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Self-similar blow-up for a diffusion–attraction problem

Ignacio A Guerra[1] **and Mark A Peletier**[2]

[1] Centre for Mathematical Modeling, Universidad de Chile, FCFM Casilla 170 Correo 3 Santiago, Chile
[2] Department of Mathematics and Computer Science, TU Eindhoven, PO Box 513 5600 MB Eindhoven, The Netherlands

E-mail: iguerra@dim.uchile.cl and Mark.Peletier@cwi.nl

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Abstract
In this paper we consider a system of equations that describes a class of mass-conserving aggregation phenomena, including gravitational collapse and bacterial chemotaxis. In spatial dimensions strictly larger than two, and under the assumptions of radial symmetry, it is known that this system has at least two stable mechanisms of singularity formation (see, e.g., Brenner M P *et al* 1999 *Nonlinearity* **12** 1071–98); one type is self-similar, and may be viewed as a trade-off between diffusion and attraction, while in the other type attraction prevails over diffusion and a non-self-similar shock wave results. Our main result identifies a class of initial data for which the blow-up behaviour is of the former, self-similar type. The blow-up profile is characterized as belonging to a subset of stationary solutions of the associated ordinary differential equation.

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1. Introduction

We consider the parabolic–elliptic system

$$n_t = \operatorname{div}\{\Theta\nabla n + n\nabla\phi\} \qquad \text{in } \Omega\times\mathbb{R}^+, \tag{1}$$

$$\Delta\phi = n \qquad \text{in } \Omega\times\mathbb{R}^+, \tag{2}$$

$$0 = (\Theta\nabla n + n\nabla\phi)\cdot\vec{\nu} \qquad \text{on } \partial\Omega\times\mathbb{R}^+, \tag{3}$$

$$\phi = 0 \qquad \text{on } \partial\Omega\times\mathbb{R}^+, \tag{4}$$

$$n(x,0) = n_0(x) \qquad \text{in } \Omega, \tag{5}$$

where $\Omega = B_1(0) = \{x \in \mathbb{R}^d : |x| \leqslant 1\}$, $d > 2$, and $\vec{\nu}$ is the outer normal vector from the boundary $\partial\Omega$. Here, $\Theta > 0$ is a constant parameter. The initial condition n_0 is chosen in $L^2(\Omega)$, radially symmetric, and such that

$$\int_\Omega n_0 \, \mathrm{d}x = 1 \quad \text{and} \quad n_0(x) \geqslant 0 \qquad \text{in } \Omega. \tag{6}$$

Equations (1)–(6) define a problem for the unknown mass density n and potential ϕ. Mass is conserved by the no-flux condition (3), and therefore (6) implies

$$\int_\Omega n(x,t) \, \mathrm{d}x = \int_\Omega n_0(x) \, \mathrm{d}x = 1. \tag{7}$$

Problem (1)–(6) is a model for the evolution of a cluster of particles under gravitational interaction and Brownian motion (see [5] and references therein). Here, n represents the mass density, ϕ the gravitational potential and Θ a rescaled temperature characterizing the Brownian motion. This model also appears in the study of the evolution of polytropic stars, by considering the evolution of self-interacting clusters of particles under frictional and fluctuating forces [29]. Finally, problem (1)–(6) also arises in the study of the motion of bacteria by chemotaxis as a simplification (see [21]) of the Keller–Segel model [2, 8, 22, 28]. Here, the variables n and ϕ represent the density of bacteria and the concentration of the chemo-attractant.

We view the problem (1)–(6) as an evolution equation in n, since by equations (2)–(3) the function ϕ is readily recovered from the solution n. It is known [6] that problem (1)–(6) has a unique local solution if $n_0 \in L^2(\Omega)$, which satisfies $n \in L^\infty(\Omega \times (\epsilon, \tilde{T}))$ for some $\tilde{T} > 0$ and for every $\epsilon > 0$. We restrict ourselves to the analysis of radially symmetric solutions and write $n(r,t) := n(x,t)$ with $r = |x| \in [0,1]$.

Since we are interested in the question of when and how the system (1)–(6) generates singularities, we define:

$$T^* = \sup\{\tau > 0 \mid \text{problem (1)–(6) has a solution } n \in L^\infty(\Omega \times (\epsilon, \tau])\}.$$

If $T^* < \infty$, then we say that blow-up occurs for (1)–(6), in which case

$$\lim_{t \to T^*} \sup_{[0,1]} n(r,t) = \infty. \tag{8}$$

Various sufficient conditions for blow-up are known [3, 4, 6, 7].

For $d = 3$, Herrero *et al* [19, 20] were the first to study the behaviour of the solution close to blow-up, using matched asymptotic expansions. Later Brenner *et al* [10] studied the problem for $2 < d < 10$. They used a numerical approach to describe solutions and proved the existence and linear stability of similarity profiles. Note, however, that no proof of convergence or characterization of blow-up in terms of initial data were given in these references. The principal types of blow-up described in [10, 19, 20] are as follows:

(a) A solution $n(r,t)$ consists of an imploding smoothed shock wave that moves towards the origin. As $t \to T^*$, the bulk of such a wave is concentrated at distances $\mathrm{O}((T^* - t)^{1/d})$ from the origin, has a width $\mathrm{O}((T^* - t)^{(d-1)/d})$, and at its peak it reaches a height of order $O((T^* - t)^{-2(d-1)/d})$. This type of blow-up has the property of concentration of mass at the origin at the blow-up time, i.e.

$$\lim_{r \to 0} \left[\lim_{t \to T^*} \int_0^r n(y,t) y^{d-1} \, \mathrm{d}y \right] = C > 0. \tag{9}$$

This situation is depicted in figure 1 (left).

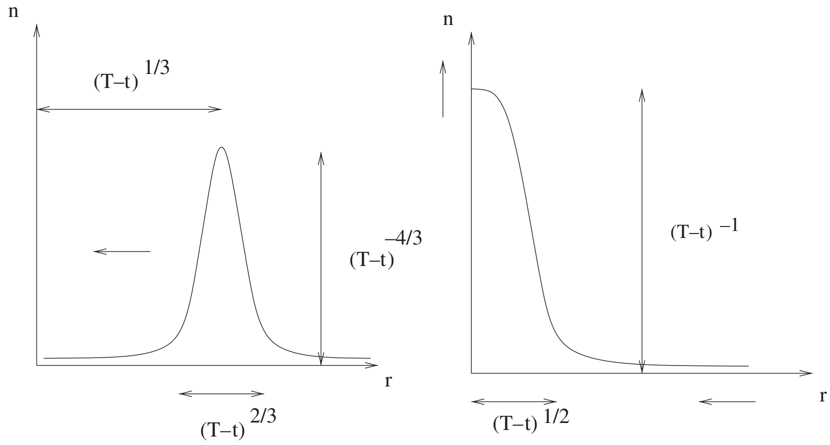

Figure 1. The profile $n(r, t)$ for blow-up with (left) and without (right) concentration of mass, with $T = T^*$.

(b) A solution $n(r, t)$ has a self-similar blow-up of the form

$$(T^* - t)n(\eta\sqrt{(T^* - t)\Theta}, t) \sim \Psi(\eta) \qquad \text{as } t \to T^*. \tag{10}$$

Note that this implies that n satisfies (9) with $C = 0$. Therefore, no concentration of mass at the origin occurs at the blow-up time. This blow-up behaviour is depicted in figure 1 (right).

The results of this paper are two-fold. First, we demonstrate rigorously that the self-similar blow-up structure (10) is an attractor for the system (1)–(6); secondly, we identify an explicit class of initial data that converges to a self-similar solution of this type. Let us elaborate on this.

Let $n_0 = n_0(r)$ be such that

$$\chi_d r^d n_0(r) \leqslant \|n_0\|_{L^1(B_r(0))} \qquad \text{for } r \in (0, 1), \tag{11}$$

$$\Theta(n_0)_r + n_0(\phi_0)_r \geqslant 0, \quad (r^d(\phi_0)_r)_r = r^d n_0 \qquad \text{in } (0, 1) \quad \text{and} \quad \phi_0(1) = 0, \tag{12}$$

where χ_d is the measure of the unit ball in $\mathbb{R}^d$. Suppose also that $\Theta \leqslant 1/(4d\chi_d)$, implying that the solution $n = n(r, t)$ of (1)–(6) blows up at finite time $T^* > 0$ and at the point $r = 0$ [4]. Finally, assume that the two functions

$$\|n_0\|_{L^1(B_r(0))} \qquad \text{and} \qquad \frac{4\Theta r^d}{2(d-2)\Theta T^* + r^2} \qquad \text{intersect exactly once in } [0, 1]. \tag{13}$$

Our main result (theorem 2.1) shows that if (11), (12) and (13) hold, then n satisfies

$$n(0, t) \leqslant \frac{2d}{(d-2)}(T^* - t)^{-1} \qquad \text{for } t \in (0, T^*)$$

and moreover has a structure near blow-up given by

$$n_*(r, t) = (T^* - t)^{-1}\Psi\left(\frac{r}{\sqrt{\Theta(T^* - t)}}\right),$$

where the function Ψ is one of a class of solutions of a steady-state problem; a class that includes the functions

$$\Psi_1(\eta) := (d-2)\frac{(2d + \eta^2)}{(d - 2 + \frac{1}{2}\eta^2)^2} \qquad \text{and} \qquad \Psi^*(\eta) := 1 \qquad \text{for } \eta > 0.$$

In particular, the initial state $n_0 \equiv 1/\chi_d$ and $\Theta \leqslant 1/(4d\chi_d)$ satisfy the conditions above (corollary 2.2). If we relax assumption (13) but assume instead that n satisfies the growth condition

$$n(0,t) \leqslant M(T^* - t)^{-1} \qquad \text{for } t \in (0, T^*),$$

for some constant $M > 0$, then n has the same structure of blow-up given above (theorem 2.3). The hypotheses on the initial data (11), (12) and (13) are more natural in the context of a transformed problem we introduce in the next section. Note, however, that $(n_0)_r \leqslant 0$ in [0, 1] implies assumption (11).

This paper is organized as follows. In section 2, we write the problem in terms of a new variable, thus transforming the system (1)–(6) into a single PDE, and then state our results in terms of this new formulation. In section 3, we discuss some non-self-similar blow-up patterns related to case (a). Sections 4, 5 and 6 provide the tools for the proofs of theorems 2.1 and 2.3, and the arguments are wrapped up in section 7. A rather technical derivation of a Lyapunov function is given in appendix A, and in appendix B we derive some linear stability results.

2. Precise statements of main results

For radial solutions, the average density function $b(r,t)$ [10] is defined by

$$b(r,t) := \frac{d\chi_d}{r^d}\int_0^r n(y,t) y^{d-1}\,\mathrm{d}y, \tag{14}$$

This variable turns out to be convenient in the analysis of this system. Note that it has the same scale invariance as $n(r,t)$, but that solutions are smoother when expressed in terms of b. For example, if for some fixed $t > 0$ the density $n(r,t)$ is a delta function at the origin with unit mass, then $b(r,t) = r^{-d}$. Let $D = (0,1)$ and set $D_T = D \times (0,T)$ for some $T > 0$. Equation (14) transforms system (1)–(6) to the form

$$b_t = \chi_d\Theta\left(b_{rr} + \frac{d+1}{r}b_r\right) + \frac{1}{d}rbb_r + b^2 \qquad \text{in } D_T, \tag{15}$$

$$b_r(0,t) = 0, \qquad b(1,t) = 1 \qquad \text{for } t \in [0,T), \tag{16}$$

$$b(0,r) = b_0(r) \qquad \text{for } r \in D. \tag{17}$$

Here, we have redefined $t := (1/\chi_d)t$. Regarding the initial condition, we assume

$$b_0 \in C^2(\overline{D}) \qquad \text{and} \qquad \frac{r}{d}(b_0)_r + b_0 \geqslant 0 \qquad \text{for } r \in D, \tag{18}$$

where the second condition is equivalent to $n_0 \geqslant 0$ in D. Note that the conservation of the mass (7) is represented by $b(1,t) = 1$ for $t \in [0,T)$. As was done for problem (1)–(6) we define $T > 0$ to be the maximal time of existence for the average density $b(r,t)$. If $T^* < \infty$ in (8), then,

$$\lim_{t\to T}\sup_{[0,1]} b(r,t) = \infty,$$

where $T = T^*/\chi_d$. Using (14), we deduce $b(r,t) \leqslant 1/r^d$ for $r \in \overline{D}$, $t > 0$; this implies single point blow-up for $b(r,t)$ at the point $r = 0$. To characterize the asymptotic behaviour near blow-up of the solution $b(r,t)$ of problem (15)–(18), we study the solutions of the associated boundary-value problem

$$\begin{aligned} &\varphi_{\eta\eta} + \frac{d+1}{\eta}\varphi_\eta + \frac{1}{d}\eta\varphi\varphi_\eta - \frac{1}{2}\eta\varphi_\eta + \varphi^2 - \varphi = 0 \qquad \text{for } \eta > 0, \\ &\varphi(0) \geqslant 1 \qquad \varphi_\eta(0) = 0. \end{aligned} \tag{19}$$

If b is a solution of (15)–(18) which blows up at time $T > 0$ and at the point $r = 0$, then we will show that it has the asymptotic form given by

$$b_*(r,t) = (T-t)^{-1}\varphi\left(\frac{r}{\sqrt{\chi_d\Theta(T-t)}}\right).$$

Equation (19) has multiple solutions for $2 < d < 10$ [10, 20]. We classify them by counting the number of times they cross the singular solution $\varphi_S(\eta) := 2d/\eta^2$. For that purpose, we introduce the set

$$\mathcal{S}_k = \{\varphi\colon\ \varphi \text{ is a solution of (19) that has } k \text{ intersections with } \varphi_S\}.$$

We shall see that $\mathcal{S}_1$ is the relevant subset of solutions of (19) for the characterization of the type of blow-up considered in this paper. Numerical evidence [10] suggests that $\mathcal{S}_1$ contains only two elements:

$$\varphi^*(\eta) = 1 \qquad \text{and} \qquad \varphi_1(\eta) := \frac{2d}{(d-2+(\eta^2/2))} \qquad \text{for } \eta \geqslant 0. \tag{20}$$

For the initial condition, we assume

$$(b_0)_r \leqslant 0 \qquad \text{for } r \in D \tag{21}$$

and

$$\chi_d\Theta\left((b_0)_{rr} + \frac{d+1}{r}(b_0)_r\right) + \frac{1}{d} r b_0\,(b_0)_r + b_0^2 \geqslant 0 \qquad \text{for } r \in D. \tag{22}$$

We will show that this implies $b_r \leqslant 0$ in D_T and $b_t \geqslant 0$ in D_T. In terms of n_0, assumption (21) becomes (11) and assumption (22) becomes (12).

