EXTREMAL POINTS OF A FUNCTIONAL ON THE SET OF CONVEX FUNCTIONS

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Abstract. We investigate the extremal points of a functional $\int f(\nabla u)$, for a convex or concave function f. The admissible functions $u : \Omega \subset \mathbf{R}^N \to \mathbf{R}$ are convex themselves and satisfy a condition $u_2 \le u \le u_1$. We show that the extremal points are exactly u_1 and u_2 if these functions are convex and coincide on the boundary $\partial\Omega$. No explicit regularity condition is imposed on f, u_1, or u_2.

Subsequently we discuss a number of extensions, such as the case when u_1 or u_2 are non-convex or do not coincide on the boundary, when the function f also depends on u, etc.

Résumé. On recherche les minima et maxima d'une fonctionnelle de la forme $\int f(\nabla u)$, avec f concave ou convexe, sur l'ensemble des fonctions $u : \Omega \subset \mathbf{R}^N \to \mathbf{R}$ qui sont elles-mêmes astreintes à être convexes et satisfont une condition de barrière de la forme $u_2 \le u \le u_1$. On montre que ces extrema sont précisément u_2 ou u_1, lorsque ces fonctions sont convexes et égales au bord de Ω. Aucune condition de régularité n'est requise sur f, u_1, u_2 ou Ω.

Différentes généralisations sont discutées également : u_1 ou u_2 non convexes, f dépendant aussi de u, etc.

1. Introduction

The aim of this paper is to characterize the solutions of the variational problems

$$\inf_{u\in\mathcal{C}} \int_\Omega f(\nabla u) \qquad \text{or} \qquad \sup_{u\in\mathcal{C}} \int_\Omega f(\nabla u) \tag{1}$$

on a set of admissible functions

$$\mathcal{C} = \{u : \overline{\Omega} \to \mathbf{R} \text{ is convex and } u_2 \le u \le u_1\},$$

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where Ω is a bounded subset of $\mathbf{R}^N$ and $u_1, u_2 : \overline{\Omega} \to \mathbf{R}^N$ are convex and coincide on $\partial\Omega$ (see Section 2 for generalizations). The main theorem, Theorem 1, establishes a connection between the convexity properties of f and the nature of the extremal points.

For the integral in (1) to be well-defined on $\mathcal{C}$ we need to introduce some hypotheses. We recall that any convex function on Ω is Lipschitz continuous on compact subsets of Ω and differentiable almost everywhere in Ω; under the additional hypothesis

$$(H_1) \qquad \nabla u_2 \text{ is bounded a.e. on } \Omega$$

combined with (H_2) below the set $\mathcal{C}$ is embedded in $W^{1,\infty}(\Omega)$. Indeed, (H_2) and the convexity imply that

$$\|\nabla u\|_{L^\infty(\Omega)} \le \|\nabla u_2\|_{L^\infty(\Omega)} \qquad \forall u \in \mathcal{C}. \tag{2}$$

Under the assumptions of Theorem 1 the function f is continuous and therefore the integral is well-defined and bounded from below on $\mathcal{C}$.

We also remark here that by the results of [5] the set $\mathcal{C}$ is compactly imbedded in $W^{1,p}_{\mathrm{loc}}(\Omega)$ for any $1 \le p < \infty$. Consequently the infimum in (3) is attained. For a complete discussion on the continuity properties of functionals of this form, even without any assumption on the convexity of the functions, we refer the reader to [5].

For our application, the problem reduces to

$$\min_{u \in \mathcal{C}} \int_\Omega f(\nabla u) \tag{3}$$

where f is convex or concave; the supremum is obtained by changing f to $-f$.

Theorem 1. *Let u_1, u_2 be convex and satisfy*

$$(H_2) \qquad u_1 = u_2 \quad on \quad \partial\Omega.$$

1. *if f is strictly convex, then $u = u_1$ is the unique solution of Problem (3);*
2. *if f is strictly concave, then $u = u_2$ is the unique solution of Problem (3).*

Remark Except for the convexity constraint on the functions u, part 1 of this theorem corresponds to a classical problem in the calculus of variations, which is usually analysed using the Euler-Lagrange equation and methods from PDE theory. The convexity constraint prompts us to use different techniques, which has some interesting consequences for the regularity of the functions involved. First, we do not need to suppose any special regularity on the function f; in particular, f need not be C^1. Second, Theorem 1 shows that the convexity constraint eliminates part of the regularizing effect that is commonly present in convex variational problems: the function u_1, unique minimizer if f and u_1 are both convex, need not be smooth; if u_1 is not convex (see generalizaton 1 below) then the minimizer u_1^{**} need not be smooth either, even in points where u_1 and u_1^{**} do not coincide.

