A supersolution for the porous media equation with non-uniform density

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Abstract. We exhibit a supersolution for the porous media equation with non-uniform density, and derive an upper bound on the interfaces of a solution.

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The purpose of this paper is to find a supersolution w of the Cauchy problem

$$\text{(P)} \quad \begin{cases} \rho(x)u_t = \Delta(u^m) & \text{in } Q = \mathbf{R}^N \times \mathbf{R}^+ & (1.1) \\ u(x,0) = u_0(x) & x \in \mathbf{R}^N & (1.2) \end{cases}$$

for the case that u_0 has compact support. The interfaces of w then constitute an upper bound (or, rather, an 'outer' bound) on the interfaces of a solution u.

We will assume that $m > 1$, and that the functions ρ and u_0 satisfy the following assumptions:

$$\rho \in C^1(\mathbf{R}) \cap L^\infty(\mathbf{R}), \quad 0 \le \rho \le 1$$
$$u_0 \in C_c(\mathbf{R}^N), \quad u_0 \ge 0.$$

When ρ is a constant, eq. (1.1) is the classical porous media equation, of which it is known (see eg. [1]) that solutions having compact support initially also are compactly supported at every finite time $t > 0$, and that the interfaces tend to infinity as t tends to infinity. For densities ρ that decay to zero for large $|x|$ the situation can be different. Kamin and Kersner [2] have proved several results about this case, of which we mention here two (u will denote any solution of problem P):

(a) For $N \ge 3$, if ρ satisfies

$$\rho(x) \cdot (1 + |x|)^{-(N-2)/m} \in L^1(\mathbf{R}^N), \tag{2}$$

then a time T_0 exists at which $\operatorname{supp} u$ can no longer be compact;

(b) For $N \ge 1$, if

$$\rho(x) \ge c(1 + |x|)^{-2} \quad \text{on } \mathbf{R}^N,$$

then u has compact support for all time.

In this paper we shall offer some new information. By exhibiting a supersolution with compact support (at least) for small time, we can use the boundaries of the support as bounds on the interfaces of the solution itself. In contrast to result (b), which is also based on a comparison function, our supersolution is defined for all smooth and bounded densities ρ; the behaviour of ρ at infinity is transported into an ordinary differential equation for the interface of the supersolution, where it directly determines the occurence or non-occurence of blow-up of the interface.

We shall prove the following theorem:

Theorem 1. *Let u be a solution of problem P. Let $\sigma : \mathbf{R}^+ \to \mathbf{R}^+$ be defined by*

$$\sigma(r) = \min_{|x|\leq r} \rho(x),$$

and let a and b be constants such that

$$a\Big(1 - \frac{|x|^2}{b^2}\Big)_+ \geq u_0(x)^{m-1}, \qquad x \in \mathbf{R}^N.$$

If the curve ζ is defined by

$$\begin{cases} \zeta'(t) = \dfrac{2ma}{m-1}\,\dfrac{1}{\zeta(t)\sigma(\zeta(t))}, & t > 0, \\ \zeta(0) = b, \end{cases} \tag{3}$$

then $\operatorname{supp} u(\cdot,t) \subset \{x \in \mathbf{R}^N : |x| \leq \zeta(t)\}$ for all t for which ζ is defined.

The information provided by this theorem is twofold: first, it offers an explicit lower bound on the time T_0, when the support ceases to be compact, as mentioned in Kamin and Kersner's theorem; second, the fact that the behaviour of the bound ζ does not depend on the dimension N, combined with the validity of the bound for $N < 3$, suggests that condition (2) might not be optimal.

Proof. We shall seek a supersolution of the form

$$w(x,t) = \Big[\frac{a}{f(t)}\Big(1 - \frac{|x|^2}{g^2(t)}\Big)_+\Big]^{1/(m-1)},$$

and we assume that the functions f and g are smooth enough to justify what shall follow. For this function to be a supersolution, three conditions must be satisfied:

$$\rho(x)w_t \geq \Delta(w^m) \qquad \text{in } \{(x,t)|w(x,t) > 0\} \tag{4a}$$

$$g'(t) \geq -\frac{m}{\rho(x)(m-1)}\frac{\partial}{\partial r}(w^{m-1}) \qquad \text{at } |x| = g(t),\ t > 0 \tag{4b}$$

