The minimum of quadratic functionals of the gradient on the set of convex functions

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Abstract

We study the infimum of functionals of the form $\int_\Omega M\nabla u \cdot \nabla u$ among all *convex* functions $u \in H_0^1(\Omega)$ such that $\int_\Omega |\nabla u|^2 = 1$. ($\Omega$ is a convex open subset of $\mathbb{R}^N$, and M is a given symmetric $N \times N$ matrix.) We prove that this infimum is the smallest eigenvalue of M if Ω is C^1. Otherwise the picture is more complicated. We also study the case of an x-dependent matrix M.

Abstract

On étudie l'infimum de fonctionnelles de la forme $\int_\Omega M\nabla u \cdot \nabla u$ parmi les fonctions *convexes* $u \in H_0^1(\Omega)$ telles que $\int_\Omega |\nabla u|^2 = 1$. ($\Omega$ est un ouvert convexe de $\mathbb{R}^N$, et M est une matrice symétrique $N \times N$ fixée.) On démontre que cet infimum est la plus petite valeur propre de M lorsque Ω est C^1; sinon la situation est plus complexe. Le cas où M dépend de $x \in \Omega$ est aussi étudié.

1 Introduction

There has been a recent surge of interest in variational problems where the class of admissible functions is characterized by a convexity condition. These are problems of the form

$$\inf \left\{ \int_\Omega f(x, u, \nabla u) \,;\, u \in \mathcal{C} \right\},$$

where

$$\mathcal{C} := \left\{ \varphi \in H_0^1(\Omega) \,;\, \varphi \text{ is convex} \right\}.$$

Such problems arise independently in different fields such as mathematical physics (Newton's problem of the body of least resistance, cf. [3], [7]), fluid mechanics (cf. [2]), and mathematical economy (cf. [4]).

Even in the case of well-behaved (convex and coercive) functionals, problems of this type with and without convexity constraint on u can be very different (cf. [6]). The convexity constraint can be expressed by Lagrange multipliers in the Euler-Lagrange equation associated with the problem; these multipliers are second derivatives of a bounded symmetric nonnegative matrix of measures [8]; however, the optimal regularity of these measures is still an open question, and since their support can be dense, the Euler-Lagrange equation is often of little practical value. This is for instance the case in Newton's problem of the body of minimal resistance (cf. [7]).

While studying this minimal resistance problem the authors were confronted with the question of the value of the infimum of a quadratic functional of the gradient (that is, $\int f(\nabla\varphi)$, where $f(V) = a + b\cdot V + MV\cdot V$) over the set

$$\left\{ v \in \mathcal{C}\,; \quad 1 = \|v\|^2_{H^1_0(\Omega)} =: \int_\Omega |\nabla v|^2 \right\}.$$

Since $\int \nabla\varphi = 0$, this reduces to the minimization of $\int M\nabla\varphi\cdot\nabla\varphi$ in this set, or equivalently to the minimization of $\int M\nabla\varphi\cdot\nabla\varphi / \int_\Omega |\nabla\varphi|^2$ in $\mathcal{C}\setminus\{0\}$ (we will implicitly ignore the zero function in the following). This is the problem considered in this paper.

It is well known that if $\Omega \subset \mathbb{R}^N$ is an open set, and M a given symmetric matrix, then the infimum

$$\inf_{\phi\in H^1_0(\Omega)} \frac{\int_\Omega M\nabla\phi\cdot\nabla\phi}{\int_\Omega |\nabla\phi|^2}$$

is not attained (except if $M \in \mathbb{R}\,\mathrm{Id}$) and equals the first eigenvalue $\lambda_1(M)$ of M. This can be proved by considering the sequence $\phi_n(x) := \eta(x)\times \sin(nx\xi_0)$ where $\eta \in C^1_0(\Omega)$ is fixed and $\xi_0 \in \mathbb{R}^N\setminus\{0\}$ satisfies $M\xi_0 = \lambda_1(M)\xi_0$; it is now easy to verify that $\int_\Omega M\nabla\phi_n\cdot\nabla\phi_n/\int_\Omega|\nabla\phi_n|^2$ converges to $\lambda_1(M)$ as $n\to\infty$.