Proof. Denote the integral in (3) by $F(u)$. We first prove the following lemma, adopting the notation $u \vee v = \max\{u, v\}$. We recall that if u and v are convex, then $u \vee v$ is, too.

Lemma 1. *Suppose that f is strictly convex. Let $u \in \mathcal{C}$ and suppose that $\theta : \Omega \to \mathbf{R}$ is an affine function such that the set $\omega = \{x \in \overline{\Omega} : \theta(x) > u(x)\}$ is non-empty and included in Ω. Then $F(u \vee \theta) < F(u)$.*

Proof of the Lemma. Since $\theta \vee u$ is equal to θ in ω and equal to u in $\Omega \setminus \omega$ we have

$$\delta F := F(\theta \vee u) - F(u) = \int_\omega [f(\nabla\theta) - f(\nabla u)].$$

Define

$$g(\xi) = f(\xi + \nabla\theta) - f(\nabla\theta) - f'(\nabla\theta) \cdot \xi, \qquad \forall \xi \in \mathbf{R}^N,$$

where for $f'(\nabla\theta)$ we take any element of the subdifferential of f at the point $\nabla\theta$ (remark that $\nabla\theta$ is constant on ω). Writing $\varphi = u - \theta$, we have

$$\delta F = -\int_\omega g(\nabla\varphi)$$

since $\varphi = 0$ on $\partial\omega$ implies $\int \nabla\varphi = 0$.

The strict convexity of f is passed on to g; by construction we also have

$$g(0) = 0 = \inf_{\xi \in \mathbf{R}^N} g(\xi),$$

which implies $g(\xi) > 0$ for any $\xi \neq 0$. Hence $\delta F < 0$ which concludes the proof of the lemma. $\square$

We now return to the proof of the theorem. Let us first assume that f is strictly convex. Then, using the lemma, we obtain $F(\theta \vee u) \leq F(u)$ for any $u \in \mathcal{C}$ and any affine function θ, the inequality being strict if $\theta \vee u \not\equiv u$ in Ω.

In particular, if v is any given function in $\mathcal{C} \setminus \{u_1\}$, there exists θ_1 such that

$$v \leq v_1 := v \vee \theta_1 \leq u_1 \quad \text{and} \quad v \not\equiv v_1 \qquad \text{in } \Omega.$$

Also, since v_1 and u_1 are convex functions satisfying $v_1 \leq u_1$, there exists a finite or infinite number of affine functions $(\theta_k)_{k\geq 2}$ such that the sequence $(v_k)_{k\geq 2}$ defined by $v_k := \theta_k \vee v_{k-1}$ satisfies $v_k \leq u_1$ and $v_k \to u_1$ in the uniform topology (and therefore in $W^{1,p}(\Omega)$, since the functions v_k are convex).

Consequently we have the sequence of inequalities

$$F(v) > F(v_1) \geq F(v_2) \geq \cdots \geq F(v_k) \to F(u_1)$$

hence $F(v) > F(u_1)$ and this concludes case 1 of the theorem, since v is arbitrary.

If f is concave, $-f$ is convex. Hence, using the lemma again, we get $F(\theta \vee u) \geq F(u)$ for any $u \in \mathcal{C}$ and any affine θ, the inequality being strict if $\theta \vee u \not\equiv u$ in Ω.

If v is any function in $\mathcal{C} \setminus \{u_2\}$, we can again define a finite or infinite sequence $(\theta_k)_{k\geq 1}$ such that $v_1 := u_2 \vee \theta_1$ satisfies $u_2 \leq v_1 \leq v$ and $u_2 \neq v_1$, and if $v_k := v_{k-1} \vee \theta_k$, the sequence (v_k) converges to v. We now have

$$F(u_2) < F(v_1) \leq F(v_2) \leq \ldots F(v_k) \to F(v)$$

and therefore $F(u_2) < F(v)$. This concludes the proof of the theorem. $\square$

2. Generalizations and improvements

We have given our main theorem in a simple and convenient form, but many generalizations and improvements are possible. We give some of them here, and also discuss the accuracy of some of the hypotheses.

1. **If u_1 is not convex,** the conclusion of the theorem is unchanged if f is concave, and should be changed to 'u_1^{**} is the unique solution' if f is convex. Here u_1^{**} is the convex regularization of u_1 (also called the Γ-regularization, [3], Proposition 4.1). This is obvious since this function is the maximal element in $\mathcal{C}$:
$$u_1^{**}(x) = \max\{u(x);\ u \in \mathcal{C}\}, \quad \forall x \in \Omega,$$
and we can therefore replace u_1 by u_1^{**} without loss of generality.
2. **If u_2 is not convex,** the situation is slightly more complicated if f is concave (though the result remains if f is convex). Indeed, in that case, it is clear from the proof that any solution u_0 of problem (3) must be 'minimal' in the following sense: for all $u \in \mathcal{C} \setminus \{u_0\}$, there exists $x \in \Omega$ such that $u(x) > u_0(x)$. On the other hand, many 'minimal' points can exist in $\mathcal{C}$, and there is no obvious way to decide which one minimizes F; it is also possible that more than one do. This situation is very similar to the usual problem of minimizing a concave function on a convex set.