Theorem 2.1. *Let $d > 2$ and b_0 satisfy (21) and (22). Let $b(r,t)$ be the corresponding solution of problem (15)–(18) that blows up at $r = 0$ and at $t = T$. If*

$$\Theta \leqslant \Theta_1 := \frac{1}{4d\chi_d} \qquad \textit{and} \qquad b_0(r) \textit{ intersects } T^{-1}\varphi_1\left(\frac{r}{\sqrt{\chi_d\Theta T}}\right) \textit{ once} \tag{23}$$

then

$$b(0,t) \leqslant M_1(T-t)^{-1} \qquad \textit{for } t \in (0,T) \tag{24}$$

with $M_1 := 2d/(d-2)$. Moreover, $T < M_1/b_0(0)$, and there exists $\varphi \in S_1$ such that

$$\lim_{t\to T}(T-t)b(\eta\sqrt{\chi_d\Theta(T-t)}) = \varphi(\eta) \tag{25}$$

uniformly on compact sets $|\eta| \leqslant C$ for every $C > 0$.

We remark that there exists a family of b_0 satisfying the conditions (18), (21) and (22), given by $b_0(r) = K_1 + K_2/(r^d + K_3)$ with positive constants K_i that satisfy $K_1 + K_2/(1+K_3) = 1$ and $\Theta < K_2/2d^2\chi_d$. Conditions (18), (21) and (22) are also satisfied for $b_0 \equiv 1$. Note that condition (23) of theorem 2.1 can be generalized by changing φ_1 for another solution φ of (19). Since these solutions are only known numerically, the counterparts of M_1 and Θ_1 cannot be given explicitly. The next corollary applies this result to $b_0 \equiv 1$.

Corollary 2.2. *Let $d > 2$, $b_0 \equiv 1$ and $\Theta < \Theta_1$. Then, $b(r,t)$, the corresponding solution of problem (15)–(17), blows up at $r = 0$ and at some time $t = T < M_1$; moreover (24) holds and there exists $\varphi \in S_1$ satisfying (25).*

Numerical simulations [10] suggest that for an open set of initial data the convergence in (25) holds for $\varphi = \varphi_1$. This self-similar behaviour may be seen roughly in figure 1 (right), by imagining $n(r,t)$ replaced by $b(r,t)$ (since n and b scale similarly). In appendix B we show that φ_1 is linearly stable (using the result in [10]) and also that φ^* is linearly unstable.

For more general initial data we have the following result.

Theorem 2.3. *Let $d > 2$ and let b_0 satisfy (21) and (22). Assume that $b(r, t)$, the corresponding solution of problem (15)–(18), blows up at $r = 0$ and at $t = T$. If b satisfies the growth condition*

$$b(0, t) \leqslant M(T - t)^{-1} \qquad \text{for } t \in (0, T) \tag{26}$$

with $M > 0$, then there exists $\varphi \in S_1$ such that the convergence (25) holds.

We now briefly discuss the structure of the proofs of these theorems. Following the scale invariance, we set

$$\tau = \log\left(\frac{T}{T-t}\right), \qquad \eta = \frac{r}{(\chi_d \Theta(T-t))^{1/2}}, \qquad \text{and} \qquad B(\eta, \tau) = (T - t)b(r, t).$$

The rectangle D_T transforms into

$$\Pi = \{(\eta, \tau) \mid \tau > 0, 0 < \eta < \ell(\tau)\} \qquad \text{where } \ell(\tau) := (\chi_d \Theta T)^{-1/2} \mathrm{e}^{\tau/2}.$$

The initial-boundary problem (15)–(18) now becomes

$$B_\tau + B + \frac{1}{2}\eta B_\eta = B_{\eta\eta} + \frac{d+1}{\eta} B_\eta + \frac{1}{d}\eta B B_\eta + B^2 \qquad \text{in } \Pi, \tag{27}$$

$$B_\eta(0, \tau) = 0, \qquad B\left(\ell(\tau), \tau\right) = \mathrm{e}^{-\tau} T \qquad \text{for } \tau \in \mathbb{R}^+, \tag{28}$$

$$B(\eta, 0) = B_0(\eta) := T b_0(\eta(\chi_d \Theta T)^{1/2}) \qquad \text{for } \eta \in \Pi(0), \tag{29}$$

where $\Pi(0) = (0, \ell(0))$. Note that a solution of (19) is a time-independent solution of (27)–(29). Therefore, the study of the blow-up behaviour of $b(r, t)$ is reduced to the analysis of the large time behaviour of solutions $B(\eta, \tau)$ of (27)–(29), and in particular stabilization towards solutions φ of (19). The proof of theorem 2.3 consists of two parts. In section 5, we first prove that $\omega \subset \mathcal{S}_1$, where

$$\omega = \{\phi \in L^\infty(\mathbb{R}^+) : \exists \tau_j \to \infty \text{ such that } B(\cdot, \tau_j) \to \phi(\cdot) \text{ as } \tau_j \to \infty \\ \text{uniformly on compact subsets of } \mathbb{R}^+\} \tag{30}$$

is the ω-limit set we introduce for (27)–(29). The proof uses the observation that equation (27), without the convection term $(1/d)\eta B B_\eta$, is the backward self-similar equation for the parabolic semilinear equation

$$\bar{b}_t = \Delta_N \bar{b} + \bar{b}^2, \tag{31}$$

where Δ_N denotes the Laplacian in $\mathbb{R}^N$ and $N = d + 2$ [15, 16]. We use the methods for the analysis of this self-similar equation to prove theorem 2.3. However, due to the presence of the convection term, a different Lyapunov functional is necessary. This functional is constructed using the method of Zelenyak [30], which yields a Lyapunov functional in implicit form. In section 6, we use intersection comparison arguments based on the ideas of Matano [23] to prove that the ω-limit set (30) is a singleton. With a result on intersection with φ_S, this completes the proof of theorem 2.3.

Note that theorem 2.3 is similar to a result for the supercritical case ($N > 6$) for equation (31), where two different kinds of self-similar blow-up behaviour may coexist [24].

Finally, to obtain theorem 2.1 and corollary 2.2, we use theorem 2.3 and comparison ideas from Samarskii *et al* [26, chapter IV].

3. Discussion on non-self-similar blow-up patterns

In this section, we discuss a family of blow-up patterns, which appears when we refine the asymptotic expansion for the profile $\varphi = \varphi^* \equiv 1$. This situation is closely related to the blow-up behaviour of (31) with $N < 6$. If a solution $\bar{b}$ of (31) with $N < 6$ blows up at $x = 0$ and $t = T$, then,

$$\lim_{t\to T}(T-t)\bar{b}(\eta\sqrt{T-t}, t) = 1$$

uniformly on compact sets $|\eta| < C$ for arbitrary $C > 0$ [15, 16]. Moreover, it has been shown (see for instance [25, 27]) that a refined description of blow-up gives the existence of two possible types of behaviour: either

$$\lim_{t\to T}(T-t)\bar{b}(\eta\sqrt{(T-t)|\log(T-t)|}, t) = \bar{\varphi}_1(\eta) \tag{32}$$

uniformly on compact sets $|\eta| < C$, with $C > 0$ arbitrary; or

$$\lim_{t\to T}(T-t)\bar{b}(\eta(T-t)^{1/2m}, t) = \bar{\varphi}_m(\eta) \qquad \text{for some } m \geqslant 2, \tag{33}$$

uniformly on compact sets $|\eta| < C$, with $C > 0$ arbitrary. Here, the family $\{\bar{\varphi}_i\}_{i\geqslant 1}$ is known explicitly. For problem (15)–(18), it was shown [20] for $d = 3$ that there exists a refined asymptotics for $\varphi^* \equiv 1$. Extending the argument to all $d > 2$, these asymptotics suggest a convergence given by either

$$\lim_{t\to T}(T-t)b(\eta\sqrt{(T-t)|\log(T-t)|^{(d-2)/d}}, t) = \tilde{\varphi}_1(\eta) \tag{34}$$

in the case of $d = 3, 4$ only, or

$$\lim_{t\to T}(T-t)b(\eta(T-t)^{1/d+1/(2(m+[d/(d-2)]-1))}, t) = \tilde{\varphi}_m(\eta) \tag{35}$$

for some $m \geqslant 2$, where $[x]$ denotes the greatest integer $\leqslant x$. An implicit formula for the family $\{\tilde{\varphi}_m\}_{m\geqslant 1}$ is given in [10, equation (43)]. The type of convergence in η towards these profiles is an open problem. In (35), we can formally take the limit $m \to \infty$ and find a non-trivial scaling,

$$\lim_{t\to T}(T-t)b(\eta(T-t)^{1/d}, t) = \tilde{\varphi}_\infty(\eta). \tag{36}$$

Note that this limit cannot be taken for the semilinear equation where (33) holds. The convergence (36) represents the convection-dominant behaviour of (15)–(18), which in terms of the density $n = n(r, t)$ describes an imploding wave moving towards the origin, as shown in figure 1 (left). The function $\tilde{\varphi}_\infty$ is discontinuous (cf [19, (3.16)]),

$$\tilde{\varphi}_\infty(\eta) = \begin{cases} \dfrac{2C^d}{\eta^d} & \text{for } \eta > C, \\ 0 & \text{for } \eta < C, \end{cases}$$

where $2C^d$ is the mass accumulated at the origin, which can be chosen arbitrarily. In [19] this type of blow-up was studied using matched asymptotic expansions. There, it was suggested that this behaviour is stable and, moreover, it was expected that there exist initial data such that (36) holds uniformly in η on compact subsets away from the shock. A result of this type was proved in [12, theorem 3] for a related equation.

4. Preliminaries

4.1. Estimates

In this section, we develop some estimates for problem (15)–(17), which in turn will imply bounds for the self-similar problem (27)–(29).

Lemma 4.1. *If b_0 satisfies (18) then*

$$\frac{r}{d}b_r + b \geqslant 0 \qquad \text{in } D_T. \tag{37}$$

Proof. The solution n of problem (1)–(6) satisfies the relation

$$n = \frac{1}{\chi_d}\left[\frac{r}{d}b_r + b\right] \qquad \text{in } D_{T^*}. \tag{38}$$

Since $n_0 \geqslant 0$ in D, an application of the maximum principle to problem (1)–(6) shows that $n \geqslant 0$ in D_{T^*}. Using this and (38) the result follows. ■

To prove the following results, we proceed as in [13] where similar estimates were found for the semilinear parabolic equation (31).

