Self-similar solutions of a fast diffusion equation that do not conserve mass

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Abstract

We consider self-similar solutions of the fast diffusion equation $u_t = \nabla \cdot (u^{-n}\nabla u)$ in $(0,\infty)\times\mathbb{R}^N$, for $N \geq 3$ and $\frac{2}{N} < n < 1$, of the form

$$u(x,t) = (T-t)^\alpha f\left(|x|\,(T-t)^{-\beta}\right).$$

Because mass conservation does not hold for these values of n, this results in a nonlinear eigenvalue problem for f and α. We employ phase plane techniques to prove existence and uniqueness of solutions (f,α), and we investigate their behaviour when $n\uparrow 1$ and when $n \downarrow \frac{2}{N}$.

1 Introduction

The equation

$$u_t = \nabla \cdot (u^{-n}\nabla u) \tag{1.1}$$

arises in many areas of applications. The case $n = 0$ corresponds to the classical heat conduction equation. When $n < 0$, the equation models the flow of a gas in a porous medium. Because the diffusion coefficient u^{-n} vanishes when $u = 0$, disturbances of $u = 0$ propagate at finite speed. Therefore (1.1) with $n < 0$ is also known as the slow diffusion equation, and has been the focus of extensive study during the past two decades. We mention the survey papers by Aronson [1] and L.A. Peletier [16] for further reference.

In this paper we consider the solutions in the case of fast diffusion, i.e. $n > 0$. For $n = 1$, equation (1.1) arises in the study of the expansion of a thermalised electron cloud [14], in gas kinetics as the central dynamical limit of Carleman's model of the Boltzman equation ([5], [8], [9], [13], [15]), and in ion exchange kinetics in cross-field convective diffusion of plasma [6]. In [10] a model is described for the diffusion of impurities in silicon, in which equation (1.1) arises for values of n between 0 and 1.

We shall consider equation (1.1) in $\mathbb{R}^N$, for a spatial dimension N larger than two, subject to an initial condition

$$u(x,0) = u_0(x) \quad \text{for} \quad x \in \mathbb{R}^N, \tag{1.2}$$

where the initial distribution u_0 is non-negative and $u_0 \in L^1(\mathbb{R}^N)$.

When $0 < n < \frac{2}{N}$, it is well-known that solutions of the initial value problem (1.1), (1.2) are smooth and exist for all time (see e.g. [16]). For values of $n \geq 1$ no solutions with finite initial mass exist [18]; in [3] it is proved that when $\frac{2}{N} < n < 1$, finite-mass solutions become identically equal to zero in finite time, due to a non-zero flux at infinity. We will be concerned with a special kind of such solutions, namely those which are of a self-similar form. In particular we take $N > 2$ and $\frac{2}{N} < n < 1$, and seek solutions u of (1.1) which vanish at a finite time T, and which are of the form

$$u(x,t) = (T-t)^{\alpha} f(\eta) \quad \text{where} \quad \eta = |x|(T-t)^{-\beta}, \tag{1.3}$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are constants that need to be determined. Such solutions were also considered by J.R. Philip [17] and in more detail by J.R. King [12], who gave a formal motivation for the existence of such solutions, and for the convergence of solutions with arbitrary initial distributions to these self-similar profiles. In this article we provide a rigorous proof of King's conjectures concerning existence and uniqueness of self-similar solutions and some of their properties. When $\beta = 0$, the solution u given in (1.3) is separable, and for this case V. A. Galaktionov and L. A. Peletier have proved convergence of general finite-mass solutions to the separable one. A similar statement on bounded domains can be found in [4].

The character of fast diffusion implies that at any time t at which a solution of (1.1) is not identically equal to zero, it is in fact strictly positive in $\mathbb{R}^N$ and smooth [7]. Hence, when looking for solutions of the form (1.3), it is no restriction to assume that $f(\eta)$ is positive and smooth for all $\eta \geq 0$.

Substituting expression (1.3) into (1.1), we find that if we choose

$$\alpha n + 2\beta = 1, \tag{1.4}$$

then f satisfies the equation

$$\eta^{1-N}(\eta^{N-1} f^{-n} f')' - \beta \eta f' + \alpha f = 0 \quad \text{for} \quad \eta > 0. \tag{1.5}$$

Symmetry and smoothness require that

$$f' = 0 \quad \text{at} \quad \eta = 0. \tag{1.6}$$

The restriction that f represent a solution of (1.1) of finite mass translates into the condition

$$\int_0^{\infty} \eta^{N-1} f(\eta)\, d\eta < \infty. \tag{1.7}$$

One can show that (1.7), when combined with (1.5), is equivalent with the statement that the flux $F(\eta) = \eta^{N-1} f^{-n} f'(\eta)$ has a finite (negative) limit at infinity. This statement is equivalent to the assertion that

$$f(\eta) \asymp \eta^{-(N-2)/(1-n)} \quad \text{as} \quad \eta \to \infty, \tag{1.8}$$

where the notation $a(t) \asymp b(t)$ signifies

$$\lim_{t\to\infty} \frac{a(t)}{b(t)} \quad \text{exists and is positive.}$$

To conclude our preliminary remarks about equation (1.5), note that the scaling

$$\bar{f}(\eta) = \gamma^{-2/n} f(\eta/\gamma) \quad \text{for} \quad \gamma > 0 \tag{1.9}$$

leaves the equation as well as both boundary conditions invariant. Throughout this article we therefore set $f(0) = 1$.

Therefore the problem to be studied in this article is: Find $f : [0, \infty) \to \mathbb{R}$, positive and smooth, and parameters $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$(P) \begin{cases} \eta^{1-N}(\eta^{N-1} f^{-n} f')' - \beta\eta f' + \alpha f = 0, \quad f > 0 \quad \text{for } \eta > 0 & (1.5) \\ f'(0) = 0 \quad \text{and} \quad f(0) = 1 & (1.6) \\ f(\eta) \asymp \eta^{-(N-2)/(1-n)} \quad \text{as } \eta \to \infty & (1.8) \\ \alpha n + 2\beta = 1. & (1.4) \end{cases}$$

The relation (1.4) between the two parameters introduced by the Ansatz (1.3) arises from the requirement that f satisfy an equation involving only η. In situations where the problem under consideration satisfies a conservation law (e.g. conservation of mass), this law supplies a second condition on α and β, thus fixing the parameters. In this case we speak of *self-similar solutions of the first kind*. Since we seek solutions that do not conserve mass, there is no second condition on α and β for Problem (P). This extra degree of freedom gives it the character of a nonlinear eigenvalue problem: the parameter α (or β) is to be determined together with the solution function f. The function f is then called a *self-similar solution of the second kind* [2].

The main results of this article are summarised in the following two theorems. The first one gives existence and uniqueness for Problem (P).

Theorem A. *For every $N > 2$ and $\frac{2}{N} < n < 1$, Problem (P) has exactly one solution (f, α, β). Moreover,*

$$0 < \alpha < \frac{N-2}{nN-2}. \tag{1.10}$$

This theorem implies that for every value of n in the given range, there exists exactly one self-similar solution of equation (1.1) of the form (1.3).

The second result concerns the behaviour of the eigenvalues α and β, as given by Theorem A and equation (1.4), when we vary the parameter n. We indicate the dependence of α and β on n by writing $\alpha(n)$ and $\beta(n)$. Let $n_0 = 4/(N+2)$. We prove the following assertions:

Theorem B.