We are interested here in the same minimization problem, under the additional constraint that $\phi \in H^1_0(\Omega)$ is *convex* (i.e., $\phi\in\mathcal{C}$). Since the set of convex functions is far from dense in H^1_0, and since the sequence ϕ_n mentioned above obviously does not belong to $\mathcal{C}$, it is quite surprising that we can prove a similar result and obtain the *same* infimum. Note that it is necessary to assume that the set Ω is convex, since otherwise $\mathcal{C} = \{0\}$.

2 Main results

Theorem 1 *Let $\Omega \subset \mathbb{R}^2$ be a convex domain, and let $M \in \mathbb{R}^{2\times 2}$ be a given symmetric matrix. Then*

$$\lambda_1(M) \leq \inf_{\substack{\phi \in H^1_0(\Omega) \\ \phi \text{ convex}}} \frac{\int_\Omega M\nabla\phi \cdot \nabla\phi}{\int_\Omega |\nabla\phi|^2} \leq \operatorname*{ess\,inf}_{x\in\partial\Omega} M\nu(x)\cdot\nu(x), \tag{1}$$

where $\lambda_1(M)$ is the smallest eigenvalue of M, and $\nu : \partial\Omega \to S^1$ is the a.e. defined map giving the normal exterior vector $\nu(x)$ at x. In particular, if Ω has a boundary of class C^1, then the previous infimum is exactly equal to $\lambda_1(M)$.

The first inequality in (1) is obvious. Moreover, if Ω is of class C^1, then the ess inf in the right-hand side of (1) equals $\lambda_1(M)$, since ν is continuous and surjective.

If Ω is not of class C^1, then the relationship between the first eigenvalue and the infimum above is an interesting open problem. It is simple to construct non-smooth boundaries such that the ess inf above is nonetheless equal to $\lambda_1(M)$, so that both inequalities reduce to equalities. In the alternative case, however, the first inequality is strict:

Theorem 2 *Under the same assumptions as Theorem 1, assume that*

$$\operatorname*{ess\,inf}_{x\in\partial\Omega} M\nu(x)\cdot\nu(x) > \lambda_1(M). \tag{2}$$

Then the first inequality in (1) is strict.

In Section 6 we present an explicit counter-example which shows that the second inequality in (1) can also be strict. The question whether in the general case the second inequality is saturated or not remains, to our knowledge, open.

Note that statements and proofs are given here in dimension $N = 2$ for the sake of simplicity. The general case is similar.

3 Proof of Theorem 1

Proof. For the length of this proof we change notation: instead of using x as a two-dimensional vectorial coordinate, we let x and y be two scalar coordinates, so that (x, y) denotes an element of $\mathbb{R}^2$ with respect to a given orthogonal basis. The differential operators ∂_x and ∂_y denote differentiation with respect to these coordinates, and $\nabla = (\partial_x, \partial_y)^T$. By choosing the basis appropriately, M takes the form of the diagonal matrix $\operatorname{diag}(\lambda_1, \lambda_2)$; moreover the decomposition

$$M = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \lambda_1 \, I + (\lambda_2 - \lambda_1) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

implies that

$$\frac{\int M\nabla\phi \cdot \nabla\phi}{\int |\nabla\phi|^2} = \lambda_1 + (\lambda_2 - \lambda_1)\frac{\int (\partial_y \phi)^2}{\int |\nabla\phi|^2}.$$