$$w(x,0) \geq u_0(x) \qquad x \in \mathbf{R}^N, \tag{4c}$$

where we have used the notation $r = |x|$. A proof of this statement can be found in [3]. The first condition is the maximum principle for the classical part of the solution; the second is the interface condition, and the third compares the functions at $t = 0$. For (4a) we find that it is sufficient to demand

$$\begin{cases} f'(t) \leq 2maN \dfrac{1}{\rho(x)g^2(t)} & \text{(5a)} \\ g'(t) \geq \dfrac{2ma}{m-1} \dfrac{1}{\rho(x)f(t)g(t)}, & \text{(5b)} \end{cases}$$

on $\{w > 0\}$, which is given by $\{(x,t) : |x| < g(t)\}$. By direct calculation one can verify that (5b) implies (4b), and (4c) is satisfied by the definition of a and b. If we define f and g—both functions of t, but not of x—by

$$\begin{cases} f' = 2maN \dfrac{1}{g^2}, \quad t > 0 \\ f(0) = 1 \end{cases} \qquad \begin{cases} g' = \dfrac{2ma}{m-1} \dfrac{1}{\sigma(g)fg}, \quad t > 0 \\ g(0) = b, \end{cases} \tag{6}$$

then conditions (5a) and (5b) are satisfied and the function w is indeed a supersolution of problem P.

The support of w is given by $|x| \leq g(t)$. We can eliminate f from (6) to obtain the following ordinary differential equation in g:

$$g'' + \left\{ \frac{\sigma'(g)}{\sigma(g)} + \frac{1}{g} + N(m-1) \frac{\sigma(g)}{g} \right\} g'^2 = 0, \quad t > 0. \tag{7}$$

By integrating (7) once we find

$$g'(t) = g'(0) \frac{\sigma(b)b}{g(t)\sigma(g(t))} e^{-N(m-1) \int_b^{g(t)} \frac{\sigma(\gamma)}{\gamma} \, d\gamma} \tag{8}$$

If we set

$$\varphi(\xi) \stackrel{\text{def}}{=} e^{-N(m-1) \int_b^{\xi} \frac{\sigma(\gamma)}{\gamma} \, d\gamma}, \qquad \xi \geq b,$$

and use (6) to rewrite $g'(0)$, then g satisfies

$$g'(t) = \frac{2ma}{m-1} \frac{\varphi(g(t))}{g(t)\sigma(g(t))}. \tag{9}$$

We can find a simpler, albeit weaker, version of (9). Since $\varphi(\xi) \leq 1$ for all $\xi \geq b$, the function $\zeta : \mathbf{R}^+ \to \mathbf{R}$, defined by

$$\begin{cases} \zeta'(t) = \dfrac{2ma}{m-1} \dfrac{1}{\zeta(t)\sigma(\zeta(t))} \\ \zeta(0) = b \end{cases} \tag{3}$$

will also be an upper bound for the interface of a solution u, but a weaker one. However, we claim that in replacing φ by 1 we do not change the qualitative behaviour, in the sense that finite time blow-up will occur either in both cases or in neither, depending on ρ. Indeed, blow-up of ζ occurs if and only if $\xi\sigma(\xi) \in L^1(b,\infty)$. If ζ blows up in finite time, then clearly $\sigma(\xi)/\xi \in L^1(b,\infty)$, and therefore $\varphi(\gamma)$ remains bounded away from zero as $\gamma \to \infty$. In that case g will also exhibit finite time blow-up. Inversely, if $\xi\sigma(\xi) \notin L^1(b,\infty)$, then

$$\frac{\xi\sigma(\xi)}{\varphi(\xi)} \geq \xi\sigma(\xi) \notin L^1(b,\infty),$$

and g will remain finite for all time. We conclude that the function ζ is an upper bound for the interface of a solution with the same qualitative behaviour as the interface g of the supersolution. •

Example. Let $\rho(x) = (1+|x|^2)^{-\alpha/2}$ for $\alpha > 0$. Then the function ζ defined by (3) is given by

$$\begin{aligned} \zeta(t)^2 &= -1 + \left[(1+b^2)^{(2-\alpha)/2} + (2-\alpha)\frac{2ma}{m-1}t\right]^{2/(2-\alpha)} && \alpha \in (0,2)\cup(2,\infty), \\ \zeta(t)^2 &= -1 + (1+b^2)e^{\frac{4ma}{m-1}t} && \alpha = 2. \end{aligned}$$

In these expressions it can be seen explicitly how ζ changes from remaining bounded on every finite time interval to blowing up in finite time, via an exponential regime, as α passes the critical value 2.

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