1. *$\alpha(n)$ and $\beta(n)$ depend continuously on n;*

2. *$\beta(n_0) = 0$; if $n > n_0$ then $\beta(n) > 0$, and if $n < n_0$ then $\beta(n) < 0$;*

3. *When $n \downarrow \frac{2}{N}$, then $\alpha(n) \to \infty$ and $\beta(n) \to -\infty$;*

4. *When $n \uparrow 1$, then $\alpha(n) \to 0$ and $\beta(n) \to \frac{1}{2}$.*

Theorem B can be interpreted in the following way. The parameter α determines the decay rate of the maximum of the solution. When n approaches one, $\alpha(n)$ tends to zero, implying that the decay of the solution near $t = T$ is very slow. On the other hand, when n tends to $\frac{2}{N}$, $\alpha(n)$ tends to infinity, signifying a very fast decay rate. The parameter β determines the spread of the profile. When $\beta < 0$, the profile of the solution spreads out as t approaches T, while for $\beta > 0$ the profile shrinks, all mass concentrating in the origin. Because $\beta(n_0) = 0$, the solution u for $n = n_0$ is separable, consisting of a fixed profile multiplied by the factor $(T-t)^{(N+2)/4}$. This situation is very similar to the one considered by Berryman and Holland in [4]. In Figure 1 the dependence of α and β on n is drawn for $N = 3$.

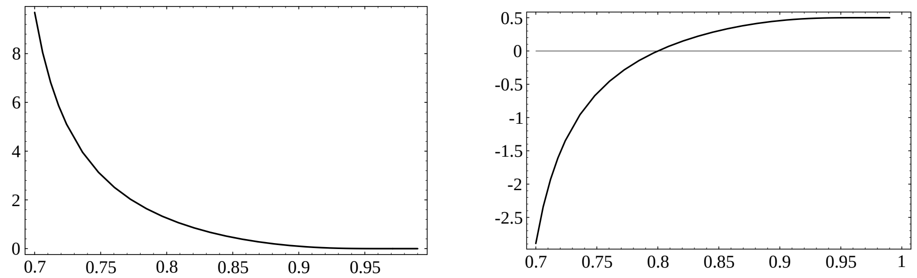

Figure 1: The dependence of α (left) and β on n, for $N = 3$ and $0.7 < n < 1$.

To prove these results we first consider in Section 2 an alternative formulation for Problem (P). In that section we also derive estimates and properties of solutions that will be used later. In Section 3 we prove the existence and uniqueness of solutions of Problem (P) (Theorem A), and in Section 4 we prove Theorem B.

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2 Preliminaries

Inspired by the analysis of J.R. King [12] we first transform equation (1.5) into a first-order autonomous system. This is the key step in our approach because it allows for an analysis in the phase plane. In particular we concentrate on the first order equation which holds along integral curves in the phase plane.

Let $f \in C^2((0,\infty)) \cap C^1([0,\infty))$ be a positive solution of Problem (P). Then introduce the functions $t, z : (0,\infty) \to \mathbb{R}$, defined by

$$t(\eta) = \tfrac{1}{2}\log(2\eta^{-2} f^{-n}(\eta)) \quad \text{and} \quad z(\eta) = -1 - \frac{n}{2}\frac{\eta f'(\eta)}{f(\eta)}.$$

They are well-defined for all $\eta > 0$, and $\{(t(\eta), z(\eta)) : 0 < \eta < \infty\}$ is a continuously differentiable curve in the t, z-plane. Remark that this curve is invariant under the scaling (1.9). Along the curve we have for $z \neq 0$

$$\frac{dz}{dt} = \left(\frac{2}{n} - 2\right) z - \left(N + 2 - \frac{4}{n}\right) - \left(N - \frac{2}{n}\right)\frac{1}{z} + e^{-2t}\left(\lambda + \frac{1}{z}\right). \tag{2.1}$$

The boundary conditions (1.6) imply

$$t \to \infty \quad \text{and} \quad z \to -1 \qquad \text{as} \quad \eta \downarrow 0, \tag{2.2}$$

and (1.8) yields

$$t \to \infty \quad \text{and} \quad z \to L \stackrel{\text{def}}{=} \frac{nN - 2}{2(1-n)} \qquad \text{as} \quad \eta \to \infty. \tag{2.3}$$

To summarise, every solution f of Problem (P) can be represented as a continuously differentiable orbit in the t, z-plane that satisfies (2.1) and connects the points $(\infty, -1)$ and (∞, L).

For brevity we introduce the notation

$$a = \frac{2}{n} - 2 \quad \text{and} \quad \lambda = 2\beta,$$

and write equation (2.1) as

$$\frac{dz}{dt} = \frac{a}{z}(z - L)(z + 1) + e^{-2t}\left(\lambda + \frac{1}{z}\right). \tag{2.4}$$

A solution of equation (2.4) is locally unique, since for every $(t, z) \in \mathbb{R}^2$, either dz/dt or dt/dz depends on t and z in a Lipschitz continuous manner.

We can immediately use this formulation to restrict the admissible values of λ:

Lemma 2.1 *Suppose there exists a $\lambda \in \mathbb{R}$ and a continuously differentiable orbit γ in the t, z-plane that satisfies (2.4) and connects the points $(\infty, -1)$ and (∞, L). Then*

$$-\frac{1}{L} < \lambda < 1.$$

Proof. We argue by contradiction. First suppose $\lambda > 1$. By the continuity of γ there exists $(t_0, z_0) \in \gamma$ such that

$$z_0 = -\frac{1}{\lambda} \quad \text{and} \quad \frac{dz}{dt} \leq 0 \quad \text{in } (t_0, z_0),$$

which contradicts equation (2.4). If $\lambda = 1$, then the line $z = -1$ is a solution curve of (2.4); we will prove in Lemma 3.2 that for fixed values of λ, an orbit with behaviour (2.2) is unique. In a similar fashion one proves the lower bound: here the contradiction is also on the line $\{z = -1/\lambda\}$, but with the crossing in the other direction. •

Solution curves that satisfy (2.4) have a simple structure. This is the content of the following lemma.

Lemma 2.2 *If f is a solution of Problem (P), and γ is the corresponding orbit in the t, z-plane, then γ intersects the t-axis exactly once. Furthermore, there exist functions $z_+(t)$ and $z_-(t)$, such that $z_+ \geq 0$ and $z_- \leq 0$, and that*

$$\gamma = \{(t, z) : z = z_+(t)\} \cup \{(t, z) : z = z_-(t)\}.$$

It follows immediately from the preceding remarks that the functions z_+ and z_- satisfy (2.4).

Proof. We can write the isocline $\Gamma = \{(t, z) : dz/dt = 0\}$ as the union of $\Gamma_+ = \{z = \phi_+(t)\}$ and $\Gamma_- = \{z = \phi_-(t)\}$, where the functions $\phi_\pm$ are given by

$$\phi_\pm(t) = -\frac{1 - L}{2} + \frac{\lambda}{2a} e^{-2t} \pm \frac{1}{2} \sqrt{\left(1 - L + \frac{\lambda}{a} e^{-2t}\right)^2 + 4\left(L - \frac{1}{a} e^{-2t}\right)}.$$

The phase plane is drawn in Figure 2; we should remark that $\phi'_+ > 0$ and $\phi'_- < 0$, and that

$$\lim_{t \to \infty} \phi_+(t) = L \quad \text{and} \quad \lim_{t \to \infty} \phi_-(t) = -1.$$