Lemma 4.2. *If b_0 satisfies (21) then*

$$b_r(r,t) < 0 \qquad \text{in } D_T. \tag{39}$$

Proof. Set $w(r,t) := r^{d+1}b_r(r,t)$. Differentiating (15), we find

$$w_t - \chi_d\Theta\left(w_{rr} - \frac{d+1}{r}w_r\right) - \frac{1}{d}rbw_r = \left(b + \frac{1}{d}rb_r\right)w. \tag{40}$$

Assume, for the moment, a stronger assumption on the initial data

$$(b_0)_r(r) < 0 \qquad \text{for } r \in (0,1) \quad \text{and} \quad (b_0)_{rr}(0) < 0. \tag{41}$$

This gives $w(0,r) = r^{d+1}b_r(0,r) < 0$. Under (41) the function $b \equiv 1$ is a sub-solution for (15)–(18), but not a solution; by Hopf's lemma, $w(1,t) = b_r(t,1) < 0$ for all $t > 0$, so that $w < 0$ on D_T, hence $b_r < 0$ on D_T. To complete the proof, we note that by the strong maximum principle, if b_0 satisfies (21), then, for each $t_1 \in (0,T)$ condition (41) holds for the function $b(r,t_1)$. This proves the result. ■

Lemma 4.3. *If b_0 satisfies (21) and assuming that blow-up occurs at time $T > 0$, then*

$$b(0,t) \geqslant (T-t)^{-1} \qquad \text{for } t \in [0,T). \tag{42}$$

Proof. Since the maximum of b in D is attained at $r = 0$ (by lemma 4.2), we have $b_{rr}(0,t) \leqslant 0$. It follows from (15) that $b_t(0,t) \leqslant b^2(0,t)$. Integrating this inequality on (t,T) gives the result. ■

Lemma 4.4. *If b_0 satisfies (22) then $b_t \geqslant 0$ for all $t \in (0,T)$.*

Proof. Condition (22) implies that b_0 is a subsolution for (15)–(17); therefore, $b(r,\epsilon) \geqslant b(r,0)$ for small $\epsilon \geqslant 0$. By the comparison principle we find $b(r,t+\epsilon) \geqslant b(r,t)$ for $t \in (0,T-\epsilon)$. It follows that $b_t \geqslant 0$ on D_T. ■

The next lemma gives a bound on $|b_r|$ in D_T.

Lemma 4.5. *Let b_0 satisfy (21) and (22). Then,*

$$\chi_d \Theta b_r^2(r,t) \leqslant \tfrac{2}{3} b(0,t)^3 \qquad \text{for } (r,t) \in D_T. \tag{43}$$

Proof. Since $b_t \geqslant 0$ and $b_r \leqslant 0$ in D_T, we multiply equation (15) by b_r and obtain

$$0 \geqslant \chi_d \Theta \int_0^r b_r b_{rr}\,\mathrm{d}s + \frac{1}{3} b^3(r,t) - \frac{1}{3} b^3(0,t)$$
$$= \tfrac{1}{2}\chi_d \Theta [b_r^2(r,t) - b_r^2(0,t)] + \tfrac{1}{3} b^3(r,t) - \tfrac{1}{3} b^3(0,t).$$

Since $b_r^2(0,t) = 0$ we obtain the desired inequality. ■

To conclude this section, we translate the properties of solutions derived above into estimates for problem (27)–(29). From hypothesis (26) and noting that $b \geqslant 1$ and $b_r \leqslant 0$ in D_T, we have the *a priori* bound

$$0 \leqslant B(\eta,\tau) \leqslant M \qquad \text{for } (\eta,\tau) \in \Pi. \tag{44}$$

Combining this with (43) and (39), we obtain

$$0 \leqslant -B_\eta(\eta,\tau) \leqslant \bar{M} \qquad \text{for } (\eta,\tau) \in \Pi, \tag{45}$$

where $\bar{M}$ depends on M. Finally, from (42), we get

$$1 \leqslant B(0,\tau) \qquad \text{for } \tau \in (0, \ell(\tau)). \tag{46}$$

4.2. The steady-state equation (19)

We begin by recalling problem (19):

$$\varphi_{\eta\eta} + \frac{d+1}{\eta}\varphi_\eta + \frac{1}{d}\eta\varphi\varphi_\eta - \frac{1}{2}\eta\varphi_\eta + \varphi^2 - \varphi = 0 \qquad \text{for } \eta > 0, \tag{47}$$

$$\varphi(0) \geqslant 1, \qquad \varphi_\eta(0) = 0. \tag{48}$$

Condition (48) is required, since $B(0,\tau) \geqslant 1$ for all $\tau \geqslant 0$. Equation (47) has three special solutions:

$$\varphi_S(\eta) = \frac{2d}{\eta^2}, \quad \varphi^*(\eta) = 1, \quad \text{and} \quad \varphi_*(\eta) = 0 \qquad \text{for } \eta > 0.$$

Note that φ_S satisfies

$$\varphi_S + \frac{1}{2}\eta(\varphi_S)_\eta = 0 \qquad \text{and} \qquad 0 = (\varphi_S)_{\eta\eta} + \frac{d+1}{\eta}(\varphi_S)_\eta + \frac{1}{d}\eta\varphi_S(\varphi_S)_\eta + (\varphi_S)^2. \tag{49}$$

For bounded non-constant solutions we have the following theorem [10, 20].

Theorem 4.6. *Let $2 < d < 10$. There exists a countable set of solutions $\{\varphi_k\}_{k\in\mathbb{N}}$ of (47)–(48) such that $\varphi_k(0) > 1$ and $\varphi_k(0) \to \infty$ as $k \to \infty$, Moreover, φ_k intersects the singular solution φ_S k times and has the asymptotic behaviour $\varphi_k(\eta)\eta^2 = \mathrm{Const}(k) > 0$.*

The proof is based on the equation for $G(\eta) := \eta^2\varphi(\eta)$,

$$G_{\eta\eta} + \left(\frac{(d-3)}{\eta} + \frac{1}{d}\frac{G}{\eta} - \frac{1}{2}\eta\right)G_\eta + \frac{2(d-2)G}{\eta^2}\left(\frac{G}{2d} - 1\right) = 0, \tag{50}$$

$$\lim_{\eta\downarrow 0} \frac{G(\eta)}{\eta^2} < \infty, \qquad \lim_{\eta\to\infty} \eta G_\eta(\eta) = 0. \tag{51}$$

Note that φ_S corresponds to $G(\eta) \equiv 2d$.

It was formally argued in [10] that for each integer $k \geqslant 2$ and $2 < d < 10$ the set

$$\mathcal{S}_k = \{\varphi\colon \varphi \text{ solution of (47)–(48) with } k \text{ intersections with } \varphi_S\}$$

is a singleton and that for $d > 2$ the set $\mathcal{S}_1$ contains only two elements. More precisely, $\mathcal{S}_1$ consists of the functions φ^* and φ_1 given in (20). If we relax condition (48) to $\varphi(0) > 0$, we conjecture that there is at least one other solution in $\mathcal{S}_1$. For $d = 3$, this was shown numerically by Brenner *et al*, who found a solution φ_1^* of (47) such that $\varphi_1^*(0) < 1$ and $(\varphi_1^*)_\eta(0) = 0$, which intersects φ_S once [10, figure 14].

5. Convergence

In this section, we prove the following convergence theorem.

Theorem 5.1. *Let conditions (21) and (22) hold. Let $B(\eta, \tau)$ be a uniformly bounded global solution of (27)–(29). Then, for every sequence $\tau_n \to \infty$ there exists a subsequence τ_n' such that $B(\eta, \tau_n')$ converges to a solution φ of (47)–(48). The convergence is uniform on every compact subset of* $[0, \infty)$.

Proof. Define $B^\sigma(\eta, \tau) := B(\eta, \sigma + \tau)$. We will first show that for any unbounded sequence $\{n_j\}$ there exists a subsequence (renamed $\{n_j\}$) such that B^{n_j} converges to a solution φ of (47)–(48) uniformly in compact subsets of $\mathbb{R}^+ \times \mathbb{R}$. Without loss of generality we assume that the sequence $\{n_j\}$ is increasing.

Let $N \in \mathbb{N}$. We take i large enough such that the rectangle $\mathcal{Q}_{2N} = \{(\eta, \tau) \in \mathbb{R}^2\colon 0 \leqslant \eta \leqslant 2N,\ |\tau| \leqslant 2N\}$ lies in the domain of B^{n_i}. The function $\tilde{B}(\xi, \tau) = B^{n_i}(|\xi|, \tau)$ is a solution of

$$\tilde{B}_\tau = \Delta_{d+2}\tilde{B} - \frac{1}{2}\xi \cdot \nabla\tilde{B} + \frac{1}{d}(\xi \cdot \nabla\tilde{B})\tilde{B} + \tilde{B}^2 - \tilde{B}$$

on the cylinder given by

$$\Gamma_{2N} = \{(\xi, \tau)\colon \mathbb{R}^{d+2} \times \mathbb{R}\colon |\xi| \leqslant 2N,\ |\tau| \leqslant 2N\}$$

and $|\tilde{B}(\xi, \tau)|$ is uniformly bounded in Γ_{2N} by (44).

By Schauder's interior estimates all partial derivatives of $\tilde{B}$ can be uniformly bounded on the subcylinder $\Gamma_N \subset \Gamma_{2N}$. Consequently, B^{n_i}, $B_\tau^{n_i}$, $B_\eta^{n_i}$ and $B_{\eta\eta}^{n_i}$ are uniformly Lipschitz on $\mathcal{Q}_N \subset \mathcal{Q}_{2N}$. By Arzela–Ascoli, there is a subsequence $\{n_j\}_1^\infty$ and a function $\bar{B}$ such that B^{n_i}, $B_\tau^{n_i}$, $B_\eta^{n_i}$, and $B_{\eta\eta}^{n_i}$ converge to $\bar{B}$, $\bar{B}_\tau$, $\bar{B}_\eta$ and $\bar{B}_{\eta\eta}$, uniformly on $\mathcal{Q}_N$.

Repeating the construction for all N and taking a diagonal subsequence, we can conclude that

$$B^{n_j} \to \bar{B}, \qquad B_\tau^{n_j} \to \bar{B}_\tau, \qquad B_\eta^{n_j} \to \bar{B}_\eta, \qquad \text{and} \qquad B_{\eta\eta}^{n_j} \to \bar{B}_{\eta\eta}, \tag{52}$$

uniformly in every compact subset in $\mathbb{R}^+ \times \mathbb{R}$. Clearly $\bar{B}$ satisfies (27) and estimates (44) and (45). Finally, it remains to prove that $\bar{B}$ is independent of τ. This implies that $\bar{B}$ is a solution of (19), since $B(0, \tau) \geqslant 1$ for all $\tau > 0$, and the result follows.

Claim. The function $\bar{B}$ is independent of τ.

To prove this, we construct a *non-explicit* Lyapunov functional in the spirit of Galaktionov [14] and Zelenyak [30].

1. *Non-explicit Lyapunov functional.* We seek a Lyapunov function of the form

$$E(\tau) = \int_0^{\ell(\tau)} \Phi(\eta, B(\eta, \tau), B_\eta(\eta, \tau))\, \mathrm{d}\eta,$$

where $\ell(\tau) = (\chi_d \Theta T)^{-1/2} e^{\tau/2}$ and $\Phi = \Phi(\eta, v, w)$ is a function to be determined. In appendix A we show that such a Lyapunov function exists; more precisely, we show that a function $\rho = \rho(\eta, v, w)$ exists such that

$$\frac{d}{d\tau} E(\tau) = -\int_0^{\ell(\tau)} \rho(\eta, B(\eta, \tau), B_\eta(\eta, \tau))(B_\tau)^2(\eta, \tau)\, d\eta + \Phi_w B_\tau|_0^{\ell(\tau)} + \frac{1}{2}\ell(\tau)\Phi(\ell(\tau), B(\ell(\tau), \tau), B_\eta(\ell(\tau), \tau)). \tag{53}$$

To identify the relevant domain of the functions Φ and ρ, we note that by estimates (44) and (45) the solution B satisfies $(\eta, B(\eta, \tau), B_\eta(\eta, \tau)) \in \tilde{\mathcal{R}}$, with

$$\tilde{\mathcal{R}} = \mathcal{R} \cap \{0 \leqslant v \leqslant M,\ 0 \leqslant -w \leqslant \bar{M}\}, \tag{54}$$

where $\mathcal{R} = \{\eta > 0, v \geqslant 0, w \leqslant 0\} \cup \{\eta = 0, v \geqslant 0, w = 0\}$.

The functions ρ and Φ are continuous in $\mathcal{R} \setminus \{\eta = \bar{\eta},\ v > 1\}$ with $\bar{\eta} > 0$ defined later and they satisfy

$$\frac{1}{C_0}\eta^{d+1} e^{-C_0\eta^2} \leqslant \rho(\eta, v, w) \leqslant \eta^{d+1} e^{-(d-2)\eta^2/4d} \qquad \text{for } (\eta, v, w) \in \tilde{\mathcal{R}}, \tag{55}$$

with $C_0 = C_0(M) > 0$ (lemma A.5), and

$$|\Phi(\eta, v, w)| \leqslant C_1 \eta^{d+1} e^{-(d-2)\eta^2/4d} \qquad \text{for } (\eta, v, w) \in \tilde{\mathcal{R}} \tag{56}$$

for some positive constants $C_1(M) > 0$ (lemma A.6).

