\rho(x)$ is a non-increasing function of x for $0 < x < \infty$, then*

$$\int_0^\infty x\rho(x)\,dx < \infty \iff \textit{finite time blow-up}$$

Note that the behaviour of ρ and u_0 towards *minus* infinity has no influence on the (qualitative) behaviour of the upper boundary of the support. We can apply this theorem once for $\{x > 0\}$ and once for $\{x < 0\}$ with independent results.

Using the Comparison Principle we can extend this result to a statement on a strip $\Omega = \boldsymbol{R} \times (-1, 1)$ with Neumann boundary conditions, with a density function ρ that does not depend on the coordinate y: $\rho(x, y) = \rho(x)$ on Ω. Consider the problem

$$\begin{cases} \rho u_t = \Delta A(u) & \text{in } Q_T = \Omega \times [0, T] \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times [0, T] \\ u = u_0 & \text{at } t = 0, \end{cases} \tag{3.1}$$

We prove the following result.

Theorem D (Blow-up in a 2d strip) *Let the initial condition u_0 be such that $u_0(x, y) = 1$ for small x and $u_0(x, y) = 0$ for large x. Let $\gamma_0(t)$ denote the interface between $\{u > 0\}$ and $\{u = 0\}$ at time t:*

$$\gamma_0(t) = \operatorname{supp} u(t) \cap \{(x, y) \in \Omega : u(x, y, t) = 0\}.$$

Then the following statements hold:

(i) If $\int_0^\infty x\rho(x)\,dx < \infty$ *then the interface* γ_0 *will run off to infinity in finite time;*

(ii) If $\int_0^\infty x\sigma(x)\,dx = \infty$ *then the interface* γ_0 *will remain bounded for all finite time.*

A similar statement holds for the interface γ_1 *between the sets* $\{u = 1\}$ *and* $\{u < 1\}$*.*

A different way of extrapolating the one-dimensional results is by considering the two-dimensional radially symmetric problem and transforming the ensuing (one-dimensional) equation to an equation of the form (1.1). In this case the auxiliary density function σ is different:

$$\sigma(r) = \min_{0\le\xi\le r} \xi^2\rho(\xi).$$

We establish the following result:

Theorem E (Blow-up in 2d, radially symmetric) *Let* u *be a solution of (1.1) with initial condition* u_0*. Suppose that both* ρ *and* u_0 *are radially symmetric, and that* $\operatorname{supp} u_0$ *is compact.*

(i) If $\int_1^\infty \rho(r) r \log r\,dr < \infty$ *and* $0 \in \operatorname{Int}(\operatorname{supp} u_0)$*, then the support of* u *ceases to be compact in finite time;*

(ii) If $\int_1^\infty \sigma(r)\frac{\log r}{r}\,dr = \infty$*, then the support of* u *will be compact for all time.*

References

[1] J. Bear. *Dynamics of Fluids in Porous Media.* Dover Publications, Inc, New York, 1972.

[2] M. Bertsch, R. Kersner, and L. A. Peletier. Positivity versus localization in degenerate diffusion equations. *Nonlinear Analysis, Theory, Methods & Applications*, 9(9):987–1008, 1985.

[3] J. H. Edelman. *Over de berekening van grondwaterstromingen.* PhD thesis, Technische Hogeschool Delft, 1947.

[4] M. Guedda, D. Hilhorst, and M. A. Peletier. Nonlinear diffusion in an inhomogeneous aquifer. Technical Report 95-44, Delft University of Technology, 1995. `ftp://ftp.twi.tudelft.nl/TWI/publications/tech-reports/1995/DUT-TWI-95-44.ps.gz`.

[5] G. de Josselin de Jong. The simultaneous flow of fresh and salt water in aquifers of large horizontal extension determined by shear flow and vortex theory. In A. Verruijt and F. B. J. Barends, editors, *Proceedings of Euromech 143*, pages 75–82, Rotterdam, 1981. A. A. Balkema.

[6] S. Kamin and R. Kersner. Disappearance of interfaces in finite time. *Meccanica*, 28:117–120, 1993.