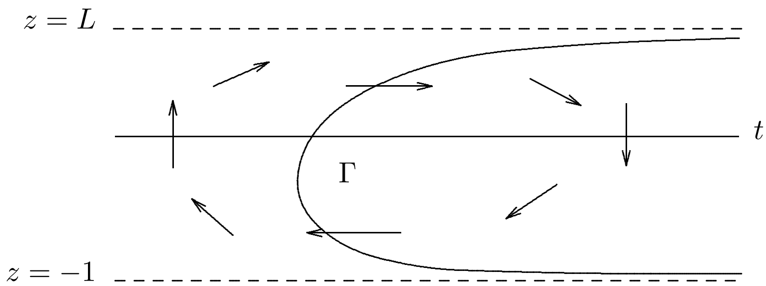

Figure 2: The phase plane

From the vector field in Figure 2 and the limiting behaviour (2.2) and (2.3) we can deduce that an orbit with more than one intersection with the t-axis has to intersect itself. This is ruled out by the local uniqueness. Any solution curve can therefore be split into two parts, one above the t-axis, and one below. Since dz/dt is finite whenever z is non-zero, the two parts can each be represented by a single-valued function of t, as in Figure 3. •

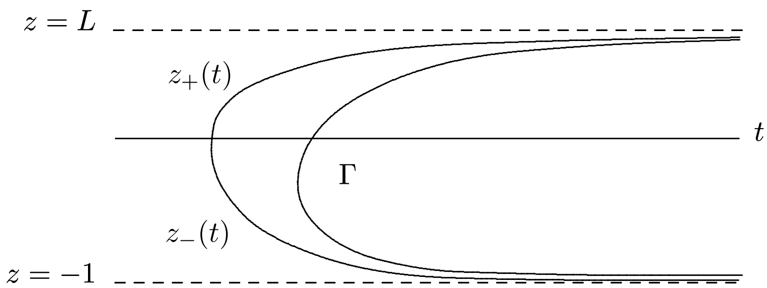

Figure 3: A typical solution

The following lemma describes how the functions z_+ and z_- approach their limits as t tends to infinity.

Lemma 2.3 *1. Let z_+ satisfy equation (2.4) and the asymptotic behaviour $z_+(t) \to L$ as $t \to \infty$. Then*

$$z_+(t) = L - Ae^{-2t} + x(t)$$

where

$$A = \frac{n}{2}\frac{1 + \lambda L}{L + 1 - n} \quad \textit{and} \quad x(t) = O\left(e^{-4t}\right) \quad \textit{as} \quad t \to \infty.$$

2. *Let z_- satisfy equation (2.4) and the asymptotic behaviour $z_-(t) \to -1$ as $t \to \infty$. Then*

$$z_-(t) = -1 + Be^{-2t} + y(t)$$

where

$$B = \frac{1-\lambda}{N} \quad \text{and} \quad y(t) = O\left(e^{-4t}\right) \quad \text{as} \quad t \to \infty.$$

Proof. We only prove the first part; the proof of the second part is similar. First remark that the isocline $\phi_+(t)$ tends to L as $L - a^{-1}e^{-2t}$, and that therefore $L - z_+$ tends to zero at least as fast as $a^{-1}e^{-2t}$. Set

$$z_+(t) = L - Ae^{-2t} + x(t) \tag{2.5}$$

and define $q(t) = x(t)e^{2t}$. By the previous remark, q remains bounded as t tends to infinity. Using (2.5) in (2.4) we find the following equation for q:

$$\begin{aligned} q' &= \left(a\frac{L+1}{L} + 2\right) q + (aq - aA + 1)\frac{L - z_+}{z_+ L}. \\ &= \kappa q + \mu(t). \end{aligned} \tag{2.6}$$

Remark that since $z_+(t) \to L$ as fast as e^{-2t}, $|\mu(t)| \le Ce^{-2t}$ for some constant C. Equation (2.6) implies

$$q(t) = -e^{\kappa t}\int_t^\infty e^{-\kappa s}\mu(s)\,ds,$$

and thus

$$|q(t)| \le \frac{C}{\kappa + 2}e^{-2t}.$$

•

We now are in a position to prove the equivalence of the two formulations that we have discussed so far.

Lemma 2.4 *With every solution f of Problem (P) correspond functions z_+ and z_-, such that*

1. *z_+ and z_- are defined on $[T, \infty)$ for some $T \in \mathbb{R}$;*

2. *z_+ and z_- satisfy (2.4) on (T, ∞), and $z_+(T) = z_-(T) = 0$;*

3. *$z_+(t) \to L$ and $z_-(t) \to -1$ as $t \to \infty$.*

Conversely, every pair of continuously differentiable functions z_+ and z_- that satisfies the above conditions defines a solution f of Problem (P).

In what follows, we shall refer to z_+ as the *upper solution* and to z_- as the *lower solution* of equation (2.4).

Proof. The first assertion was shown in Lemma 2.2; we only need to prove the inverse case. First let us remark that we can choose a parametrisation $(\tilde{t}(\xi), \tilde{z}(\xi))$ of the union of the two curves $S = \{(t, z) : z = z_+(t)\} \cup \{(t, z) : z = z_-(t)\}$, in such a way that

$$\tilde{t}(0) = T \quad \text{and} \quad \tilde{z}(0) = 0; \tag{2.7}$$

$$\tilde{z}(\xi) = z_+(\tilde{t}(\xi)) \quad \text{if } \xi \geq 0, \text{ and } \quad \tilde{z}(\xi) = z_-(\tilde{t}(\xi)) \quad \text{if } \xi \leq 0; \tag{2.8}$$

$$\tilde{t}'(\xi) = \tilde{z}(\xi). \tag{2.9}$$

Indeed, with any point $(\tau, \zeta) \in S$ we associate the parameter value ξ as follows:

$$\xi = \begin{cases} \displaystyle\int_T^\tau \frac{ds}{z_+(s)} & \text{if } \zeta \geq 0 \\ \displaystyle\int_T^\tau \frac{ds}{z_-(s)} & \text{if } \zeta < 0. \end{cases} \tag{2.10}$$

From equation (2.4) we deduce that

$$\lim_{t \downarrow T} \frac{d}{dt} z_+^2(t) = \lim_{t \downarrow T} \frac{d}{dt} z_-^2(t) = -2aL + 2e^{-2T} \tag{2.11}$$

which implies that $1/z_+(t)$ and $1/z_-(t)$ are integrable near $t = T$. Therefore the integrals in (2.10) are well defined. Observation (2.11) also implies that the orbit thus obtained is continuously differentiable for all $\xi \in \mathbb{R}$.

We can then construct the solution f of Problem (P) by defining

$$\eta = e^\xi \quad \text{and} \quad f(\eta)^n = 2e^{-2\tilde{t}(\xi)} \eta^{-2}. \tag{2.12}$$

From differentiation of (2.12) it follows that f is a solution of equation (1.5). It remains to prove that boundary conditions (1.6) and (1.8) are satisfied. It follows from (2.12) that

$$\frac{d}{d\eta}(f^n)(\eta) = -4\eta^{-3}(1 + \tilde{z}(\xi)) e^{-2\tilde{t}(\xi)}.$$

Using the limiting behaviour of z_- (Lemma 2.3) we find that $f'(0) = 0$. This proves (1.6). For the boundary condition at $\eta = \infty$, we calculate

$$\eta^{\frac{n(N-2)}{1-n}} f(\eta)^n = 2e^{2(L\xi - \tilde{t}(\xi))}.$$

The limiting behaviour of z_+ implies that

$$\frac{d}{d\xi}\left(L\xi - \tilde{t}(\xi)\right) = L - \tilde{z}(\xi) \leq 2Ae^{-2\tilde{t}(\xi)} \leq 2Ae^{-L\xi},$$

where the second inequality is true if ξ is large enough, and therefore

$$\lim_{\xi\to\infty}\left(L\xi - \tilde{t}(\xi)\right)$$

is finite. This concludes the proof. $\bullet$

3 Existence and uniqueness

Problem (P) is a nonlinear eigenvalue problem: the number β is to be determined together with the solution function f. In this section we prove the following theorem:

Theorem 3.1 *For every $N > 2$ and $\frac{2}{N} < n < 1$, Problem (P) has exactly one solution (f, α, β). Moreover,*

$$-\frac{1-n}{nN-2} < \beta < \frac{1}{2}. \tag{3.1}$$

Note that statement (3.1) is an immediate consequence of Lemma 2.1.