2. *Proof of the claim.* An integration over the interval (a, b) of (53) gives

$$\int_a^b \int_0^{\ell(\tau)} \rho(\eta, B(\eta, \tau), B_\eta(\eta, \tau)) B_\tau^2(\eta, \tau)\, d\eta\, d\tau = E(a) - E(b) + \psi(a, b), \tag{57}$$

where

$$\psi(a, b) := \int_a^b \frac{1}{2}\ell(\tau)\Phi(\ell(\tau), B(\ell(\tau), \tau), B_\eta(\ell(\tau), \tau))\, d\tau + \int_a^b B_\tau(\ell(\tau), \tau)\left[\int_0^{B_\eta(\ell(\tau), \tau)} \rho(\ell(\tau), B(\ell(\tau), \tau), s)\, ds\right] d\tau. \tag{58}$$

Since $B_\tau(\ell(\tau), \tau) = -B(\ell(\tau), \tau) - \frac{1}{2}\ell(\tau) B_\eta(\ell(\tau), \tau)$,

$$B_\tau(\ell(\tau), \tau) = -T e^{-\tau} - \tfrac{1}{2} b_r(1, T(1 - e^\tau)).$$

Applying (37) at $r = 1$ gives $|b_r(1, T(1 - e^\tau))| \leqslant d$ and, consequently, B_τ is uniformly bounded as $\tau \to \infty$. Employing this bound on B_τ and the estimates (55) and (56) we find

$$\lim_{a \to \infty} \{\sup_{b > a} \psi(a, b)\} = 0. \tag{59}$$

By (52), we have that there exists a sequence $n_j \to \infty$ such that $B^{n_j}(\eta, \tau)$ converges to $\bar{B}$ uniformly in compact subsets of $(\mathbb{R}^+)^2$. For any fixed N we will prove for a subsequence satisfying $\lim_{j \to \infty}(n_{j+1} - n_j) = \infty$ that

$$\lim_{n_j \to \infty} \int_{\mathcal{Q}_N} \rho(\eta, B^{n_j}(\eta, \tau), B_\eta^{n_j}(\eta, \tau))(B_\tau^{n_j})^2(\eta, \tau)\, d\eta\, d\tau = 0, \tag{60}$$

where we recall that $\mathcal{Q}_N = \{(\eta, \tau) : \mathbb{R}^2 : 0 \leqslant \eta \leqslant N,\ |\tau| \leqslant N\}$. Since ρ is bounded from below on bounded subsets of $\tilde{\mathcal{R}}$, it then follows that

$$\int_{\mathcal{Q}_N} \bar{B}_\tau^2\, d\eta\, d\tau = \lim_{n_j \to \infty} \int_{\mathcal{Q}_N} (B_\tau^{n_j})^2(\eta, \tau)\, d\eta\, d\tau = 0,$$

proving the claim. For all j sufficiently large,

$$N \leqslant (\chi_d \Theta T)^{-1/2} e^{1/2(n_j - N)} \qquad \text{and} \qquad n_{j+1} - n_j \geqslant 2N.$$

Consequently, using (57), we find

$$\int_{-N}^{N}\int_{0}^{N} \rho(\eta, B^{n_j}(\eta,\tau), B^{n_j}_\eta(\eta,\tau))(B^{n_j}_\tau)^2(\eta,\tau)\,\mathrm{d}\eta\,\mathrm{d}\tau$$

$$\leqslant \int_{-N}^{-N+n_{j+1}-n_j}\int_{0}^{(\chi\Theta^*T)^{-1/2}\mathrm{e}^{(n_j-N)/2}} \rho(\eta, B^{n_j}(\eta,\tau), B^{n_j}_\eta(\eta,\tau))(B^{n_j}_\tau)^2(\eta,\tau)\,\mathrm{d}\eta\,\mathrm{d}\tau$$

$$\leqslant \int_{n_j-N}^{n_{j+1}-N}\int_{0}^{(\chi\Theta^*T)^{-1/2}\mathrm{e}^{(n_j-N)/2}} \rho(\eta, B(\eta,\tau), B_\eta(\eta,\tau))(B_\tau)^2(\eta,\tau)\,\mathrm{d}\eta\,\mathrm{d}\tau$$

$$\leqslant E(n_j-N) - E(n_{j+1}-N) + \psi(n_j-N, n_{j+1}-N).$$

Hence, applying (59), we find

$$\int_{Q_N} \rho(\eta, B^{n_j}(\eta,\tau), B^{n_j}_\eta(\eta,\tau))(B^{n_j}_\tau)^2(\eta,\tau)\,\mathrm{d}\eta\,\mathrm{d}\tau \leqslant \limsup_{j\to\infty}[E(n_j-N)-E(n_{j+1}-N)].$$

Next, we divide the expression $E(n_j-N)-E(n_{j+1}-N)$ into three integrals, choosing K arbitrarily large:

$$E(n_j-N)-E(n_{j+1}-N)$$

$$= \int_0^K [\Phi(\eta, B^{n_j}(\eta,-N), B^{n_j}_\eta(\eta,-N)) - \Phi(\eta, B^{n_j}(\eta,-N), B^{n_j}_\eta(\eta,-N)]\,\mathrm{d}\eta \tag{61}$$

$$+ \int_K^{T^{-1/2}\mathrm{e}^{(n_j-N)/2}} \Phi(\eta, B^{n_{j+1}}(\eta,-N), B^{n_{j+1}}_\eta(\eta,-N))\,\mathrm{d}\eta \tag{62}$$

$$+ \int_K^{T^{-1/2}\mathrm{e}^{(n_{j+1}-N)/2}} \Phi(\eta, B^{n_j}(\eta,-N), B^{n_j}_\eta(\eta,-N))\,\mathrm{d}\eta. \tag{63}$$

Integral (61) tends to zero as $j\to\infty$. In fact, by the continuity of Φ in the second and third arguments we obtain pointwise convergence and by the bounds (56) on Φ, we apply the dominated convergence theorem to conclude. Expressions (62) and (63) can be made arbitrarily small since they can be bounded by

$$C\int_K^\infty \eta^{d+1}\mathrm{e}^{-(d-2)\eta^2/4d}\,\mathrm{d}\eta,$$

where C is a positive constant, and K can be chosen arbitrarily large. Thus, we have proved (60), concluding the proof of the theorem. ■

6. Comparison results

6.1. Comparison with the singular solution φ_S

This section closely follows [1]. From section 4.2, we recall that solutions φ of (47)–(48) are classified by their intersections with φ_S. In this section, we study the intersections of solutions B of (27)–(29) with φ_S. Our results are closely related to the ones found in [1], where equation (31) was studied.

We first see that for $\Theta < 1/(2d\chi_d)$ a solution B of (27)–(29) intersects the singular solution φ_S at least once in $\Pi(0)$ since

$$\varphi_S(0) = \infty > B(0,0) \qquad \text{and} \qquad \varphi_S((\chi_d\Theta T)^{-1/2}) < B((\chi_d\Theta T)^{-1/2}, 0) = T.$$

On the other hand, for $\Theta \geqslant 1/(2d\chi_d)$ it can also be shown that B intersects φ_S at least once in $\Pi(0)$. Assuming the contrary, suppose that $B(\cdot, 0) < \varphi_S(\cdot)$ in $\Pi(0)$. By the maximum principle, we obtain $B < \varphi_S$ in Π. Therefore, in the limit $\tau \to \infty$, thanks to theorem 5.1 and since $B(0, \tau) \geqslant 1$ for all $\tau > 0$, we find a solution φ of (19) such that $\varphi < \varphi_S$. However, we can show that every bounded non-zero solution φ of (19) has to cross φ_S. This is equivalent to proving that there exists no solution G of (50)–(51) such that $G(\eta) < 2d$ for $\eta \geqslant 0$. To check this, we assume that such a solution exists; we examine two cases. Suppose that for some η^*, we have $G_\eta(\eta^*) = 0$ and $G(\eta^*) < 2d$. By (50), G has a strict minimum at η^*, which contradicts the boundary condition (51). On the other hand, if $G(\eta)$ is increasing for all $\eta > 0$, then, for large η, equation (50) implies that $G_{\eta\eta} > 0$, which also contradicts (51).

We conclude that there exists $\eta_1 \in \Pi(0)$ such that $B(\eta_1, 0) = \varphi_S(\eta_1)$ and $B(\eta, 0) < \varphi_S(\eta)$ for $\eta < \eta_1$.

Lemma 6.1. *Under the assumptions (21) and (22), there exists a continuously differentiable function $\eta_1(\tau)$ with domain $[0, \infty)$ such that $\eta_1(0) = \eta_1$ and $B(\eta_1(\tau), \tau) = \varphi_S(\eta_1(\tau))$ for all $\tau \geqslant 0$.*

Proof. Define $H(\eta, \tau) := B(\eta, \tau) - \varphi_S(\eta)$. We first claim that H, H_η, and H_τ do not vanish simultaneously. Using lemma 4.4 and the strong maximum principle we find

$$b_t = (T - t)^{-2}\big(B_\tau + B + \tfrac{1}{2}\eta B_\eta\big) > 0 \qquad \text{in } D_T. \tag{64}$$

Suppose there exists a point in Π where $H_\eta = H_\tau = H = 0$. Then, $H_\tau = 0$ implies $B_\tau = 0$, and condition $H_\eta = 0$ combined with $H = 0$ gives

$$B + \tfrac{1}{2}\eta B_\eta = 0 \qquad \text{in } \Pi,$$

using (49). This implies that $b_t = 0$ at some point of D_T, which contradicts (64). Secondly, we claim that $H_\eta \neq 0$ at any point $(\bar\eta, \bar\tau) \in \Pi$ where $H(\bar\eta, \bar\tau) = 0$ and, moreover, $H(\eta, \bar\tau) < 0$ in a left neighbourhood of $\bar\eta$. A proof of this can be given as in [1]. Moreover, from the proof, we find that $H_\eta(\bar\eta, \bar\tau) > 0$.

Now, we prove that $H_\eta(\eta_1, 0) > 0$. This follows from the equation satisfied by $H(\eta, 0)$. To the left of η_1, we find

$$H_{\eta\eta}(\eta, 0) + \frac{d+1}{\eta} H_\eta(\eta, 0) + \frac{1}{2d}\eta H_\eta(\eta, 0)(B(\eta, 0) + \varphi_S) + \frac{1}{2d}\eta H(\eta, 0)(B(\eta, 0) + \varphi_S)_\eta \geqslant 0. \tag{65}$$

Since $(B(\eta, 0) + \varphi_S)_\eta \leqslant 0$ and $H(\eta_1, 0) = 0$, we can apply Hopf's lemma to obtain that $H_\eta(\eta_1, 0) > 0$. Finally, to conclude the proof of the lemma, we use the implicit function theorem as in [1]. ■

Define the set $\Pi_1 = \{(\eta, \tau) \mid 0 < \eta < \eta_1(\tau)\}$ and the function

$$\eta_2(\tau) = \mathrm{e}^{\tau/2} \cdot \sup\{\eta \in \eta_1, (\chi_d \Theta T)^{-1/2}]\colon H(s, 0) \geqslant 0 \text{ for } s \in [\eta_1, \eta]\}.$$

Since $H(\eta_1, 0) = 0$ and $H_\eta(\eta_1, 0) > 0$, the above supremum is finite. Define the set

$$\Pi_2 = \{(\eta, \tau) \mid \eta_1(\tau) < \eta < \eta_2(\tau)\}.$$

Let $F(\tau) = H(\eta_2(\tau), \tau)$. By definition of η_2, $F(0) \geqslant 0$. Also,

$$\frac{\mathrm{d}}{\mathrm{d}\tau} F(\tau) = H_\tau(\eta_2(\tau), \tau) + \frac{1}{2}\eta_2(\tau) H_\eta(\eta_2(\tau), \tau).$$

Using (64), we have $\mathrm{d}[\mathrm{e}^\tau F(\tau)]/\mathrm{d}\tau \geqslant 0$. An integration yields $F(\tau) \geqslant 0$ for $\tau \geqslant 0$.

As was done in [1], applying the maximum principle, using lemma 6.1, and noting that $H(\eta_2(\tau), \tau) \geqslant 0$ for $\tau \geqslant 0$, we can prove the following lemma and its corollary.

Lemma 6.2. *The function $H(\eta,\tau) = B(\eta,\tau) - \varphi_S(\eta)$ satisfies $H < 0$ in Π_1 and $H > 0$ in Π_2.*

Corollary 6.3. *Assume the conditions in lemma 6.1. For each $N > 0$ there is $\tau_N > 0$ such that for $\tau > \tau_N$, $B(\eta,\tau)$ intersect $\varphi_S(\eta)$ at most once in $\eta \in (0, N)$.*

6.2. Intersection comparison

In this section, we derive comparison results, which will be used to prove that ω, the limit set (30), is a singleton.

We start by considering the following linear equation with inhomogeneous boundary conditions:

$$\begin{aligned} &v_t = v_{rr} + \frac{d+1}{r} v_r + a(r,t)v \qquad \text{for } 0 < r < 1, T_1 < t < T_2, \\ &v_r(0,t) = 0 \qquad \text{for } T_1 < t < T_2, \\ &v(1,t) = h(t) \qquad \text{for } T_1 < t < T_2, \end{aligned} \tag{66}$$

where T_1, T_2 are positive constants and

$$a \in L^\infty([0,1] \times (T_1,T_2)), \qquad h \in C^1((T_1,T_2)), \tag{67}$$

are given functions. Moreover we assume

$$h(t) > 0 \qquad \text{for } T_1 < t < T_2. \tag{68}$$

The zero number functional of (66) is defined by

$$z[v(\cdot,t)] = \#\{r \in [0,1]\colon v(r,t) = 0\} \tag{69}$$

and the following lemma provides some properties of this zero number functional.