Let $\partial\Omega$ be twice differentiable at $(x_0, y_0) \in \partial\Omega$, with normal $\nu(x_0, y_0) = \nu_0 = (\nu_{0x}, \nu_{0y})$. Since

$$M\nu_0 \cdot \nu_0 = \lambda_1 + (\lambda_2 - \lambda_1)\nu_{0y}^2,$$

the estimate (which we shall prove)

$$\inf_{\substack{\phi \in H_0^1(\Omega) \\ \phi \text{ convex}}} \frac{\int (\partial_y \phi)^2}{\int |\nabla\phi|^2} \leq \nu_{0y}^2 \tag{3}$$

implies that

$$\inf_{\substack{\phi \in H_0^1(\Omega) \\ \phi \text{ convex}}} \frac{\int_\Omega M\nabla\phi \cdot \nabla\phi}{\int_\Omega |\nabla\phi|^2} \leq M\nu(x_0, y_0) \cdot \nu(x_0, y_0). \tag{4}$$

The assertion of the theorem follows from (4) by remarking that every convex function—in particular, the boundary $\partial\Omega$—is twice differentiable almost everywhere [5, Section 6.4], and that therefore (4) is valid for almost every $(x_0, y_0) \in \partial\Omega$.

It is therefore sufficient to prove (3) for this choice of $(x_0, y_0) \in \partial\Omega$. Pick $\varepsilon,\ \eta > 0$; define

$$\Gamma = \big\{ (x, y) \in \partial\Omega : -\eta < (x - x_0, y - y_0) \cdot \nu_0 < 0 \big\},$$

and its translation

$$\Gamma_\varepsilon = \Gamma - \varepsilon\nu_0,$$

(see Figure 1 (a)). If ε and η are sufficiently small, then $\Gamma_\varepsilon \subset \Omega$.

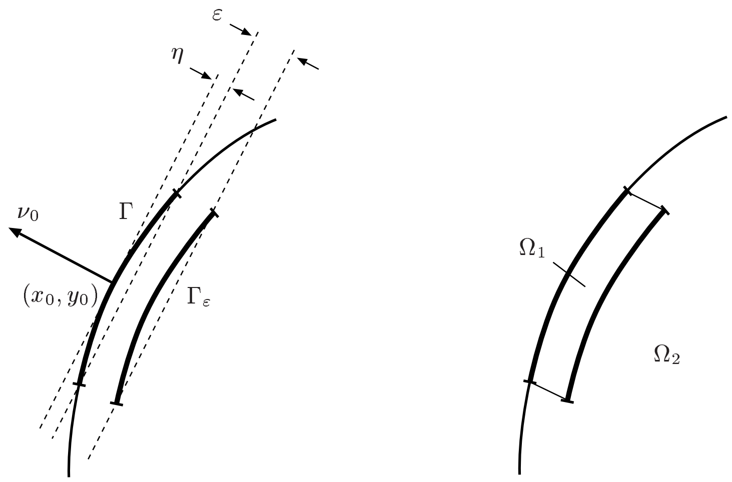

Figure 1: (a) The sets Γ and Γ_ε; (b) The partition of Ω.

We define the function $\phi_\varepsilon : \overline{\Omega} \to \mathbb{R}$ as the largest convex function satisfying $\phi_\varepsilon = -1$ on Γ_ε and $\phi_\varepsilon = 0$ on $\partial\Omega$. We next partition Ω (Figure 1 (b)):

$$\Omega_1 = \bigcup_{\lambda\in(0,1)} \lambda\Gamma_\varepsilon + (1-\lambda)\Gamma$$

$$\Omega_2 = \Omega \setminus \Omega_1.$$

If Γ is differentiable at $(\bar{x}, \bar{y}) \in \Gamma$, then

$$\nabla\phi_\varepsilon(x, y) = \varepsilon^{-1}(\nu_0 \cdot \nu(\bar{x}, \bar{y}))^{-1}\, \nu(\bar{x}, \bar{y}) \qquad \text{for all} \qquad (x, y) \in (\bar{x}, \bar{y}) - \lambda\varepsilon\nu_0,\ \lambda \in (0, 1). \tag{5}$$