By Lemma 2.4, the assertion of Theorem 3.1 is equivalent to the existence and uniqueness of a number $\lambda \in \mathbb{R}$ and functions z_+ and z_- as described in the Lemma. The proof shall proceed as follows: for every $-1/L < \lambda < 1$ we show that there exist functions z_+ and z_-, solutions of (2.4), which have the prescribed behaviour at $t = \infty$. Both functions intersect the t-axis, but in general at different values of t. For exactly one value of λ, the two half-orbits connect in a continuous way, and therefore define a solution of Problem (P).

Lemma 3.2 *For every $-1/L < \lambda < 1$, the following statements hold:*

1. *There exist unique solutions $z_+(t)$ and $z_-(t)$ of (2.4), defined for t large enough, such that*

$$z_+(t) \to L \quad \text{and} \quad z_-(t) \to -1$$

   as t tends to infinity;

2. *The solutions z_+ and z_- can be uniquely continued for decreasing t as long as they remain non-zero.*

Proof. If we choose t_0 sufficiently large, then $\phi_+(t_0) > 0$, and the part of the phase plane to the right of $t = t_0$ will have a structure as shown in Figure 4.

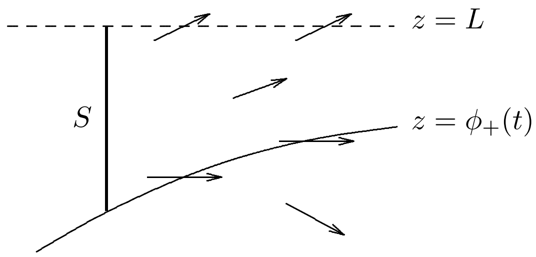

Figure 4: The existence proof

Let $\gamma_+(\tau, \zeta)$ denote the orbit in the t, z-plane that starts in (τ, ζ) and continues for increasing t. Define the set $S = \{(t, z) : t = t_0,\ \phi_+(t_0) \leq z \leq L\}$ and the subsets

$$\begin{aligned} S_1 &= \{(t, z) \in S : \gamma_+(t, z) \text{ intersects the line } z = L\} \\ S_2 &= \{(t, z) \in S : \gamma_+(t, z) \text{ intersects the curve } z = \phi_+(t)\}. \end{aligned}$$

By a classical argument it can be shown that S_1 and S_2 are disjoint, and that both are non-empty and open relative to S. It follows that there exists an $s \in S \setminus (S_1 \cup S_2)$. The orbit $\gamma_+(s)$ then remains between $z = \phi_+(t)$ and $z = L$ for all $t > t_0$. Let z_+ be defined by

$$\gamma_+(s) = \{(t, z) : z = z_+(t),\ t \geq t_0\};$$

since $\phi_+(t) \to L$ as t tends to infinity, it follows that $z_+(t) \to L$ as $t \to \infty$ as well.

To prove the uniqueness of z_+, consider two solutions z_+ and $\bar{z}_+$, and suppose that $\bar{z}_+ > z_+$ on $t > t_0$ (local uniqueness does not permit that solution curves intersect). If we subtract the equations (2.4) for z_+ and $\bar{z}_+$ and integrate the result from $t_1 > t_0$ to $t_2 > t_1$, we find that

$$\left[\bar{z}_+^2 - z_+^2\right]_{t_1}^{t_2} \geq \tfrac{1}{2} a \int_{t_1}^{t_2} (\bar{z}_+^2 - z_+^2)\, dt,$$

provided t_1 is large enough. Letting t_2 tend to infinity yields

$$-\left\{\bar{z}_+(t_1)^2 - z_+(t_1)^2\right\} \geq \tfrac{1}{2} a \int_{t_1}^{\infty} (\bar{z}_+^2 - z_+^2)\, dt.$$

Hence $\bar{z}_+$ and z_+ are equal on $t \geq t_1$.

Because solutions of (2.4) are locally unique as long as z remains non-zero, we can continue z_+ for decreasing t in a unique manner as long as $z_+(t) > 0$.

This proves the theorem as far as z_+ is concerned. The result for z_- is derived in a similar way. •

The uniqueness shown above implies that when $\lambda = -1/L$, the only orbit in the phase plane for which z tends to L as $\xi \to \infty$ is the line $z = L$. Obviously, this orbit can never match up with a lower solution z_-. In a similar way, a solution can not have $\lambda = 1$, either. This proves the strictness of the inequalities of Lemma 2.1.

Define the functions $T_+(\lambda)$ and $T_-(\lambda)$ as follows:

$$T_\pm(\lambda) = \inf\{t \in \mathbb{R} : z_\pm(t) > 0\}.$$

A priori these functions need not be finite, and there is no reason why $T_+(\lambda)$ should be equal to $T_-(\lambda)$ for any λ. The next Lemma leads the way to the conclusion that there exists exactly one value of λ such that $T_+(\lambda) = T_-(\lambda)$.

Lemma 3.3 *1. For all $-1/L < \lambda < 1$, $T_+(\lambda)$ and $T_-(\lambda)$ are finite;*

2. *T_+ is a strictly increasing function of λ, and T_- a strictly decreasing one;*

3. *We have the following upper bounds:*

$$T_+(\lambda) \le \hat{T}_+(\lambda) \quad \textit{and} \quad T_-(\lambda) \le \hat{T}_-(\lambda),$$

where $\hat{T}_+$ and $\hat{T}_-$ are defined by

$$e^{-2\hat{T}_+(\lambda)} = \begin{cases} \dfrac{2}{\lambda^2}\{\lambda L - \log(1+\lambda L)\} & \textit{for } \lambda \neq 0 \\ L^2 & \textit{for } \lambda = 0 \end{cases}$$

$$e^{-2\hat{T}_-(\lambda)} = \begin{cases} -\dfrac{2}{\lambda^2}\{\lambda + \log(1-\lambda)\} & \textit{for } \lambda \neq 0 \\ 1 & \textit{for } \lambda = 0; \end{cases}$$

4. $\begin{cases} T_+(\lambda) \to -\infty \ \textit{as} \ \lambda \downarrow -\dfrac{1}{L}; \\ T_-(\lambda) \to -\infty \ \textit{as} \ \lambda \uparrow 1; \end{cases}$

5. *$T_+(\lambda)$ and $T_-(\lambda)$ are continuous in λ.*

Proof. We shall only prove the assertions for T_+, as the extension to T_- is straightforward. Assume the converse of part 1 of the lemma: $z_+(t)$ exists and is positive for all $t \in \mathbb{R}$. Since $z_+'(t) > 0$ for all t, this implies $z_+'(t) \downarrow 0$ as $t \to -\infty$. This contradicts equation (2.4).

For part two, suppose that $T_+(\lambda_1) \ge T_+(\lambda_2)$ while $\lambda_1 < \lambda_2$. From Lemma 2.3 we conclude that for t_0 large enough, $z_+(t_0, \lambda_1) > z_+(t_0, \lambda_2)$. Between $t = T_+(\lambda_1)$ and $t = t_0$, the solutions $z_+(t, \lambda_1)$ and $z_+(t, \lambda_2)$ must intersect in such a way that

$$\frac{d}{dt} z_+(t, \lambda_1) \ge \frac{d}{dt} z_+(t, \lambda_2).$$

Again we find a contradiction with equation (2.4).