Lemma 6.4 ([24]). *Let $v = v(r,t)$ be a nontrivial classical solution of (66) and assume that (67) and (68) hold. Then, the following properties hold true:*

(i) $z[v(\cdot,t)] < \infty$ *for any* $T_1 < t < T_2$;
(ii) $z[v(\cdot,t)]$ *is nonincreasing in time;*
(iii) if $v(r_0,t_0) = v_r(r_0,t_0) = 0$ *for some* $r_0 \in [0,1]$ *and* $t_0 > T_1$, *then* $z[v(\cdot,t)]$ *drops strictly at* $t = t_0$, *that is,* $z[v(\cdot,t_1)] > z[v(\cdot,t_2)]$ *for any* $T_1 < t_1 < t_0 < t_2 < T_2$.

From this lemma we deduce a property of intersection between a solution φ of (19) and a solution B of (27)–(29).

Lemma 6.5. *Let B be a bounded solution of (27)–(29) and let φ be a solution of (47). Denote $Z(\tau) = \#\{r \in [0,\ell(\tau)]\colon B(\eta,\tau) = \varphi(\eta)\}$. Then, the following properties hold true:*

(i) $Z(\tau) < \infty$ *for any* $\tau > \tau^*$;
(ii) $Z(\tau)$ *is nonincreasing in time;*
(iii) if $B(\eta_0,\tau_0) = \varphi(\eta_0)$ *and* $B_\eta(\eta_0,\tau_0) = \varphi_\eta(\eta_0)$ *for* $\tau_0 > \tau_1$, *and* $\eta_0 \leqslant \ell[\tau]$ *then* $Z(\tau_1) > Z(\tau_2)$ *for any* $\tau_1 < \tau_0 < \tau_2$.

Proof. Writing $\bar V = U - b$, where $U(r,t) = (T-t)^{-1}\varphi(r/(\chi_d\Theta(T-t))^{1/2})$, we have

$$\begin{aligned} &\bar V_t = \bar V_{rr} + \left(\frac{d+1}{r} + \frac{r}{d}U\right)\bar V_r + \left(\frac{r}{d}b_r + b + U\right)\bar V \qquad \text{for } 0 < r < 1, 0 < t < T, \\ &\bar V_r(0,t) = 0, \quad \bar V(1,t) = U(1,t) - b(1,t) \qquad \text{for } 0 < t < T. \end{aligned} \tag{70}$$

Let $T_1 < T_2 < T$. For the variable $V(r,t) = \exp((1/2d)\int_0^r yU(y,t)\,\mathrm{d}y)\bar{V}(r,t)$, we find

$$V_t = V_{rr} + \frac{d+1}{r}V_r + A(r,t)V \qquad \text{for } 0 < r < 1, T_1 < t < T_2,$$
$$V_r(0,t) = 0 \qquad \text{for } T_1 < t < T_2,$$
$$V(1,t) = (U(1,t)-1)\exp\left(\frac{1}{2d}\int_0^1 yU(y,t)\,\mathrm{d}y\right) \qquad \text{for } T_1 < t < T_2,$$

where

$$A(r,t) = \frac{r}{d}b_r + b + U + \frac{1}{2d}\int_0^r yU_t(y,t)\,\mathrm{d}y - \frac{1}{4d^2}r^2U^2 - \frac{1}{2d}(U + rU_r) - \frac{d+1}{2d}U.$$

Note that $A \in L^\infty([0,1]\times(T_1,T_2))$ since $b, b_r, U, U_t, U_r \in L^\infty([0,1]\times(T_1,T_2))$. If we show that $V(1,t)$ does not change sign for $t > t_0$, then, setting $T_1 = t_0$ and using lemma 6.4, we have proved the lemma.

We claim that there exists $\bar{t}_0$ such that $U_t(1,t)$ does not change sign for $t > \bar{t}_0$. By definition of V, this implies that there exists $t_0 \geqslant \bar{t}_0$ such that $V(1,t)$ does not change sign for $t > t_0$.

Since $U_t(r,t) = (T-t)^{-2}(\eta^2\varphi)_\eta/(2\eta)$, if $r = 1$ and $t > t^*$, then,

$$U_t(1,t) = (T-t)^{-2}\frac{1}{2\eta}(\eta^2\varphi)_\eta \qquad \text{for } t > t^* \quad \text{and} \quad \eta > \eta^*(t^*), \tag{71}$$

where $\eta^*(t^*) := (\chi_d\Theta(T-t^*))^{-1/2}$. From [9, lemma A.1], we know that for a given $a \in (0,4d)$, any solution φ of (47) satisfying

$$\eta^2\varphi(\eta) \to a \qquad \text{as } \eta \to \infty \tag{72}$$

is such that there exists $\bar{\eta}_0 = \bar{\eta}_0(a)$ so that the sign of $(\eta^2\varphi)_\eta$ does not change on $[\bar{\eta}_0, \infty)$. Using (71), this implies that there exists $\bar{t}_0 = \bar{t}_0(\bar{\eta}_0)$ such that the claim holds. ■

7. Proofs of main results

We start by proving that the ω-limit set of problem (27)–(29) is a singleton.

Theorem 7.1. *Assume the hypotheses of theorem 2.3. Then, the set ω defined in (30) is a singleton.*

Proof. For this proof we extend a solution B of (27)–(29) to all $(\mathbb{R}^+)^2$ by setting $B(\eta,\tau) = \mathrm{e}^{-\tau}T$ for $(\eta,\tau) \in (\mathbb{R}^+)^2\backslash\Pi$. We also define the weight function $\rho^*(\eta) = \mathrm{e}^{-\eta^2/4}$ for $\eta > 0$.

The hypothesis (26) implies that B is uniformly bounded; theorem 5.1, therefore, states that ω is non-empty, and that each $\varphi \in \omega$ is a solution of (47)–(48).

We claim that for each $\varphi \in \omega$ there exists $\tau^* > 0$ such that $B(0,\tau) - \varphi(0)$ never changes sign in $[\tau^*,\infty)$. By contradiction, we assume that there exists a sequence τ_k, such that $\tau_k \to \infty$, and $B(0,\tau_k) = \varphi(0)$. Since $B_\eta(0,\tau_k) = \varphi_\eta(0) = 0$, by lemma 6.5 the function $Z(\tau)$ has to decrease at least by one. However, this cannot happen an infinite number of times. This proves the claim.

Suppose, now, that ω is not a singleton. Since the ω-limit set is connected, closed, and non-empty, it contains an infinite number of elements. We select three different elements $\varphi_1, \varphi_2, \varphi_3$ in the ω-limit set. Since these functions are different and each solves (19), we may assume that $\varphi_1(0) < \varphi_2(0) < \varphi_3(0)$. By the claim above, $B(0,\tau) - \varphi_2(0)$ never changes sign in $[\tau^*, \infty)$. This contradicts the fact that φ_1 and φ_3 are elements of ω; it follows that ω is a singleton. ■

We now conclude the proof of theorems 2.3 and 2.1, and corollary 2.2.

Proof of theorem 2.3. By the previous theorem, ω is a singleton, say $\{\bar{B}\}$. From corollary 6.3, we find that for every $N > 0$ there exists a $\tau_N > 0$ such that the solution $B(\eta, \tau)$ intersects $\varphi_S(\eta)$ at most once in $\eta \in [0, N]$ for each $\tau > \tau_N$. This implies that in the limit $\tau \to \infty$, $\bar{B}$ intersects φ_S at most once, concluding the proof. ■

Proof of theorem 2.1. Since b and $U_1(r,t) = (T-t)^{-1}\varphi_1(r/(\chi_d\Theta(T-t))^{1/2})$ are solutions of (15) with the same blow-up time, $\bar{V} = b - U_1$ satisfies equation (70). Using the fact that $U_1(r,t) = 2d/((d-2)(T-t) + r^2/(2\chi_d\Theta))$, we find

$$\bar{V}(1,t) = (1 - U_1(1,t)) > 0 \qquad \text{if } \Theta \leqslant \frac{1}{4d\chi_d}, \text{ for any } t < T.$$

The functions U_1 with b necessarily intersect exactly once for all t, since non-intersection implies that the solutions must have different times of blow-up [26, p 271]. It follows that $b(0,0) < U_1(0)$, and one finds $(T-t)b(0,t) \leqslant 2d/(d-2)$. An application of theorem 2.3 proves the theorem. ■

Proof of corollary 2.2. If $b_0 \equiv 1$ and $\Theta < 1/(2(d+2)\chi_d)$, we know from [7, theorem 2] that the corresponding solution b blows up. Now, assuming $\Theta \leqslant 1/(4d\chi_d) < 1/(2(d+2)\chi_d)$, we can apply theorem 2.1 to conclude. ■

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Appendix A. The Lyapunov functional

In this appendix, we construct the Lyapunov functional E satisfying (53), with suitable properties of ρ and Φ, to prove theorem 5.1. We start with a formal construction of the functional. This requires solving a first-order equation for ρ after which Φ can be expressed in terms of ρ. Finally, we explain how to use smooth approximations of Φ to obtain a rigorous derivation of (53).

Appendix A.1. Formal derivation of a Lyapunov functional

Assume that Φ and ρ are regular. To find such functions satisfying (53), we compute

$$\frac{\mathrm{d}}{\mathrm{d}\tau}E(\tau) = \int_0^{\ell(\tau)} \Phi_v B_\tau \,\mathrm{d}\eta + \int_0^{\ell(\tau)} \Phi_w B_{\tau\eta}\,\mathrm{d}\eta + \frac{\ell(\tau)}{2}\Phi(\ell(\tau), B(\ell(\tau),\tau), B_\eta(\ell(\tau),\tau)). \tag{A.1}$$

Wherever possible we omit the arguments of Φ and ρ, for clarity. Integrating by parts the second integral in (A.1) becomes

$$\int_0^{\ell(\tau)} \Phi_w B_{\tau\eta}\,\mathrm{d}\eta = -\int_0^{\ell(\tau)} [\Phi_{\eta w} + \Phi_{vw}B_\eta + \Phi_{ww}B_{\eta\eta}]B_\tau\,\mathrm{d}\eta + \Phi_w B_\tau|_0^{\ell(\tau)}.$$

Defining

$$f(\eta, v, w) = \frac{d+1}{\eta} w - \frac{\eta}{2} w + \frac{1}{d} \eta v w + v^2 - v,$$

equation (27) takes the form $B_\tau = B_{\eta\eta} + f(\eta, B, B_\eta)$, by which equation (A.1) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\tau} E(\tau) = \int_0^{\ell(\tau)} \{[\Phi_v - \Phi_{\eta w} - \Phi_{vw} B_\eta + \Phi_{ww} f] B_\tau - \Phi_{ww} (B_\tau)^2\} \,\mathrm{d}\eta$$
$$+ \Phi_w B_\tau|_0^{\ell(\tau)} + \frac{\ell(\tau)}{2} \Phi(\ell(\tau), B(\ell(\tau), \tau), B_\eta(\ell(\tau), \tau)).$$

Now, if functions $\rho = \rho(\eta, v, w) > 0$ and $\Phi = \Phi(\eta, v, w)$ exist, which satisfy the system of equations

$$-\Phi_v + \Phi_{\eta w} + w \Phi_{vw} = \rho f \qquad \text{and} \qquad \Phi_{ww} = \rho, \tag{A.2}$$

then E has the form of a Lyapunov functional with a contribution on the boundary, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\tau} E(\tau) = - \int_0^{\ell(\tau)} \rho(\eta, B, B_\eta)(B_\tau)^2 \,\mathrm{d}\eta + \Phi_w B_\tau|_0^{\ell(\tau)}$$
$$+ \frac{\ell(\tau)}{2} \Phi(\ell(\tau), B(\ell(\tau), \tau), B_\eta(\ell(\tau), \tau)). \tag{A.3}$$

Therefore, we may obtain this formula by solving system (A.2), which we do by transforming it to a first-order equation for ρ,

$$w \rho_v + \rho_\eta - f \rho_w = f_w \rho. \tag{A.4}$$

If we supplement a given solution ρ of this equation with the function Φ given by

$$\Phi(\eta, v, w) = \int_0^w (w - s) \rho(\eta, v, s) \,\mathrm{d}s - \int_0^v \rho(\eta, \mu, 0) f(\eta, \mu, 0) \,\mathrm{d}\mu, \tag{A.5}$$

then the pair (ρ, Φ) solves (A.2). In order to find the pair (ρ, Φ) we therefore only need to solve equation (A.4).

