Convergence of a gradient flow to a non-gradient-flow

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(joint work with Mikola Schlottke)

We study the limit $\varepsilon \to 0$ for the family of Fokker-Planck equations in one dimension defined by

$$\partial_t \rho_\varepsilon = \varepsilon\tau_\varepsilon \left[\partial_{xx}\rho_\varepsilon + \partial_x\left(\rho_\varepsilon \frac{1}{\varepsilon}\partial_x V\right)\right], \quad \text{on } \mathbb{R}_+ \times \mathbb{R}. \tag{1}$$

Here we take an *asymmetric* double-well potential $V : \mathbb{R} \to \mathbb{R}$ as depicted in Figure 1.

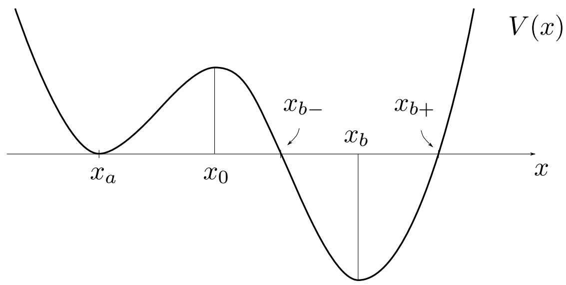

Figure 1. A typical asymmetric potential $V(x)$.

A typical solution $\rho_\varepsilon(t, x)$ is displayed in Figure 2, showing mass flowing from left to right: as time increases, the mass shifts from the left to the right well.

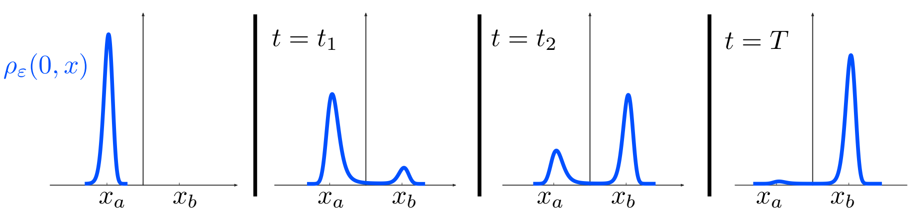

Figure 2. The time evolution of a solution $\rho_\varepsilon(t, x)$ to (1) whose initial distribution is supported on the left. Time increases from left to right. The smaller the value of ε, the sharper the equilibrium distribution concentrates around the global minimum x_b.

There are two parameters, $\varepsilon > 0$ and $\tau_\varepsilon > 0$. The parameter ε controls how sharply the mass is concentrated around x_a and x_b, and how fast mass can move between the potential wells. The second parameter τ_ε sets the global time scale, and is chosen such that transitions from the local minimum x_a to the global minimum x_b happen at rate of order one as $\varepsilon \to 0$:

$$\tau_\varepsilon := \frac{2\pi}{\sqrt{V''(x_a)|V''(x_0)|}} \exp\{\varepsilon^{-1}(V(x_0) - V(x_a))\}.$$

As $\varepsilon \to 0$, we expect the mass to be concentrated at x_a and x_b, and we expect the limiting dynamics to be characterized by mass being transferred at rate one from the local minimum x_a to the global minimum x_b, with no mass moving in the opposite direction. In terms of the solution ρ_ε, therefore, we expect

$$\rho_\varepsilon \rightharpoonup \rho_0 = z\delta_{x_a} + (1-z)\delta_{x_b},$$

where the limit density $z = z(t)$ of particles at x_a satisfies $\partial_t z = -z$, corresponding to left-to-right transitions happening at rate 1. This is illustrated in Figure 3.

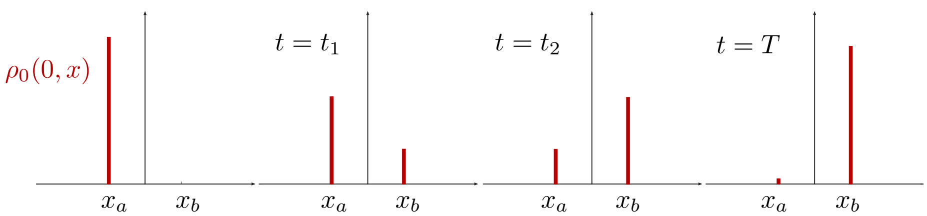

Figure 3. The time evolution of ρ_0, defined as the $\varepsilon \to 0$ limit of the solution $\rho_\varepsilon(t,x)$ to (1).

Equation (1) is a Wasserstein gradient flow [AGS08] of the functional

$$E_\varepsilon(\rho) := \mathcal{H}(\rho|\gamma_\varepsilon) \quad \text{where} \quad \gamma_\varepsilon(\mathrm{d}x) := \frac{1}{Z_\varepsilon} e^{-V(x)/\varepsilon}\,\mathrm{d}x,$$

where $\mathcal{H}(\mu|\nu)$ is the relative entropy of μ with respect to ν. This structure was used in [AMPSV12, LMPR17] to prove 'EDP-convergence' of this gradient system to a limiting gradient system, in the case of a *symmetric* potential V.