Part three is proved by considering the solution ζ of the problem

$$\begin{cases} \zeta'(t) = e^{-2t}\left(\lambda + \dfrac{1}{\zeta(t)}\right) & \text{for } t \in \mathbb{R} \\ \zeta(\infty) = L. \end{cases}$$

The function ζ can be calculated explicitly:

$$e^{-2t} = \frac{2}{\lambda^2}\left\{\lambda(L - \zeta(t)) + \log\frac{1+\lambda\zeta(t)}{1+\lambda L}\right\}. \tag{3.2}$$

From (3.2) we calculate that ζ tends to L as $L - (1+\lambda L)/(2L)e^{-2t}$, which implies by Lemma 2.3 that $z_+(t) > \zeta(t)$ for large t. Then, if t_0 is the largest value of t for which the graphs of z_+ and ζ intersect, we have $z_+'(t_0) \geq \zeta'(t_0)$. This is contradicted by equation (2.4).

Part four follows from the observation that

$$\lim_{\lambda\downarrow -1/L} \frac{2}{\lambda^2}\left\{\lambda L - \log(1+\lambda L)\right\} = \infty.$$

To prove the continuity of T_+ with respect to λ, suppose that for some $-1/L < \lambda_0 < 1$, $\ell_\ell = \lim_{\lambda\uparrow\lambda_0} T_+(\lambda)$ and $\ell_r = \lim_{\lambda\downarrow\lambda_0} T_+(\lambda)$ do not coincide. In proving part 2 of this Lemma we not only showed that T_+ decreases, but also that the function $z_+(t)$ increases when λ increases. We can therefore define the limit functions $w_\ell(t) = \lim_{\lambda\uparrow\lambda_0} z_+(\lambda, t)$ and $w_r(t) = \lim_{\lambda\downarrow\lambda_0} z_+(\lambda, t)$. By considering a weak formulation of (2.4) and passing to the limit in λ, we find that w_ℓ and w_r both satisfy equation (2.4) for $\lambda = \lambda_0$. Since they both lie between $z \equiv L$ and $z = \phi_+(\lambda_0, t)$, and therefore they both tend to L as $t \to \infty$, this is in contradiction with the uniqueness of z_+. •

To conclude the proof of Theorem 3.1, let us draw T_+ and T_- in one diagram (Figure 5). Lemma 3.3 guarantees that there is exactly one value of λ such that $T_+(\lambda) = T_-(\lambda)$. For this value of λ, z_+ and z_- match up continuously at $t = T_\pm(\lambda)$. Using Lemma 2.4 we conclude that there exists exactly one solution (f, α, β) of Problem (P). •

4 Qualitative properties.

In the previous section we have proved that for every value of n between $\frac{2}{N}$ and 1, there exists exactly one solution (f, α, β) of Problem (P). In this section we study the behaviour of β, or equivalently, $\lambda = 2\beta$, as we vary n. We will write $\lambda^*(n)$ for the value of λ given by Theorem 3.1.

First we prove continuity of λ^* with respect to n.

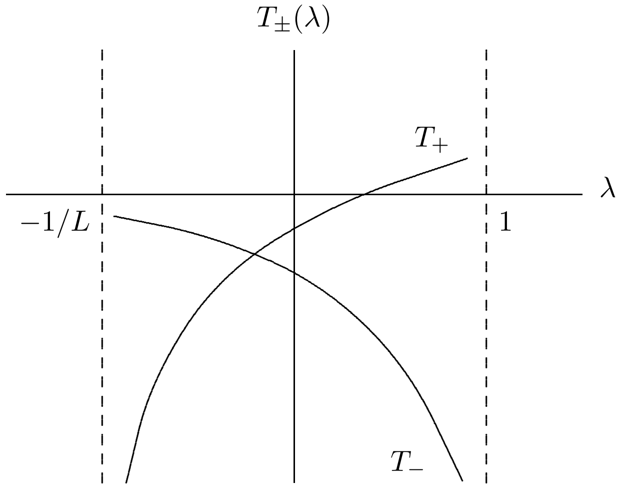

Figure 5: The functions T_+ and T_-

Lemma 4.1 *λ^* is a continuous function of n.*

Proof. In Lemma 3.3 it was proved that for fixed n, $T_+(\lambda)$ and $T_-(\lambda)$ are continuous functions of λ. One can extend this result in a straightforward way to state that the functions T_+ and T_- are continuous in the variable pair (λ, n) for all (λ, n) in the appropriate range.

Now suppose that λ^* is discontinuous in $\tilde{n}$. Then we can choose a sequence $\{n_i\}$ converging to $\tilde{n}$ such that $\lambda_i = \lambda^*(n_i) \to \lambda_1 \neq \lambda^*(\tilde{n})$. Therefore, by definition, $T_+(\lambda_i, n_i) = T_-(\lambda_i, n_i)$, and

$$0 = \lim_{i\to\infty} \{T_+(\lambda_i, n_i) - T_-(\lambda_i, n_i)\} = T_+(\lambda_1, \tilde{n}) - T_-(\lambda_1, \tilde{n}),$$

which implies that there exists a solution (f_1, λ_1) other than the one given by Theorem 3.1. This is contradicted by the uniqueness. •

It has been known for some time (see [16] or [11]) that when n equals

$$n_0 \stackrel{\text{def}}{=} \frac{4}{N+2},$$

the solution of Problem (P) can be calculated explicitly:

$$f(\eta) = \left(1 + \frac{\eta^2}{4N}\right)^{-\frac{1}{2}N-1}.$$

By substituting f into equation (1.5) one finds that $\lambda^*(n_0) = 0$. The values of λ^* are ordered with respect to $n = n_0$:

Lemma 4.2 *If $n < n_0$ then $\lambda^*(n) < 0$, and if $n > n_0$ then $\lambda^*(n) > 0$.*

Proof. Suppose that $n < n_0$; when $n > n_0$ the argument is similar. We shall show that $T_+(0) > T_-(0)$. This implies by the monotonicity of T_+ and T_- that $\lambda^* < 0$ (see Figure 5).

Let z_+ and z_- be the upper and lower solution of equation (2.4) in which we have set $\lambda = 0$. Then $z_-(t) \to -1$ as $t \to \infty$. Set $\hat{z}_- = -z_-$. Then $\hat{z}_-(t) \to 1$ as $t \to \infty$, and since $L < 1$ because $n < n_0$, it follows that $\hat{z}_-(t) > z_+(t)$ when t is sufficiently large. Plainly, it is enough to show that $\hat{z}_-(t) > z_+(t)$ for all $T_+(0) < t < \infty$.

To prove that this is indeed the case, suppose to the contrary that

$$\tau = \inf\{t > T_+(0) : \hat{z}_- > z_+ \text{ on } (t,\infty)\} > T_+(0).$$

Then

$$\hat{z}_-(\tau) = z_+(\tau) \quad \text{and} \quad \hat{z}'_-(\tau) \geq z'_+(\tau). \tag{4.1}$$

Hence, from (2.4) we deduce that at $t = \tau$,

$$\begin{aligned}
\hat{z}'_- &= a\hat{z}_- - a(1-L) - (aL - e^{-2\tau})\frac{1}{\hat{z}_-} \\
&< a\hat{z}_- + a(1-L) - (aL - e^{-2\tau})\frac{1}{\hat{z}_-} \\
&= az_+ + a(1-L) - (aL - e^{-2\tau})\frac{1}{z_+} = z'_+,
\end{aligned}$$

which contradicts (4.1). •

The ordering given by Lemma 4.2 has an important consequence for the behaviour of the solution u. When $n > n_0$, $\lambda^*(n) > 0$, which is equivalent with $\beta(n) > 0$, and therefore the solution u given by (1.3) contracts as t approaches T. When $n < n_0$, the profile of u spreads out when $t \to T$. When n equals n_0, η is in fact equal to $|x|$, and the solution u is given by

$$u(x,t) = (T-t)^{(N+2)/4}\left(1 + \frac{|x|^2}{4N}\right)^{-(N+2)/2}.$$

The remainder of this section is devoted to the calculation of the two limits

$$\lim_{n\downarrow\frac{2}{N}} \lambda^*(n) \quad \text{and} \quad \lim_{n\uparrow 1} \lambda^*(n).$$

To simplify the notation, we shall drop the superscript '$*$' from λ^*, and write $T = T(\lambda(n)) = T(n)$ for the common vanishing point of z_+ and z_-.