Appendix A.2. The first-order equation for ρ

We solve equation (A.4) by the method of characteristics. Characteristic curves of equation (A.4) are curves $\boldsymbol{x} = (\eta, v, w)$ in $\mathbb{R}^3$, which we consider parametrized by η, along which

$$\frac{\mathrm{d}}{\mathrm{d}\eta} v = w \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}\eta} w = -f. \tag{A.6}$$

If a curve $\boldsymbol{x}(\eta) = (\eta, v^1(\eta), w^1(\eta))$ satisfies these equations, then equation (A.4) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}\eta} \rho(\boldsymbol{x}(\eta)) = f_w(\boldsymbol{x}(\eta)) \rho(\boldsymbol{x}(\eta)). \tag{A.7}$$

In order to solve the system of ODEs (A.6) and (A.7), we select a vector $(\eta_0, v_0, w_0) \in \mathbb{R}^+ \times \mathbb{R}^2$ and define $\phi(\xi) = \phi(\xi; \eta_0, v_0, w_0)$ to be the solution of the initial value problem

$$\phi'' + f(\xi, \phi, \phi') = 0 \qquad \text{with } \phi|_{\xi=\eta_0} = v_0 \quad \text{and} \quad \phi'|_{\xi=\eta_0} = w_0, \tag{A.8}$$

where $' = \partial/\partial\xi$. If the curve $\boldsymbol{x}$ passes through (η_0, v_0, w_0), i.e. if $\boldsymbol{x}(\eta_0) = (\eta_0, v_0, w_0)$, then, this curve can be identified with $\phi(\cdot; \eta_0, v_0, w_0)$, since $\boldsymbol{x}(\eta) = (\eta, v^1(\eta), w^1(\eta))$ where

$$v^1(\eta) = \phi(\eta; \eta_0, v_0, w_0) \qquad \text{and} \qquad w^1(\eta) = \phi'(\eta; \eta_0, v_0, w_0). \tag{A.9}$$

Since $f_w = (d+1)/\eta - (\eta/2) + (1/d)\eta v$, we may integrate (A.7) to find

$$\rho(\eta, v, w) = \rho(\eta_0, v_0, w_0) \exp\left\{\int_{\eta_0}^{\eta} \left[\frac{d+1}{\xi} - \frac{\xi}{2} + \frac{1}{d}\xi v^1(\xi)\right] \mathrm{d}\xi\right\}$$
$$= \rho(\eta_0, v_0, w_0) \frac{\eta^{d+1}}{\eta_0^{d+1}} \mathrm{e}^{-\eta^2/4+\eta_0^2/4} \exp\left\{\frac{1}{d}\int_{\eta_0}^{\eta} \xi v^1(\xi)\,\mathrm{d}\xi\right\}. \qquad \text{(A.10)}$$

To prove theorem 5.1, we need to define ρ in the set $\tilde{\mathcal{R}} \subset \mathcal{R}$ given by (54),

$$\mathcal{R} = \{\eta > 0, v \geqslant 0, w \leqslant 0\} \cup \{\eta = 0, v \geqslant 0, w = 0\}$$

$$\tilde{\mathcal{R}} = \mathcal{R} \cap \{0 \leqslant v \leqslant M,\ 0 \leqslant -w \leqslant \bar{M}\}.$$

We do so in the following way: for each $(\eta, v, w) \in \mathcal{R}$, we define $\rho(\eta, v, w)$ by following the characteristic curve through (η, v, w) to a reference point (η_0, v_0, w_0) for which $\rho(\eta_0, v_0, w_0)$ is fixed by choice; the value of $\rho(\eta, v, w)$ is then given by (A.10). To select an appropriate set of reference points, we study some of the properties of the solutions ϕ of (A.8), since they define the characteristic curves.

It follows from standard ODE theory that solutions of (A.8) are locally smooth and continuous under changes of (η_0, v_0, w_0). In general, however, we cannot extend these solutions to the whole of $\mathbb{R}^+$; in fact, for each $(\eta, v, w) \in \mathcal{R}$, there may exist $0 \leqslant \xi_1 < \eta$ and/or $\xi_2 > \eta$ such that

$$\phi(\xi_1; \eta, v, w) = \infty \qquad \text{and/or} \qquad \phi(\xi_2; \eta, v, w) = -\infty.$$

Partly because of this difficulty, we choose to only use forward solutions of (A.8) to define the characteristic curves. The next result details the behaviour of a forward solution ϕ of (A.8).

Lemma A.1. *Let $(\eta, v, w) \in \mathcal{R}$, and let $\phi(\xi) = \phi(\xi; \eta, v, w)$ be the solution of (A.8). For $\xi \geqslant \eta$, exactly one of the following three alternatives holds:*

(i) $\phi \equiv 1$ or $\phi \equiv 0$;
(ii) there exists $\eta^* > \eta$ such that $\phi(\eta^*) = 0$ and $\phi(\xi) < 0$ for $\xi > \eta^*$;
(iii) $\phi(\xi) \to 0$ as $\xi \to \infty$ and there exists a constant $C > 0$ such that $\phi(\xi)\xi^2 \to C$ as $\xi \to \infty$.

Proof. See [17, p 95]. The proof is based on results from [20]. ■

Since we need to define ρ with the appropriate estimates, we introduce a parameter $\bar{\eta}$ in the following lemma.

Lemma A.2. *There exists $\bar{\eta} > 0$ such that for every $\eta_1 \geqslant \bar{\eta}$ any solution ϕ of (A.8) with $\phi(\eta_1) = 1$ and $\phi'(\eta_1) \leqslant 0$ satisfies*

$$\phi'(\eta_2) < -1 \qquad \text{for all } \eta_2 > \eta_1 \text{ with } \phi(\eta_2) \in [0, 1/2]. \qquad \text{(A.11)}$$

Corollary A.3. *For every $\eta_2 \geqslant \bar{\eta}$, we have*

$$\phi(\xi; \eta_2, \epsilon, -\bar{\epsilon}) < 1 \qquad \text{for } \xi \in [\bar{\eta}, \eta_2] \qquad \text{(A.12)}$$

for all $0 \leqslant \epsilon \leqslant 1/2$ and $0 < \bar{\epsilon} \leqslant 1$.

Proof of corollary A.3. A violation of (A.12) implies the existence of $\eta_1 \in [\bar{\eta}, \eta_2)$ with $\phi(\eta_1) = 1$ and $\phi'(\eta_1) \leqslant 0$; then (A.11) contradicts the condition $\phi'(\eta_2; \eta_2, \epsilon, -\bar{\epsilon}) = -\bar{\epsilon} \geqslant -1$. ■

Proof of lemma A.2. We fix $\eta_1 \gg 1$ and define the variable $y = \xi/\eta_1$. Changing variables, equation (A.8) transforms into

$$0 = \frac{1}{\eta_1^2}\left(\ddot{\phi} + \frac{d+1}{y}\dot{\phi}\right) - \frac{y}{2}\dot{\phi} + \frac{1}{d}y\phi\dot{\phi} + \phi^2 - \phi \qquad \text{for } y > 1, \tag{A.13}$$

$$\dot{\phi}(1) = -D\eta_1, \qquad \phi(1) = 1, \tag{A.14}$$

where $\dot{} = \mathrm{d}/\mathrm{d}y$. Define

$$y_0 = \sup\{y > 1 : \ddot{\phi}(y) < 0 \text{ and } \phi(y) > 0\}$$

and note that $y_0 > 1$ if η_1 is large. On $[1, y_0]$, $\dot{\phi} \leqslant -D\eta_1$; therefore, $y_0 \leqslant 1 + 1/(D\eta_1)$, and consequently, on $[1, y_0]$,

$$-\frac{1}{d}(y-1)\phi\dot{\phi} \leqslant \frac{1}{dD\eta_1}|\dot{\phi}| \leqslant \frac{1}{4}\left(\frac{1}{2} - \frac{1}{d}\right)|\dot{\phi}| \qquad \text{if } \eta_1 \geqslant \frac{8d}{d-2}\frac{1}{dD}.$$

Similarly,

$$-\frac{d+1}{y\eta_1^2}\dot{\phi} \leqslant \frac{1}{4}\left(\frac{1}{2} - \frac{1}{d}\right)|\dot{\phi}| \qquad \text{if } \eta_1^2 \geqslant \frac{8d}{d-2}(d+1).$$

Therefore,

$$\frac{1}{\eta_1^2}\ddot{\phi} \leqslant \frac{1}{2}\left(\frac{1}{2} - \frac{1}{d}\phi\right)\dot{\phi} - \phi^2 + \phi \qquad \text{on } [1, y_0],$$

for large η_1. Estimating $|\dot{\phi}|$ by $D\eta_1$, we find

$$\frac{1}{\eta_1^2}\ddot{\phi} \leqslant -\frac{1}{2}\left(\frac{1}{2} - \frac{1}{d}\right)D\eta_1 + \frac{1}{4} \qquad \text{on } [1, y_0]$$

and since the right-hand side of this expression is negative for large η_1 it follows that $\ddot{\phi} < 0$ on $[1, y_0]$; therefore, y_0 may be redefined as

$$y_0 = \sup\{y > 1 : \phi(y) > 0\}.$$

It follows that on $[1, y_0]$,

$$\begin{aligned}\dot{\phi}(y) &\leqslant -D\eta_1 - \eta_1^2\frac{d}{4}\left[\left(\frac{1}{2} - \frac{1}{d}\phi\right)^2 - \left(\frac{d-2}{2d}\right)^2\right] + \int_1^y (\phi - \phi^2) \\ &\leqslant -D\eta_1 - \eta_1^2\frac{d}{4}\left[\left(\frac{1}{2} - \frac{1}{d}\phi\right)^2 - \left(\frac{d-2}{2d}\right)^2\right] + \frac{1}{4D\eta_1}.\end{aligned}$$

When $0 \leqslant \phi(y) \leqslant 1/2$, this expression is bounded from above by $-\eta_1^2/64$ for large η_1. In terms of the original variable ξ we obtain $\phi'(\xi) \leqslant -\eta_1/64$, thus proving the lemma. ■

Appendix A.3. Definition of ρ in $\mathcal{R}$

The general idea is to use $\eta_0 = \bar{\eta}$ as a reference point. In this way, owing to corollary A.3, we can obtain the required estimates for ρ. It can happen, however, that the function $\phi(\xi, ; \eta, v, w)$

is not defined at $\xi = \bar{\eta}$. In such a situation, to define ρ, we introduce functions representing the intersection of $\phi(\cdot; \eta, v, w)$ with the lines $\phi = 0$ for $\eta < \bar{\eta}$ and $\phi = 1$ for $\eta > \bar{\eta}$. Thus, it is useful to define the following subsets of $\mathcal{R}$:

$\mathcal{R}_1 = \{(\eta, v, w) \in \mathcal{R}\colon \phi(\xi; \eta, v, w)$ satisfies (i) in lemma A.1$\}$;

$\mathcal{R}_2 = \{(\eta, v, w) \in \mathcal{R}\colon \phi(\xi; \eta, v, w)$ satisfies (ii) in lemma A.1$\}$, with

$\mathcal{R}_{2a} = \mathcal{R}_2 \cap \{\eta \leqslant \bar{\eta}\}$ and $\mathcal{R}_{2b} = \mathcal{R}_2 \cap \{\eta > \bar{\eta}\}$;

$\mathcal{R}_3 = \{(\eta, v, w) \in \mathcal{R}\colon \phi(\xi; \eta, v, w)$ satisfies (iii) in lemma A.1$\}$.

We treat the cases in turn.

Case $\mathcal{R}_3$. Fix a point $(\eta, v, w) \in \mathcal{R}_3$. We choose $\eta_0 = \bar{\eta}$, $v_0 = \phi(\bar{\eta}; \eta, v, w)$ and $w_0 = \phi'(\bar{\eta}; \eta, v, w)$. Note that this choice is well defined: since $\phi(\xi)\xi^2 \to C > 0$ as $\xi \to \infty$, there exists $\eta_\epsilon > \bar{\eta}$ such that $\phi(\eta_\epsilon; \eta, v, w) = \epsilon < 1/2$, and $-\phi'(\eta_\epsilon; \eta, v, w) = \bar{\epsilon} < 1$, with $\bar{\epsilon} \sim 2\epsilon/\eta_\epsilon$. Then, corollary A.3 implies that the solution $\phi(\cdot; \eta, v, w)$ can be continued to $\bar{\eta}$, even if $\bar{\eta} < \eta$. Setting $\rho(\eta_0, v_0, w_0) = \eta_0^{d+1}\mathrm{e}^{-\eta_0^2/4}$, we find (cf (A.10))

$$\rho(\eta, v, w) = \eta^{d+1}\mathrm{e}^{-\eta^2/4} \exp\left\{\frac{1}{d}\int_{\bar{\eta}}^{\eta} \xi\phi(\xi; \bar{\eta}, \phi(\bar{\eta}; \eta, v, w), \phi'(\bar{\eta}; \eta, v, w))\,\mathrm{d}\xi\right\}. \tag{A.15}$$

The choice of $\eta_0 = \bar{\eta}$ also allows us to estimate the value of ϕ for $\xi > \bar{\eta}$, which in turn permits us to control ρ for large η, since the bound $\phi(\xi) \leqslant 1$ for $\xi > \bar{\eta}$ implies an exponential decay for ρ as $\eta \to \infty$.