Note that the assumption that η is small implies that $(\nu_0 \cdot \nu(\bar{x}, \bar{y}))^{-1}$ is bounded on Γ. Since Γ is differentiable almost everywhere, (5) gives an a.e. definition of $\nabla\phi_\varepsilon$ in Ω_1; it follows that

$$\int_{\Omega_1} |\nabla\phi_\varepsilon|^2 \geq c\varepsilon^{-1} \qquad \text{and} \qquad \frac{\displaystyle\int_{\Omega_1} (\partial_y\phi_\varepsilon)^2}{\displaystyle\int_{\Omega_1} |\nabla\phi_\varepsilon|^2} \leq m^2, \tag{6}$$

for some $c > 0$, and where $m = \operatorname{ess\,sup}_{(x,y)\in\Gamma} |\nu_y(x, y)|$.

We now turn to the estimate of the integral on Ω_2. We can describe the graph of ϕ_ε as a collection of line segments in $\mathbb{R}^3$, on which $\nabla\phi_\varepsilon$ is constant [1]; after projection onto the x, y-plane each of these segments connects $\partial\Omega$ to the convex envelope of Γ_ε.

If $|\nabla\phi_\varepsilon(x,y)| = \gamma$, then the line segment that passes through (x,y) has length $1/\gamma$. When γ is large, there are two possibilities: either $(x,y) \in \Omega_1$, or the line segment connects one of the end points of Γ_ε to $\partial\Omega$. In the latter case, (x,y) lies in a circle of radius $1/\gamma$ around either of the two end points of Γ_ε. This implies that

$$\left|\left\{(x,y) \in \Omega \setminus \Omega_1 : |\nabla\phi_\varepsilon(x,y)| \geq \gamma\right\}\right| \leq 2\pi\gamma^{-2}, \qquad \text{for large } \gamma.$$

This estimate implies that

$$\int_{\{\gamma \leq |\nabla\phi_\varepsilon| \leq 2\gamma\}\setminus\Omega_1} |\nabla\phi_\varepsilon|^2 \leq 8\pi,$$

and by concatenating estimates of this type we find that there exists $C > 0$, independent of ε, such that

$$\int_{\{|\nabla\phi_\varepsilon| \leq \gamma\}\setminus\Omega_1} |\nabla\phi_\varepsilon|^2 \leq C \log\gamma \qquad \text{for large } \gamma.$$

Since $\operatorname{dist}(\Gamma_\varepsilon, \partial\Omega) \geq C'\varepsilon$ for some $C' > 0$, the gradient $\nabla\phi$ is bounded on Ω by $(C'\varepsilon)^{-1}$; therefore we can estimate the integral over Ω_2 by

$$\int_{\Omega_2} |\nabla\phi_\varepsilon|^2 \leq C'' \left|\log\varepsilon\right| \qquad \text{for small } \varepsilon.$$

Combining this estimate with (6) we find that

$$\frac{\displaystyle\int_\Omega (\partial_y\phi_\varepsilon)^2}{\displaystyle\int_\Omega |\nabla\phi_\varepsilon|^2} \leq m^2 + C''c^{-1}\varepsilon\left|\log\varepsilon\right|.$$

In the limit $\varepsilon \to 0$ the right-hand side of this expression converges to m^2. It follows that

$$\inf_{\substack{\phi\in H_0^1(\Omega)\\ \phi \text{ convex}}} \frac{\int (\partial_y\phi)^2}{\int |\nabla\phi|^2} \leq m^2,$$

and in the limit $\eta \to 0$, which implies $m \to \nu_{0y}$, this expression yields (3). This concludes the proof of Theorem 1. $\square$

4 Case of a varying matrix

It is also possible to prove a result similar to Theorem 1 with a quadratic form depending on x.