First we consider the limit process $n \downarrow \frac{2}{N}$. Recall that

$$a = \frac{2}{n} - 2 > 0 \quad \text{and} \quad L = \frac{nN-2}{2(1-n)} > 0.$$

Hence $n \downarrow \frac{2}{N}$ implies that

$$a \uparrow \underline{a} \stackrel{\text{def}}{=} N - 2 \quad \text{and} \quad L \downarrow 0.$$

Therefore the upper half of the phase plane 'collapses': the line $z = L$ descends to zero. In addition, the isocline $\phi_+(t)$ vanishes for the value of t given by

$$aL = e^{-2t}, \tag{4.2}$$

and this vanishing point clearly 'runs off' to plus infinity when $n \downarrow \frac{2}{N}$. These observations suggest the following scaling of z_+:

$$e^{-2t} = aLe^{-2\sigma} \iff \sigma = t + \tfrac{1}{2}\log(aL) \quad \text{and} \quad w(\sigma) = \frac{1}{L} z_+(t). \tag{4.3}$$

For every n, the function w tends to 1 as σ tends to infinity, and satisfies the following equation (in which primes denote differentiation with respect to σ):

$$\frac{L}{a} ww' = (Lw+1)(w-1) + e^{-2\sigma}(1-\gamma w), \tag{4.4}$$

where we have written γ for $-\lambda L$. Note that by Lemmas 2.1 and 4.2, $0 < \gamma < 1$. The coefficient of the derivative w' in (4.4) tends to zero as $n \downarrow \frac{2}{N}$. We therefore introduce a second scaling of the independent variable. Define Σ as the vanishing point of w (the analogue of T in the variable σ):

$$\Sigma = T + \tfrac{1}{2}\log(aL),$$

and set

$$\sigma = \Sigma + aL\tau \quad \text{and} \quad x(\tau) = w(\sigma).$$

We find that x satisfies the following equation (where the prime now denotes differentiation with respect to τ):

$$\frac{1}{a^2} xx' = (Lx+1)(x-1)e^{-2\Sigma - 2aL\tau}(1-\gamma x) \quad \text{for } \tau > 0. \tag{4.5}$$

Before we can continue with this equation, we have to consider the lower part of the phase plane. We shall see later that $\gamma \to 1$, and therefore $\lambda = -\gamma/L \to -\infty$. We can rid equation (2.4) of this parameter blow-up by introducing a scaling of z_- which is different from the one we use for z_+:

$$-\lambda e^{-2t} = e^{-2s} \iff s = t - \tfrac{1}{2}\log(-\lambda) \quad \text{and} \quad y(s) = z_-(t),$$

which results in the equation

$$yy' = a(y - L)(y + 1) - e^{-2s}\left(y + \frac{1}{\lambda}\right). \tag{4.6}$$

We also define

$$S = T - \tfrac{1}{2}\log(-\lambda).$$

Note that S and Σ are linked in the following way:

$$S = \Sigma - \tfrac{1}{2}\log(a\gamma). \tag{4.7}$$

Now we are in a position to formulate our result. To facilitate the notation, the functions x and y are defined equal to zero outside of their domain of definition.

Lemma 4.3 *Let $n \downarrow \frac{2}{N}$. Then*

1. $\gamma \to 1$*;*
2. $S \to \underline{S} = -\frac{1}{2}\log N$*, which is equivalent to* $\Sigma \to \underline{\Sigma} = \frac{1}{2}\log\dfrac{N-2}{N}$*;*
3. *x tends to the solution of the problem*
$$\begin{cases} \underline{x}\,\underline{x}' = \dfrac{2}{N-2}(1 - \underline{x}) & \textit{for } \tau > 0 \\ \underline{x}(\tau) = 0 & \textit{for } \tau \le 0; \end{cases}$$
4. *y tends to the limit function $\underline{y}$ given by*
$$\underline{y}(s) = \begin{cases} -1 + \dfrac{1}{N}e^{-2s} & \textit{for } s > \underline{S} \\ 0 & \textit{for } s \le \underline{S}; \end{cases} \tag{4.8}$$

Here the convergence of x and y is uniform on compact subsets of the real line.

Proof. In the same way as the existence and uniqueness of solutions was shown by matching the upper half of the phase plane with the lower half, we prove this lemma by studying, separately, first the functions y and then the functions x, and then combining the results.

Step 1: The lower half of the phase plane. Throughout step one *we shall assume that λ is bounded away from zero as $n \downarrow \frac{2}{N}$.* In step two we shall prove, independently of these results, that $\gamma \to 1$ and therefore $\lambda = -\gamma/L \to -\infty$, thereby justifying this assumption.

First we prove that S can not tend to plus infinity as $n \downarrow \frac{2}{N}$. This follows from Lemma 3.3, in the following way:

$$e^{-2S} = -\lambda e^{-2T} \geq -\lambda e^{-2\hat{T}_-(\lambda)} = 2 + \frac{2}{\lambda}\log(1-\lambda),$$

and since we assume that λ stays bounded away from zero, this last expression is positive and bounded away from zero. This implies that S is bounded from above.

Now choose a sequence $\{n_i\}$, converging to $\frac{2}{N}$, such that $S \to \underline{S} \in [-\infty, \infty)$ and $\lambda \to \underline{\lambda} \in [-\infty, 0]$ along that sequence. Equation (4.6) implies that the sequence of functions y^2 is equicontinuous. By the Arzela-Ascoli theorem we can extract a subsequence of $\{n_i\}$ such that y^2 converges uniformly on compact subsets of $\mathbb{R}$ along that subsequence. The same holds for the sequence of functions y, because the function $t \mapsto \sqrt{t}$ is uniformly continuous, and the limit function $\underline{y}$ is continuous on $\mathbb{R}$. We integrate equation (4.6) from $s_1 > \underline{S}$ to $s_2 > s_1$:

$$\tfrac{1}{2}y^2(s_2) - \tfrac{1}{2}y^2(s_1) = a\int_{s_1}^{s_2} \left\{(y-L)(y+1) - e^{-2\tilde{s}}\left(y + \frac{1}{\lambda}\right)\right\} d\tilde{s},$$

and by passing to the limit we deduce that the limit function $\underline{y}$ satisfies

$$\begin{cases} \underline{y}\underline{y}' = \underline{a}\underline{y}(\underline{y}+1) - e^{-2s}\left(\underline{y} + \frac{1}{\underline{\lambda}}\right) & \text{for } s > \underline{S} \\ \underline{y}(s) = 0 & \text{for } s \leq \underline{S} \text{ (if } \underline{S} > -\infty). \end{cases} \tag{4.9}$$

If, for the moment, we assume that $\lambda \to \infty$, then we can integrate the equation for $\underline{y}$ to obtain

$$\underline{y}(s) = \begin{cases} -1 + \dfrac{1}{N}e^{-2s} & \text{for } s > \underline{S} \\ 0 & \text{for } s \leq \underline{S}. \end{cases} \tag{4.10}$$

The continuity of $\underline{y}$ implies that $\underline{S}$ is equal to either $-\infty$ or $-\frac{1}{2}\log N$. Since all y are positive, the former is ruled out. We conclude that $S \to -\frac{1}{2}\log N$.