Case $\mathcal{R}_1$. Points in $\mathcal{R}_1$ are of the form $(\eta, 1, 0)$ and $(\eta, 0, 0)$. We again choose $\eta_0 = \bar{\eta}$; substituting $\phi \equiv 1$ and $\phi \equiv 0$ into formula (A.15) gives

$$\rho(\eta, 1, 0) = \eta^{d+1}\mathrm{e}^{-((d-2)\eta^2)/4d}\mathrm{e}^{-\bar{\eta}^2/2d} \qquad \text{and} \qquad \rho(\eta, 0, 0) = \eta^{d+1}\mathrm{e}^{-\eta^2/4}. \tag{A.16}$$

Case $\mathcal{R}_{2a}$. Fix a point $(\eta, v, w) \in \mathcal{R}_{2a}$. Let η^* be given by lemma A.1 and define the function $L_0\colon \mathcal{R}_{2a} \to \mathbb{R}^+$ such that $L_0(\eta, v, w) = \min\{\eta^*, \bar{\eta}\}$. Note that the function L_0 is continuous and equals either the point η^* where $\phi(\eta^*; \eta, v, w)$ vanishes or $\bar{\eta}$ if $\phi(\bar{\eta}; \eta, v, w) \geqslant 0$. To find ρ, we choose $(\eta_0, v_0, w_0) = (\eta^*, 0, \phi'(\eta^*, \eta, v, w))$, and set $\rho(\eta_0, v_0, w_0) = \eta_0^{d+1}\mathrm{e}^{-\eta_0^2/4}$. This gives

$$\rho(\eta, v, w) = \eta^{d+1} \exp\left\{-\frac{\eta^2}{4} + I_0\right\}, \tag{A.17}$$

where

$$I_0 = \int_{L_0(\eta,v,w)}^{\eta} \frac{1}{d}\xi\phi\left(\xi; L_0(\eta, v, w), \phi(L_0(\eta, v, w); \eta, v, w), \phi'(L_0(\eta, v, w); \eta, v, w)\right)\mathrm{d}\xi.$$

Case $\mathcal{R}_{2b}$. Here, it is convenient to define for any $(\eta, v, w) \in R_{2b}$ the function $L_1\colon \mathcal{R}_{2b} \to \mathbb{R}^+$, by

$$L_1(\eta, v, w) = \begin{cases} \max\{\bar{\eta}, \max\{\xi \in (0, \eta) \mid \phi(\xi; \eta, v, w) \geqslant 1\}\} & \text{if } v < 1, \\ \min\{\xi \in (\eta, \infty) \mid \phi(\xi; \eta, v, w) \leqslant 1\} & \text{if } v \geqslant 1. \end{cases} \tag{A.18}$$

The function L_1 is well defined for $v < 1$ since if $\phi(\tilde{\xi}; \eta, v, w) = 0$ for some $\tilde{\xi} \in (\bar{\eta}, \eta)$, then, $\phi < 1$ in $(\bar{\eta}, \eta)$ by corollary A.3 and ϕ has to attain a local maximum in $(\bar{\eta}, \eta)$, which is a contradiction with equation (A.8). For $v \geqslant 1$, L_1 is well-defined by lemma A.1.

Note that $\phi(L_1(\eta, v, w); \eta, v, w) \leqslant 1$. The function L_1 is continuous and equals either η_* where $\phi(\eta_*; \eta, v, w) = 1$ or $\bar{\eta}$ if $\phi(\bar{\eta}; \eta, v, w) \in (0, 1)$.

Now, fix a point $(\eta, v, w) \in \mathcal{R}_{2b}$, choose $\eta_0 = L_1(\eta, v, w)$ and set $\rho(\eta_0, v_0, w_0) = \eta_0^{d+1} \mathrm{e}^{-(d-2)\eta_0^2/4d} \mathrm{e}^{-\bar{\eta}^2/2d}$. Using (A.10), we find that

$$\rho(\eta, v, w) = \eta^{d+1} \exp\left\{-\frac{\eta^2}{4} + \frac{\eta_0^2}{2d} - \frac{\bar{\eta}^2}{2d} + I_1\right\}, \tag{A.19}$$

where

$$I_1 = \int_{L_1(\eta,v,w)}^{\eta} \frac{1}{d}\xi\phi(\xi, L_1(\eta, v, w), \phi(L_1(\eta, v, w); \eta, v, w), \phi'(L_1(\eta, v, w), \eta, v, w))\,\mathrm{d}\xi$$

and

$$\rho(\eta, v, w) = \eta^{d+1} \exp\{-(d-2)\eta^2/4d - \bar{\eta}^2/2d + I_1'\}, \tag{A.20}$$

where

$$I_1' = \int_{L_1(\eta,v,w)}^{\eta} \frac{1}{d}\xi[\phi(\xi, L_1(\eta, v, w), \phi(L_1(\eta, v, w); \eta, v, w), \phi'(L_1(\eta, v, w), \eta, v, w)) - 1]\,\mathrm{d}\xi.$$

Appendix A.4. Properties of ρ and Φ

In the previous section, we have found a solution ρ of (A.4). Here, we show that this solution, together with the function Φ given by (A.5), satisfies the properties required for the proof of theorem 5.1. We start by stating a result which provides a lower bound for ρ in $\mathcal{R}_{2b}$.

Lemma A.4. *Let M and $\bar{M}$ be the constants in estimates (44) and (45), and let L_1 be defined as in (A.18). Then, there exists a large constant $\bar{\eta}_0$ such that the function $G\colon [\bar{\eta}_0, \infty) \to \mathbb{R}^+$ given by*

$$G(\eta) = \max\{L_1(\eta, a, -b) \mid 1 \leqslant a \leqslant M \text{ and } 0 \leqslant b \leqslant \bar{M}\} \qquad \text{for } \eta \geqslant \bar{\eta}_0,$$

satisfies $G(\eta) \leqslant C\eta$ for some constant $C = C(M) > 0$.

Proof. We take $\bar{\eta}_0$ large and we fix $\eta \geqslant \bar{\eta}_0$. Using the continuity of L_1, we have that $G(\eta) = L_1(\eta, \bar{a}, -\bar{b})$ for some $\bar{a} \in [1, M]$, and $\bar{b} \in [0, \bar{M}]$. Now, we define the variable $y = \xi/\eta \geqslant 1$; the result is proved if we show that $\sup\{y \geqslant 1 : \phi(y) > 1\} \leqslant C(M)$.

As in the proof of lemma A.2, equation (A.8) transforms into

$$0 = \frac{1}{\eta^2}\left(\ddot{\phi} + \frac{d+1}{y}\dot{\phi}\right) - \frac{y}{2}\dot{\phi} + \frac{1}{d}y\phi\dot{\phi} + \phi^2 - \phi \qquad \text{for } y > 1, \tag{A.21}$$

$$\dot{\phi}(1) = -\bar{b}\eta \qquad \text{and} \qquad \phi(1) = \bar{a}. \tag{A.22}$$

Note that for $\phi > 1$ we have $\dot{\phi}(y) < 0$ for all $y > 1$, since $\dot{\phi}(\bar{y}) = 0$ implies that $\bar{y}$ can only be a maximum, which contradicts equation (A.8).

We prove the claim in two steps. In the first step, we consider the case $\bar{a} > d/2 - \delta > 1$, where $\delta = (d+1)d/\eta^2$. Define $y_1 = \sup\{y > 1 : \phi(y) > d/2 - \delta\}$. We write (A.21) as

$$\frac{1}{\eta^2}\ddot{\phi} = -yA_2(y)\dot{\phi} - A_1(y) \qquad \text{for } y > 1, \tag{A.23}$$

where

$$A_1(y) = \phi^2 - \phi \qquad \text{and} \qquad A_2(y) = \left(\frac{1}{d}\phi - \frac{1}{2} + \frac{d+1}{y^2\eta^2}\right).$$

Since $\phi(\cdot) \in [d/2 - \delta, \bar{a}]$ on $[1, y_1]$, A_2 is non-negative and bounded by $\bar{A}_2 := \bar{a}/d$. The function A_1 is positive and bounded from below:

$$A_1(y) \geqslant \underline{A_1} := \left(\frac{d}{2} - \delta\right)^2 - \left(\frac{d}{2} - \delta\right) > 0.$$

Integrating equation (A.23), we have

$$\dot\phi(y) = -\bar b\eta\, \mathrm{e}^{-\eta^2\int_1^y tA_2(t)\,\mathrm{d}t} - \eta^2\int_1^y A_1(s)\mathrm{e}^{-\eta^2\int_s^y tA_2(t)\,\mathrm{d}t}\,\mathrm{d}s \qquad \text{for } y > 1. \tag{A.24}$$

We observe that

$$\eta^2\int_1^y A_1(s)\mathrm{e}^{-\eta^2\int_s^y tA_2(t)\,\mathrm{d}t}\,\mathrm{d}s \geqslant \underline{A_1} f(y;\eta) \qquad \text{for } 1\leqslant y\leqslant y_1, \tag{A.25}$$

where $f(y;\eta) = \eta^2\int_1^y \mathrm{e}^{-\eta^2\bar A_2(y^2-s^2)/2}\,\mathrm{d}s$ is a positive bounded function satisfying $yf(y;\eta)\to 1/\bar A_2$ as $y\to\infty$ (the latter claim follows from considering the integrand close to $s=y$), and more precisely,

$$yf(y;\eta)\geqslant \frac{1}{2\bar A_2} \qquad \text{for } y\geqslant 2 \text{ and for sufficiently large } \eta.$$

Therefore, the primitive function

$$F(y;\eta) = \int_1^y f(s;\eta)\,\mathrm{d}s$$

satisfies

$$F(y;\eta)\geqslant \frac{1}{2\bar A_2}(\log y - \log 2). \tag{A.26}$$

Integrating (A.24) on $[1, y_1]$ and using (A.25), we obtain

$$\phi(y_1)\leqslant \bar a - \bar b\eta(y_1-1) - \underline{A_1}F(y_1;\eta).$$

To obtain a bound on y_1, we use $\phi(y_1) = d/2-\delta$ and conclude that

$$\underline{A_1}F(y_1;\eta)\leqslant \bar a\leqslant M,$$

from which it follows that $y_1\leqslant C(M)$ by (A.26).

For the second step, we replace η by $y_1\eta$ in the rescaling above, by which we can assume that we are in the same situation: $\phi(1)=\bar a$, $\dot\phi(1)=\bar b\eta$, but this time $1\leqslant\bar a\leqslant d/2-\delta$.

Similarly, define $y_2 = \sup\{y\geqslant 1 : \phi(y)>1\}$. Since $1\leqslant\phi(\cdot)\leqslant d/2-\delta$ on $[1, y_2]$, the function $A_2(\cdot)$ in (A.23) is negative, so that ϕ satisfies the differential inequality

$$\frac{1}{\eta^2}\ddot\phi\leqslant -\phi^2+\phi < -2(\phi-1). \tag{A.27}$$

Let the function ψ solve

$$\frac{1}{\eta^2}\ddot\psi = -2(\psi-1) \qquad \text{with } \psi(1)=\bar a \quad\text{and}\quad \dot\psi(1)=0.$$

The solution of this equation is $\psi(y) = 1+(\bar a-1)\cos(\eta\sqrt{2}(y-1))$, and note that $\psi(\tilde y_2)=1$ for $\tilde y_2 := \pi/(2\eta\sqrt{2})$. From (A.27), $\phi(1+) < \psi(1+)$; if $\phi(y)=\psi(y)$ for some $y\in(1,\tilde y_2)$, then, by the comparison principle (which the operator $u\mapsto \ddot u/\eta^2+2(u-1)$ satisfies on intervals of length less than $\tilde y_2$), we find $\phi\geqslant\psi$ on the interval $[1, y]$, which contradicts the previous remark.

In conclusion, we find that $y_2\leqslant\tilde y_2$, thus proving the lemma. ■

We now derive estimates for ρ and Φ in $\tilde{\mathcal{R}}$ and $\mathcal{R}$.

Lemma A.5. *The function ρ is continuous in $\mathcal{R}\backslash\{\eta=\bar\eta,\ v>1\}$; for $(\eta, v, w)\in\mathcal{R}$, one finds*

$$\rho(\eta, v, w)\leqslant \eta^{d+1}\mathrm{e}^{-(d-2)\eta^2/4d}. \tag{A.28}$$

In addition, if $(\eta, v, w) \in \tilde{\mathcal{R}}$*, then,*

$$\rho(\eta, v, w) \geqslant \frac{1}{C_0}\eta^{d+1}\mathrm{e}^{-C_0\eta^2} \tag{A.29}$$

for some constant $C_0 = C_0(M) > 0$.