Corollary 3 *Let $\Omega \subset \mathbb{R}^2$ be a convex domain, and let $M : \overline{\Omega} \to \mathbb{R}^{2\times 2}$ be a measurable map of symmetric matrices. If $\partial\Omega$ is differentiable at $x_0 \in \partial\Omega$ and there exists $\lambda \in \mathbb{R}$ such that*

$$M(x)\nu(x_0) \cdot \nu(x_0) \le \lambda \qquad \text{for a.e. } x \in \Omega, \tag{7}$$

then

$$\inf_{\substack{\phi \in H_0^1(\Omega) \\ \phi \text{ convex}}} \frac{\int_\Omega M(x)\nabla\phi \cdot \nabla\phi}{\int_\Omega |\nabla\phi|^2} \le \lambda.$$

Note that in condition (7) the vector $\nu(x_0)$ is fixed.

Proof. Fix $\varepsilon > 0$. We claim that there exists a constant symmetric matrix $\widetilde{M}$ satisfying

$$\widetilde{M}\nu(x_0) \cdot \nu(x_0) \le \lambda + \varepsilon$$

that majorizes M, i.e. such that $\widetilde{M} - M$ is positive semidefinite. Applying Theorem 1 to the matrix $\widetilde{M}$ we find

$$\inf_{\substack{\phi \in H_0^1(\Omega) \\ \phi \text{ convex}}} \frac{\int_\Omega M(x)\nabla\phi \cdot \nabla\phi}{\int_\Omega |\nabla\phi|^2} \le \inf_{\substack{\phi \in H_0^1(\Omega) \\ \phi \text{ convex}}} \frac{\int_\Omega \widetilde{M}\nabla\phi \cdot \nabla\phi}{\int_\Omega |\nabla\phi|^2} \le \operatorname*{ess\,inf}_{x\in\partial\Omega} \widetilde{M}\nu(x) \cdot \nu(x).$$

Since $\partial\Omega$ is differentiable at x_0,

$$\operatorname*{ess\,inf}_{x\in\partial\Omega} \widetilde{M}\nu(x) \cdot \nu(x) \le \widetilde{M}\nu(x_0) \cdot \nu(x_0) \le \lambda + \varepsilon.$$

The corollary follows from the observation that ε is an arbitrary positive number.

To prove the claim made above, we choose as a basis of $\mathbb{R}^2$ the vectors $e_1 = \nu(x_0)$ and $e_2 = e_1^\perp$, and write $M(x)$ as

$$M(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & d(x) \end{pmatrix},$$

with respect to this basis. The inequality (7) implies that $a(x) \le \lambda$ a.e.

We now choose $\widetilde{M} = \mathrm{diag}(\lambda + \varepsilon, K)$ for some large $K > 0$. Omitting the dependence on the variable x,

$$\begin{aligned}\det(\widetilde{M} - M) &= (\lambda + \varepsilon - a)(K - d) - b^2 \\ &= ad - b^2 - d(\lambda + \varepsilon) + K(\lambda + \varepsilon - a).\end{aligned}$$

Since $\lambda + \varepsilon - a \geq \varepsilon$ a.e. we can make this expression positive by choosing K large enough. This proves the claim and concludes the proof of the corollary. $\square$

5 A strict inequality: Proof of Theorem 2

Proof. Let us note $J_\Omega(\phi) := \int_\Omega M\nabla\phi \cdot \nabla\phi \,/\, \int_\Omega |\nabla\phi|^2$ for short. As in the proof of Theorem 1, we can assume that $M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, therefore $J_\Omega(\phi) := \int_\Omega (\partial_y\phi)^2 \,/\, \int_\Omega |\nabla\phi|^2$, and $\lambda_1(M) = 0$.

The assumption (2) implies that there are two points $A = (x_A, y_A)$, $B = (x_B, y_B)$ in $\partial\Omega$, and a scalar $C > 0$ such that for $(x, y) \in \Omega$,

$$x_A < x < x_B \quad \text{and} \quad \max\left\{\left|\frac{y - y_A}{x - x_A}\right|, \left|\frac{y - y_B}{x - x_B}\right|\right\} \leq C. \tag{8}$$

We write I_x for $\{y \in \mathbb{R}, (x, y) \in \Omega\}$.