3. **It is not necessary for f to be *strictly* convex or concave,** except that if it is not, the minimizer is usually not unique (but u_1 or u_2 are minimizers as stated in the theorem; this can be easily proved by using an approximation argument on f, for instance). The extreme case is obtained when f is affine: here F is constant on the set $\mathcal{C}$, since $u = u_1$ on $\partial\Omega$ implies $\int \nabla u = \int \nabla u_1$.

   If f is neither convex nor concave, as happens for the Newton functional for instance, the situation is much more complicated. We refer the reader to [2] for the statement of the problem, and to [1] for an interesting non-symmetry result. A first contribution towards a complete characterization of the solution is to be given in [4].
4. **One can also consider f depending also on u,** for instance by minimizing $\int f(u, \nabla u)$ on $\mathcal{C}$. A simple generalization of the main theorem is as follows:

   Lemma 2. *Assume hypothesis (H_2) and consider the problem*
   $$\inf_{u\in\mathcal{C}} \int_\Omega f(u, \nabla u).$$
   Then,
   (a) *if $f(u,p)$ is nondecreasing with respect to u and convex with respect to p, then $u = u_1$ is a minimizer;*
   (b) *if $f(u,p)$ is nonincreasing with respect to u and concave with respect to p, then $u = u_2$ is a minimizer;*
   In both cases, the minimizer is unique if $f(u,p)$ is strictly monotone with respect to u or strictly convex or concave with respect to p.

   This follows directly with the same proof, since Lemma 1 is still true in these cases.
5. **Ω may be unbounded;** in that case, condition (H_2) should be changed to either
   $$\lim_{x\in\Omega, x\to\infty} |u_1(x) - u_2(x)| = 0$$
   which is the natural generalization of (H_2), or
   $$\lim_{x\in\Omega, x\to\infty} |\nabla u_2(x)| = +\infty.$$
   If this last condition is satisfied, then for any affine function θ and any $u \in \mathcal{C}$ the set $\{u < \theta\}$ is bounded so the proof is similar.
6. **The hypothesis (H_2) is crucial,** since for instance if we just assume u_1 and u_2 to be affine functions, one may have either $u_1 > u_2$ or $u_1 < u_2$ in Ω just by adding a constant to one of them;

but adding a constant to u does not change $F(u)$, hence there is no way to compare $F(u_1)$ and $F(u_2)$ in this example. (Many thanks to A. Damlamian for this counterexample.)

Hence, if $u_1 \neq u_2$ on some part of $\partial\Omega$, a minimizer u different from u_1 and u_2 could exist in $\mathcal{C}$. It still possible to give some information if f is strictly convex. Since the inequality of u_1 and u_2 on the boundary implies that ∇u is not bounded on $\mathcal{C}$, we need to impose restrictions on f in order to ensure that $F(u)$ is finite for all $u \in \mathcal{C}$. One can suppose that f is bounded from below; but it appears that the condition

$$\exists c_1, c_2 > 0,\ \forall \xi \in \mathbf{R}^N, \qquad f(\xi) > -c_1 - c_2\,|\xi| \tag{4}$$

is sufficient even to ensure $\inf F > -\infty$, since for any convex function u,

$$\int_\Omega |\nabla u| \leq 2N \mathrm{diam}\,(\Omega)^{N-1} \operatorname*{osc}_{\overline{\Omega}} u$$

where we adopt the notation $\operatorname{osc} u = \sup u - \inf u$. This follows from considering the integral of $|\partial u/\partial x_1|$ in the direction of x_1. Since u is convex the oscillation of $u \in \mathcal{C}$ is bounded by $\sup u_1 - \inf u_2$.

Lemma 3. *Assume that f is strictly convex, but hypothesis (H_2) is not satisfied. Then for any minimizer u of F on the set $\mathcal{C}$ and any given point $x \in \Omega$, either $u(x) = u_1^{**}(x)$ or for any affine function θ satisfying $\theta(x) = u(x)$ and $\theta \leq u$ we have*

$$\exists y \in \partial\Omega, \qquad u_1(y) > \theta(y) > u_2(y). \tag{5}$$

This lemma generalizes the main theorem (for f convex) and can be proved the same way: just observe that if condition (5) is not satisfied, then there exists an $\epsilon > 0$ such that the affine function $\theta + \epsilon$ satisfies the conditions of Lemma 1 with respect to u, hence u is not minimal.

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