Step 2: The upper half of the phase plane. For all n, the solution z_+ lies above the isocline ϕ_+, and therefore (4.2) implies that $T \leq -\frac{1}{2}\log(aL)$, or $\Sigma \leq 0$. Choose a sequence $\{n_i\}$, converging to $\frac{2}{N}$, such that $\Sigma \to \underline{\Sigma} \in [-\infty, 0]$ along that sequence. Then integrate (4.5) from $\tau_1 > 0$ to $\tau_2 > \tau_1$:

$$\frac{1}{2a^2}\left(x^2(\tau_2) - x^2(\tau_1)\right) = -\int_{\tau_1}^{\tau_2} (Lx+1)(1-x)\,d\tau - e^{-2\Sigma}\int_{\tau_1}^{\tau_2} e^{-2aL\tau}(\gamma x-1)\,d\tau. \tag{4.11}$$

First we use (4.11) to prove that Σ does not tend to minus infinity. Suppose that it does. Then $e^{-2\Sigma}$ becomes very large, while the first two terms in (4.11) remain bounded. Since x and γ both are less than or equal to one, this implies that $\gamma \to 1$ as $n \downarrow \frac{2}{N}$. But then $\lambda = -\gamma/L$ tends to minus infinity, and we have previously calculated that in that case S tends to $-\frac{1}{2}\log N$. Using (4.7), we find that

$$\Sigma \to -\tfrac{1}{2}\log N + \tfrac{1}{2}\log \underline{a} > -\infty,$$

which contradicts the assumption. Therefore Σ is bounded from below (and also from above, since $\Sigma \le 0$).

It follows that γ can not tend to zero. For if it did, then using (4.7) and the boundedness of Σ, S would tend to plus infinity, a contradiction. This implies that $\lambda = -\gamma/L$ indeed tends to infinity.

To prove the convergence of x, we pass to the limit in equation (4.11). Since γ is bounded between zero and one, we can extract a subsequence such that $\gamma \to \underline{\gamma} \in (0,1]$. We find the following differential equation for the limit function $\underline{x}$:

$$\begin{cases} \frac{1}{\underline{a}^2}\underline{x}\,\underline{x}' = -(1-\underline{x}) - e^{-2\underline{\Sigma}}(\underline{\gamma}\underline{x} - 1) & \text{for } \tau > 0 \\ \underline{x}(\tau) = 0 & \text{for } \tau \le 0. \end{cases} \tag{4.12}$$

It follows from (4.12) that

$$\underline{x} \to \frac{e^{-2\underline{\Sigma}} - 1}{\underline{\gamma}e^{-2\underline{\Sigma}} - 1} \quad \text{as} \quad \tau \to \infty. \tag{4.13}$$

Note that

$$\Sigma \to -\tfrac{1}{2}\log N + \tfrac{1}{2}\log(\underline{a}\underline{\gamma}) \le \tfrac{1}{2}\log\frac{N-2}{N} < 0,$$

and therefore $e^{-2\underline{\Sigma}} > 1$. The limit value (4.13) can only be less or equal to one (as is necessary, since all x are less or equal to one) if $\underline{\gamma} = 1$.

A final remark to conclude the proof. We have liberally taken subsequences to arrive at this result. Because of the exact characterisation of the limit functions $\underline{x}$ and $\underline{y}$ and of the limit value 1 of γ, however, the assertions automatically apply to any sequence such that $n \downarrow \frac{2}{N}$. •

In Figure 6 the convergence of γ towards 1 is plotted from numerical calculations.

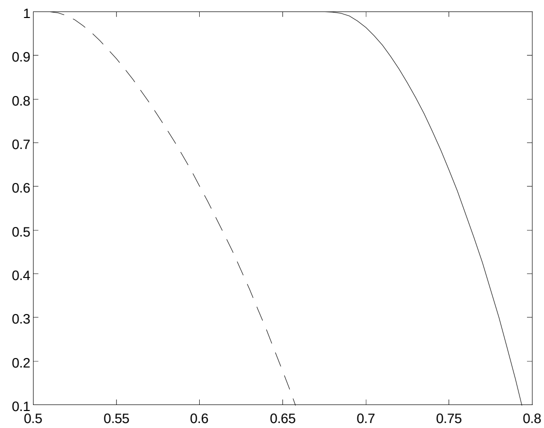

Figure 6: Plot of γ against n, for two values of N. The continuous line is for $N = 3$, the dashed line for $N = 4$.

If we translate the results of Lemma (4.3) back in terms of z_+ and z_-, we find the following statements:

Theorem 4.4 *Let* $n \downarrow \frac{2}{N}$, *and write* $\varepsilon(n) = n - \frac{2}{N}$. *Then*

1. $\varepsilon(n)\lambda(n) \to -2\dfrac{N-2}{N^2}$;
2. $T(n) + \frac{1}{2}\log\varepsilon(n) \to \frac{1}{2}\log 2\dfrac{N-2}{N^3}$;
3. $\dfrac{1}{\varepsilon(n)}\, z_+\left(T(n) + \varepsilon(n)\dfrac{N}{n}\tau\right) \to \dfrac{N^2}{2(N-2)}\,\underline{x}(\tau)$ *for all* $\tau \in \mathbb{R}$;
4. $z_-\left(s + \frac{1}{2}\log(-\lambda(n))\right) \to \underline{y}(s)$ *for all* $s \in \mathbb{R}$.

The convergence is uniform on compact subsets of $\mathbb{R}$ *in the variables* τ *and* s.

Let us now direct our attention towards the other limit, $n \uparrow 1$. As n approaches 1, the parameter $a = \frac{2}{n} - 2$ tends to zero and $L = (nN-2)/2(1-n) \to \infty$. Note that $aL \to N - 2$. In the previous limit, λ converged to its lower bound ($-\infty$); here we therefore expect λ to tend to its upper bound, one. We shall show that this is indeed the case.

Since $n \geq n_0$, the values of λ are confined to the interval $[0, 1]$, and we can choose a sequence $\{n_i\}$, converging to one, such that $\lambda \to \bar{\lambda} \in [0, 1]$ along that sequence. When L tends to infinity, $\hat{T}_+(\lambda)$—as defined in Lemma 3.3—tends to minus infinity uniformly in λ, thereby forcing $T_+(\lambda) = T_-(\lambda)$ to minus infinity, too. We shall write $T = T(n) = T_\pm(\lambda(n))$.

With this remark in mind we introduce the following variable transformations:

$$\alpha = e^{2T}, \quad t = T + \alpha\sigma, \quad \text{and} \quad y(\sigma) = z_-(t).$$

This leads to

$$yy' = \alpha a(y - L)(y + 1) + e^{-2\alpha\sigma}(\lambda y + 1) \quad \text{for} \quad \sigma > 0, \tag{4.14}$$

while $y(0) = 0$. Define $y(\sigma) = 0$ for all $\sigma < 0$, too. Equation (4.14) implies that the sequence of functions y^2 is equicontinuous as $n \uparrow 1$. The Arzela-Ascoli theorem then implies that we can extract a subsequence such that the functions y^2, and therefore also the functions y, converge uniformly on compact sets. The limit function $\bar{y}$ is continuous and by passing to the limit in the equivalent integral equation we find that $\bar{y}$ satisfies

$$\begin{cases} \bar{y}' = \bar{\lambda} + \dfrac{1}{\bar{y}} & \text{for } \sigma > 0 \\ \bar{y}(\sigma) = 0 & \text{for } \sigma \leq 0. \end{cases} \tag{4.15}$$

From (4.15) it follows that the limit function $\bar{y}$ tends to $-\bar{\lambda}^{-1}$ as $\sigma \to \infty$, and by the fact that $y \geq -1$ for all σ and n, we conclude that $\bar{\lambda}$ must be equal to one.