If $\phi \in C^1_0(\Omega)$ is convex, then the tangent plane at $(x_0, y_0) \in \Omega$ has the functional expression

$$p(x, y) = \phi(x_0, y_0) + (x - x_0)\,\partial_x\phi(x_0, y_0) + (y - y_0)\,\partial_y\phi(x_0, y_0).$$

By convexity, $p(x_A, y_A) \leq 0$ and $p(x_B, y_B) \leq 0$, so that

$$\begin{aligned}\frac{\phi(x_0, y_0) + (y_A - y_0)\,\partial_y\phi(x_0, y_0)}{x_0 - x_A} &\leq \partial_x\phi(x_0, y_0) \leq \\ &\leq \frac{\phi(x_0, y_0) + (y_B - y_0)\,\partial_y\phi(x_0, y_0)}{x_0 - x_B}.\end{aligned}$$

Therefore, using (8),

$$\begin{aligned}\int_{I_{x_0}} \partial_x\phi(x_0, y)^2\,dy \leq 2\min\left(\frac{1}{x_0 - x_A}, \frac{1}{x_B - x_0}\right)^2 \int_{I_{x_0}} \phi(x_0, y)^2\,dy + \\ + 2C^2 \int_{I_{x_0}} \partial_y\phi(x_0, y)^2\,dy.\end{aligned}$$

With the Poincaré inequality,

$$\int_{I_{x_0}} \phi(x_0,y)^2\,dy \le \frac{|I_{x_0}|^2}{\pi^2}\int_{I_{x_0}} \partial_y\phi(x_0,y)^2\,dy,$$

we find, since by (8) we have $|I_{x_0}| \le 2C\min(x_0-x_A, x_B-x_0)$,

$$\int_{I_{x_0}} \partial_x\phi(x_0,y)^2\,dy \le 2C^2\Big(1+\frac{4}{\pi^2}\Big)\int_{I_{x_0}} \partial_y\phi(x_0,y)^2\,dy,$$

and therefore,

$$\int_\Omega |\nabla\phi|^2 \le \Big[1+2C^2\Big(1+\frac{4}{\pi^2}\Big)\Big]\int_\Omega \partial_y\phi^2.$$

This proves that

$$\inf J_\Omega(\phi) > 0 = \lambda_1(M).$$

$\square$

6 An example of non-saturation

The second inequality in (1) is always saturated if Ω is smooth; but for non-smooth Ω, one can construct a situation in which the inequality is strict.

Let Ω be the square

$$\Omega = \{(x,y)\in\mathbb{R}^2 : |x-y|<1 \text{ and } |x+y|<1\},$$

so that $\nu = \frac{1}{2}\sqrt{2}\,(\pm1,\pm1)^T$, and let

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $M\nu\cdot\nu = 0$ on $\partial\Omega$.

To construct a convex function ϕ such that

$$\frac{\int_\Omega M\nabla\phi\cdot\nabla\phi}{\int_\Omega |\nabla\phi|^2} < 0,$$

we define

$$\phi_0(x,y) = \max(x+y, x-y, -x+y, -x-y) - 1.$$

This function ϕ_0 is convex, belongs to $H_0^1(\Omega)$, and $\nabla\phi_0 = (\pm 1, \pm 1)^T$ in Ω; therefore $M\nabla\phi_0 \cdot \nabla\phi_0 = 0$ in Ω.

The function ϕ is defined as

$$\phi(x, y) = \max(\phi_0(x, y), \lambda(y - 1)),$$

for some $0 < \lambda < 1$. Setting $\Omega' = \{(x, y) \in \Omega : \phi(x, y) > \phi_0(x, y)\}$, we have $\phi = \phi_0$ in $\Omega \setminus \Omega'$, so that $M\nabla\phi \cdot \nabla\phi = 0$; in the set Ω', we have $\nabla\phi = (0, \lambda)^T$, so that $M\nabla\phi \cdot \nabla\phi = -\lambda^2 < 0$. Consequently

$$\int_\Omega M\nabla\phi \cdot \nabla\phi < 0,$$

which implies that the inequality in (1) is strict.

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