For a non-symmetric potential V we show in [PS21] that no EDP-convergence is possible. The main reason for this is that the gradient structure is 'lost' in the limit. One indication of this is the singular behaviour of the energy E_ε:

$$E_\varepsilon \xrightarrow{\Gamma} E_0(\rho) := \mathcal{H}(\rho|\delta_{x_b}) = \begin{cases} 0 & \text{if } \rho = \delta_{x_b} \\ +\infty & \text{otherwise,} \end{cases}$$

while

$$\varepsilon E_\varepsilon \xrightarrow{\Gamma} \int \rho V.$$

This shows that for any $\rho \neq \delta_{x_b}$, $E_\varepsilon(\rho)$ diverges at rate $1/\varepsilon$, and any gradient system driven by the limit functional E_0 only admits constant solutions $\rho(t) = \delta_{x_b}$ for all t.

Instead, we follow the idea of variational convergence but for more general formulations. The starting point is the following reformulation of equation (1). For pairs $(\rho_\varepsilon, j_\varepsilon)$ satisfying the continuity equation $\partial_t\rho_\varepsilon + \partial_x j_\varepsilon = 0$, equation (1) can formally be written as $\mathcal{I}_\varepsilon(\rho_\varepsilon, j_\varepsilon) \le 0$, where

$$\mathcal{I}_\varepsilon(\rho_\varepsilon, j_\varepsilon) := \frac{1}{2}\int_0^T \int_{\mathbb{R}} \frac{1}{\varepsilon\,\tau_\varepsilon}\frac{1}{\rho(t,x)}\big|j(t,x) - J_\varepsilon^\rho(t,x)\big|^2\,\mathrm{d}x\mathrm{d}t$$

$$\text{and } J_\varepsilon^\rho := -\tau_\varepsilon\left[\varepsilon\,\partial_x\rho_\varepsilon + \rho_\varepsilon\partial_x V\right].$$

In the context of EDP-convergence, one would split this functional into energetic and dissipation parts,

$$\mathcal{I}_\varepsilon(\rho_\varepsilon, j_\varepsilon) \;=\; E_\varepsilon(\rho)\Big|_{t=0}^{t=T} + \underbrace{\int_0^T \Big[R_\varepsilon(\rho, j) + R_\varepsilon^*\big(\rho, -\mathrm{D}E_\varepsilon(\rho)\big)\Big]\,\mathrm{d}t}_{(*)}$$

and one would proceed to characterize the Γ-convergence of the terms separately. Because of the singularity of E_ε this fails, and both $E_\varepsilon(\rho)|_{t=0}^{t=T}$ and the term $(*)$ diverge as $1/\varepsilon$. Instead we keep the terms together, and study the Γ-convergence of the combined functional $\mathcal{I}_\varepsilon$.

Theorem 1 (Main result)**.** *Let V satisfy a number of technical assumptions that encode the 'two-well' nature. Then*

(1) *Sequences $(\rho_\varepsilon, j_\varepsilon)$ for which there exists a constant C such that*

$$\mathcal{I}_\varepsilon(\rho_\varepsilon, j_\varepsilon) \le C \qquad \textit{and} \qquad E_\varepsilon(\rho_\varepsilon(0)) \le \frac{C}{\varepsilon}$$

are compact in a distributional sense;

(2) *Along sequences $(\rho_\varepsilon, j_\varepsilon)$ satisfying*

$$\rho_\varepsilon(t=0) \rightharpoonup \rho_0^\circ(\mathrm{d}x) := z^\circ\delta_{x_a}(\mathrm{d}x) + (1-z^\circ)\delta_{x_b}(\mathrm{d}x) \qquad \textit{as } \varepsilon \to 0,$$

the functional $\mathcal{I}_\varepsilon$ Γ-converges to a limit $\mathcal{I}_0$.

The limit functional $\mathcal{I}_0$ is defined by

$$\mathcal{I}_0(\rho, j) := 2\int_0^T S(j(t)|z(t))\,\mathrm{d}t,$$

provided $\rho(t,\mathrm{d}x) = z(t)\delta_{x_a}(\mathrm{d}x) + (1-z(t))\delta_{x_b}(\mathrm{d}x)$ with $z(0) = z^\circ$, and $j(t,x) = j(t)\mathbb{1}_{(x_a,x_b)}(x)$. Here the function $S : \mathbb{R}^2 \to [0,\infty]$ is given by

$$S(a|b) := \begin{cases} a\log\dfrac{a}{b} - a + b, & a, b > 0,\\ b, & a = 0,\ b > 0,\\ +\infty, & \text{otherwise.}\end{cases}$$

Lemma 2 (Characterization of minimizers of $\mathcal{I}_0$)**.** *If $\mathcal{I}_0(\rho, j) = 0$, then z satisfies $z'(t) = -z(t)$ for all t.*

Details can be found in the forthcoming publication [PS21].

References

[AGS08] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient Flows in Metric Spaces and in the Space of Probability Measures*. Lectures in Mathematics ETH Zürich. Birkhäuser, 2008.

[AMPSV12] S. Arnrich, A. Mielke, M. A. Peletier, G. Savaré, and M. Veneroni. Passing to the limit in a Wasserstein gradient flow: From diffusion to reaction. *Calculus of Variations and Partial Differential Equations*, 44:419–454, 2012.

[LMPR17] M. Liero, A. Mielke, M. A. Peletier, and D. R. M. Renger. On microscopic origins of generalized gradient structures. *Discrete and Continuous Dynamical Systems-Series S*, 10(1):1, 2017.

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