The behaviour of z_+ can be retrieved with the following scaling:

$$ae^{-2t} = e^{-2\tau} \iff \tau = t - \tfrac{1}{2}\log a \quad \text{and} \quad x(\tau) = \frac{1}{L}z_-(t),$$

leading to

$$xx' = a(x-1)\left(x + \frac{1}{L}\right) + \frac{1}{aL}e^{-2\tau}\left(\lambda x + \frac{1}{L}\right), \tag{4.16}$$

for $\tau > T - \frac{1}{2}\log a$. Again we set $x(\tau)$ equal to zero for $\tau \leq T - \frac{1}{2}\log a$. The Arzela-Ascoli theorem yields the convergence of a subsequence of the functions x, uniform on compact subsets of $\mathbb{R}$, to a continuous limit function $\bar{x}$. Define

$$\bar{\tau} = \limsup_{n\uparrow 1}\,(T - \tfrac{1}{2}\log a).$$

It follows from the upper bound $\hat{T}_+$ defined in Lemma 3.3 that $\bar{\tau} < \infty$. On $\{\tau > \bar{\tau}\}$ we can pass to the limit in equation (4.16), finding

$$\bar{x}' = \frac{1}{N-2}e^{-2\tau},$$

which results in

$$\bar{x}(\tau) = 1 - \frac{1}{2(N-2)}e^{-2\tau} \quad \text{for all} \quad \tau > \bar{\tau}.$$

The continuity of the limit function $\underline{x}$ now implies that $\bar{\tau} = -\frac{1}{2}\log 2(N-2)$. It follows from the explicitness of this value that $T - \frac{1}{2}\log a$ converges to $-\frac{1}{2}\log 2(N-2)$ along every sequence $n \uparrow 1$.

Let us summarise our results in the

Lemma 4.5 *Let $n \uparrow 1$. Then*

1. $\lambda \to 1$;
2. $T - \frac{1}{2}\log a \to -\frac{1}{2}\log 2(N-2)$;
3. *x tends to the limit function*

$$\begin{cases} \bar{x}(\tau) = 1 + \dfrac{1}{2(N-2)}e^{-2\tau} & \textit{for } \tau > -\frac{1}{2}\log 2(N-2) \\ \bar{x}(\tau) = 0 & \textit{for } \tau \le -\frac{1}{2}\log 2(N-2) \end{cases}$$

4. *y tends to the function $\bar{y}$ given by*

$$\begin{cases} \bar{y}(\sigma) - \log(1+\bar{y}(\sigma)) = \sigma & \textit{for } \sigma > 0 \\ \bar{y}(\sigma) = 0 & \textit{for } \sigma \le 0 \end{cases}$$

Here convergence is uniform on compact subsets of the real line.

Figure 7 shows the convergence of λ to 1 as n tends to 1.

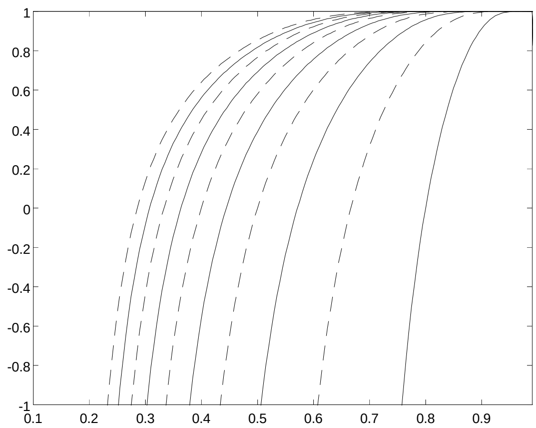

Figure 7: Graph of λ as a function of n. The curves are for $N = 3, 4, \ldots, 12$, where the dimension increases from right to left.

We conclude with the translation of these assertions into the original variables. We again define $z_+(t) = z_-(t) = 0$ for all $t < T(n)$.

Theorem 4.6 *Let $n \uparrow 1$. Then*

1. $\lambda(n) \to 1$;

2. $T(n) - \frac{1}{2}\log(1-n) \to -\frac{1}{2}\log(N-2)$;

3. $(1-n)z_+(\tau + \frac{1}{2}\log(1-n)) \to (N-2)\bar{x}(\tau - \frac{1}{2}\log 2)$ *for all* $\tau \in \mathbb{R}$;

4. $z_-(T(n) + \alpha\sigma) \to \bar{y}(\sigma)$ *for all* $\sigma \in \mathbb{R}$.

The convergence is uniform on compact subsets of $\mathbb{R}$ *in the variables* τ *and* σ*.*

References

[1] D. G. Aronson. The porous medium equation. In A. Fasano and M. Primicerio, editors, *Some Problems in Nonlinear Diffusion.* Lecture Notes in Mathematics, No. 1224, Springer-Verlag, 1986.

[2] G. I. Barenblatt. *Similarity, Self-similarity, and Intermediate Asymptotics.* Consultants Bureau, New York, 1979.

[3] Ph. Bénilan and M. G. Crandall. The continuous dependence on ϕ of solutions of $u_t - \Delta\phi(u) = 0$. *Ind. Univ. Math. J.*, 30:161–177, 1981.

[4] J. G. Berryman and C. J. Holland. Stability of the separable solution for fast diffusion. *Arch. Rational Mech. Anal.*, 74:379–388, 1980.

[5] T. Carleman. *Problèmes mathématiques dans la théorie cinétique des gaz.* Almqvist and Wilsell, Uppsala, 1957.

[6] F. Hellfferich and M. S. Plesset. Ion exchange kinetics: A nonlinear diffusion problem. *J. Chem. Phys.*, 28(3):418–424, 1958.

[7] M. A. Herrero and M. Pierre. The Cauchy problem for $u_t = \Delta u^m$ when $0 < m < 1$. *Trans. Amer. Math. Soc.*, 291:145–158, 1985.

[8] H. G. Kaper and G. K. Leaf. Initial value problems for the Carleman equation. *Nonl. Anal.*, 4(2):343–362, 1980.

[9] H. G. Kaper, G. K. Leaf, and S. Reich. Convergence of semigroups with an application to the Carleman equation. *Math. Meth. Appl. Sci.*, 2:303–308, 1980.

[10] J. R. King. Extremely high concentration dopant diffusion in silicon. *IMA J. Appl. Math.*, 40:163–181, 1988.

[11] J. R. King. Exact polynomial solutions to some nonlinear diffusion equations. *Physica*, 64:35–65, 1993.

[12] J. R. King. Self-similar behaviour for the equation of fast nonlinear diffusion. *Phil. Trans. R. Soc. Lond. A*, 343:337–375, 1993.

[13] T. G. Kurtz. Convergence of sequences of semigroups of nonlinear operators with an application to gas kinetics. *Trans. Amer. Math. Soc.*, 186:259–272, 1973.

[14] K. E. Lonngren and A. Hirose. Expansion of an electron cloud. *Physics Lett. A*, 59(4):285–286, 1976.

[15] H. P. McKean. The central limit theorem for Carleman's equation. *Israel J. Math.*, 21:54–92, 1976.

[16] L. A. Peletier. The porous medium equation. In H. Amann, N. Bazley, and K. Kirchgässner, editors, *Applications of Nonlinear Analysis in the Physical Sciences*. Pitman Advanced Publishing Program, 1981.

[17] J. R. Philip. Fast diffusion with loss at inifinity. *J. Austral. Math. Soc. Ser. B*, 36:1–22, 1994.

[18] J. L. Vázquez. Nonexistence of solutions for nonlinear heat equations of fast-diffusion type. *J. Math. Pures Appl.*, 71:503–526, 1992.