Proof. We start by proving (A.28)–(A.29). Let $\tilde{\mathcal{R}}_i = \tilde{\mathcal{R}} \cap \mathcal{R}_i$ for $i = 1, 2, 3$. If $(\eta, v, w) \in \mathcal{R}_1$, then the estimates (A.28)–(A.29) follow by definition. If $(\eta, v, w) \in \mathcal{R}_{2a}$, then as $\phi > 0$ on $(\eta, L_0(\eta, v, w))$ the integral in (A.17) is negative. This gives

$$\rho(\eta, v, w) \leqslant \eta^{d+1}\mathrm{e}^{-\eta^2/4} \qquad \text{for } (\eta, v, w) \in \mathcal{R}_{2a}.$$

Now, for $(\eta, v, w) \in \tilde{\mathcal{R}}_{2a}$, we have that $[\eta, L_0(\eta, v, w)] \subset [0, \bar{\eta}]$, $v \in [0, M]$ and $w \in [-\bar{M}, 0]$. Then, the continuity of ϕ on $[\eta, L_0(\eta, v, w)]$ implies that

$$|\phi(\cdot; L_0(\eta, v, w), \phi(L_0(\eta, v, w); \eta, v, w), \phi'(L_0(\eta, v, w); \eta, v, w))|_{C^0([\eta, L_0(\eta,v,w)])} \leqslant \bar{C}_0,$$

where $\bar{C}_0 = \bar{C}_0(M, \bar{M}, \bar{\eta})$. Using this bound to estimate I_0 in (A.17), we find that

$$C(M)\eta^{d+1}\mathrm{e}^{-\eta^2/4} \leqslant \rho(\eta, v, w) \qquad \text{for } (\eta, v, w) \in \tilde{\mathcal{R}}_{2a}$$

with $C(M) < 1$, since we have integrated backwards.

For any $(\eta, v, w) \in \mathcal{R}_{2b}$, we use (A.20) and find the upper bound

$$\rho(\eta, v, w) \leqslant \eta^{d+1}\mathrm{e}^{-(d-2)\eta^2/4d}\mathrm{e}^{-\bar{\eta}^2/2d}.$$

This estimate follows from the negative sign of the integral I_1' in (A.20). In fact, for $\xi \in (\eta, L_1(\eta, v, w))$, we have

$$\phi\left(\xi; L_1(\eta, v, w), \phi(L_1(\eta, v, w); \eta, v, w), \phi'(L_1(\eta, v, w); \eta, v, w)\right) - 1 > 0 \qquad \text{if } v \geqslant 1,$$
$$\phi\left(\xi; L_1(\eta, v, w), \phi(L_1(\eta, v, w); \eta, v, w), \phi'(L_1(\eta, v, w); \eta, v, w)\right) - 1 < 0 \qquad \text{if } v < 1.$$

Next, for $(\eta, v, w) \in \tilde{\mathcal{R}}_{2b}$, we find

$$\rho(\eta, v, w) \geqslant \eta^{d+1}\mathrm{e}^{-\eta^2/4} \qquad \text{for } v \leqslant 1,$$
$$\rho(\eta, v, w) \geqslant \eta^{d+1}\mathrm{e}^{-(d-2)\eta^2/4d}\mathrm{e}^{-\bar{\eta}^2/2d}\mathrm{e}^{-\bar{C}(M)\eta^2} \qquad \text{for } v > 1,$$

where $\bar{C}(M) > 0$. The estimate for the case when $v \leqslant 1$ follows directly from (A.19). To obtain the estimate for ρ when $v > 1$, we use (A.20). In fact, noting that ϕ is nonincreasing in $[\eta, L_1(\eta, v, w)]$, we have that

$$|\phi(\cdot; L_1(\eta, v, w), \phi(L_1(\eta, v, w); \eta, v, w), \phi'(L_1(\eta, v, w); \eta, v, w))|_{C^0([\eta, L_1(\eta,v,w)])} \leqslant M.$$

Using this bound together with the estimate $L_1(\eta, v, w) \leqslant C(M)\eta$ (see lemma A.4), we find that I_1' in (A.20) satisfies $-I_1' \leqslant \bar{C}(M)\eta^2$, which gives the desired estimate.

To prove (A.28) for $(\eta, v, w) \in \mathcal{R}_3$, we examine two cases: if $\eta \leqslant \bar{\eta}$ then the estimate for $\mathcal{R}_{2a}$ holds and for $\eta > \bar{\eta}$ the estimate for $\mathcal{R}_{2b}$ holds. Finally, to obtain (A.29) for $(\eta, v, w) \in \tilde{\mathcal{R}}_3$, we also check two cases: if $\eta \leqslant \bar{\eta}$ then the estimate for $\tilde{\mathcal{R}}_{2a}$ holds and for $\eta > \bar{\eta}$ the estimate for $\tilde{\mathcal{R}}_{2b}$ with $v \leqslant 1$ holds.

Claim. ρ is continuous in $\mathcal{R} \backslash \{\eta = \bar{\eta},\ v > 1\}$.

Before we prove this, note that $\mathcal{R}_2$ is an open set and $\mathcal{R}_1$ and $\mathcal{R}_3$ are closed.

We first see that ρ is continuous within $\mathcal{R}_{2a}$ and $\mathcal{R}_{2b}$, by continuity of L_0 and L_1. For the elements in $\mathcal{R}_1$, the definition of ρ is as for $\mathcal{R}_2$; therefore, there is continuity of ρ between $\mathcal{R}_2$ and $\mathcal{R}_1$.

The delicate part is to prove the continuity between $\mathcal{R}_3$ and $\mathcal{R}_2$. Taking a sequence $(\eta_n, v_n, w_n) \in \mathcal{R}_2$, we associate a solution $\phi_n(\cdot, \eta_n, v_n, w_n)$. Suppose that

$(\eta_n, v_n, w_n) \to (\eta, v, w) \in \mathcal{R}_3$. Now, if $\phi(\cdot, \eta, v, w)$ is the solution of (A.8), then $\phi_n \to \phi$ in compact subsets of $\mathbb{R}^+$. Therefore, by corollary A.3, for $n \geqslant n_0 \in \mathbb{N}$, we find $\phi_n(\bar{\eta}) \in (0, 1)$. Then $(\eta_n, v_n, w_n) \in \mathcal{R}_2$, for $n \geqslant n_0$, have the same definition of ρ as for $(\eta, v, w) \in \mathcal{R}_3$. Finally, if $v \leqslant 1$ and $\eta = \bar{\eta}$, then ρ is continuous. If η is close enough to $\bar{\eta}$, then we have that $\eta_0 = \bar{\eta}$. So, the computation of ρ uses the same formula, independent of the subset of $\mathcal{R}$ to which (η, v, w) belongs. ∎

For Φ we deduce the following lemma, which implies (56).

Lemma A.6. *The function Φ is continuous in $\mathcal{R}\backslash\{\eta = \bar{\eta},\ v > 1\}$ and if $(\eta, v, w) \in R$, then,*

$$\Phi(\eta, v, w) \leqslant \left\{ w^2 + \frac{v^2}{2} \right\} \eta^{d+1} \mathrm{e}^{-(d-2)\eta^2/4d}$$

and

$$\Phi(\eta, v, w) \geqslant -\left\{ \frac{v^3}{3} - \frac{v^2}{2} \right\} \eta^{d+1} \mathrm{e}^{-(d-2)\eta^2/4d}.$$

Proof. Follows directly from the definition (A.5) of Φ and uses the upper bound (A.28) of ρ. ∎

Appendix A.5. Regularizing argument

At the beginning of this appendix, we formally constructed a Lyapunov functional $E(\tau)$ with Φ and ρ satisfying (A.3). In the previous section, we obtained a solution ρ of (A.4) and Φ given by (A.2). Moreover, these functions satisfy the properties found in lemmas A.5 and A.6. From these results we do not obtain enough regularity to derive (A.3). To do this, we introduce a regularization of Φ using standard mollifiers and a translation function to avoid the singularity of f at $\eta = 0$. See the details of the proof in [17, p 102].

Appendix B. Linear stability of blow-up profiles

In this appendix, we study the linear stability of the blow-up profiles φ_1 and φ^*; see (20).

Let B be a solution of (27)–(29) and let φ be a solution of (19). The idea is to study the linearized equation for the difference $\Phi(\eta, \tau) := B(\eta, \tau) - \varphi(\eta)$, i.e.

$$\Phi_\tau = \Phi_{\eta\eta} + \frac{d+1}{\eta}\Phi_\eta + \left(\frac{1}{d}\varphi - \frac{1}{2}\right)\eta\Phi_\eta + \left(\frac{1}{d}\eta\varphi_\eta + 2\varphi - 1\right)\Phi. \tag{B.1}$$

Here, we have implicitly assumed that sufficiently close to blow-up only the linear terms play a role in describing the singularity formation.

For the stability analysis, let $\lambda > 0$ and consider a solution of (B.1) of the form $\psi_\lambda(\eta)\mathrm{e}^{\lambda\tau}$. By (B.1), $\langle\psi_\lambda(\eta), \lambda\rangle$ satisfies

$$(\psi_\lambda)_{\eta\eta} + \frac{d+1}{\eta}(\psi_\lambda)_\eta + \left(\frac{1}{d}\varphi - \frac{1}{2}\right)\eta(\psi_\lambda)_\eta + \left(\frac{1}{d}\eta\varphi_\eta + 2\varphi - 1 - \lambda\right)\psi_\lambda. \tag{B.2}$$

For the analysis of boundary conditions we first consider $\varphi = \varphi_1$. We note that at $\eta = 0$ we have either $\psi_\lambda \sim 1$ or $\psi_\lambda \sim 1/\eta^d$. To have ψ_λ bounded near 0, we impose

$$(\psi_\lambda)_\eta(\eta) \to 0 \qquad \text{as } \eta \to 0. \tag{B.3}$$

For large η, we can either have $\psi_\lambda \sim \eta^{-(2\lambda+3)}e^{\eta^2/4}$ or $\psi_\lambda \sim \eta^{2\lambda-2}$. We see that both types of behaviour diverge with η; however, the second asymptotic is bounded in terms of r and t as $t \to T$. Therefore, to have polynomial behaviour at infinity, we prescibe

$$\psi_\lambda(\eta)e^{-\eta} \to 0 \qquad \text{as } \eta \to \infty. \tag{B.4}$$

Now solving equation (B.2) together with (B.3) and (B.4), we find a sequence of solutions of (B.1) given by $\{e^{\lambda_n\tau}\psi_n(\eta)\}_{n\in\mathbb{N}\cup\{0\}}$, with $\lambda_0 > \lambda_1 > \cdots$, where $\psi_n := \psi_{\lambda_n}$. If the blow-up time $T > 0$ is chosen correctly in the definition of η and τ, we can eliminate, see [10], the first mode ($n = 0$) corresponding to change of blow-up and write

$$B(\eta,\tau) = \varphi(\eta) + \psi_1(\eta)e^{\lambda_1\tau} + O(e^{\lambda_2\tau}).$$

Therefore, from the sign of λ_1 we obtain the linear stability of φ.

In [10], Brenner *et al* proved, using (B.1), the following stability result for various blow-up profiles.

Theorem A.7. *Every solution φ of (47) satisfying $\eta\varphi_\eta/\varphi \to 2$ as $\eta \to \infty$ has an unstable mode corresponding to changing the blow-up time. Also, a blow-up profile with k intersections with the singular solution φ_S has at least $k - 1$ additional unstable modes.*

In addition, the authors in [10] found numerically that $\lambda_1 < 0$ when $\varphi = \varphi_1$ and $d > 2$. In particular, they computed $\lambda_1 = -0.272\ldots$ for $d = 3$. This implies that φ_1 is linearly stable for $d > 2$.

For $\varphi = \varphi^*$, we can proceed as above and solve the eigenvalue problem for (B.1). Considering (B.2) with $\varphi = \varphi^*$, we find that $\langle\psi_\lambda, \lambda\rangle$ satisfies

$$(\psi_\lambda)_{\eta\eta} + \left(\frac{d+1}{\eta} - \frac{d-2}{2d}\eta\right)(\psi_\lambda)_\eta + (1-\lambda)\psi_\lambda = 0 \tag{B.5}$$

with (B.3) and (B.4). These boundary conditions are chosen by the same arguments for $\varphi = \varphi_1$; however, in the current case we have either $\psi_\lambda \sim \eta^{(2d/(d-2))(\lambda-1)-d-2}e^{((d-2)/4d)\eta^2}$ or $\psi_\lambda \sim \eta^{2d/(d-2)(1-\lambda)}$ as $\eta \to \infty$. Note that on changing η to $(-\eta)$ the equation remains invariant, so only solutions consisting of even powers are allowed. Then, we construct a sequence of solutions of the form

$$\psi_n(\eta) = \sum_{i=0}^{n} A_i\eta^{2i} \qquad \text{for any } n = 0, 1, 2, 3, \ldots,$$

where the coefficients are given by $A_i(2i(2i-1) + (d+1)2i) = A_{i-1}(1 - \lambda - 2i(d-2)/2d)$ for $i = 1, 2, \ldots$ and A_0 is an arbitrary constant. This means that when $(1 - \lambda - 2(n+1)(d-2)/2d) = 0$, we find an explicit polynomial solution of degree $2n$, where λ is given by

$$\lambda_n = \frac{d - n(d-2)}{d}. \tag{B.6}$$

Consequently, we have obtained an explicit sequence of solutions $\{\langle\psi_n, \lambda_n\rangle\}_{n\in\mathbb{N}\cup\{0\}}$ for the eigenvalue problem (B.5). The eigenvalue $\lambda_0 = 1$ corresponds to the unstable mode of change of blow-up time and since $\lambda_1 > 0$ for all $d > 2$, by (B.6), this means that φ^* is linearly unstable.

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