ON THE EXISTENCE AND UNIQUENESS OF A SOLUTION FOR AN ELLIPTIC PROBLEM

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Abstract. Existence, uniqueness and a maximum principle for the solution of a nonlinear elliptic problem is obtained. The underlying ideea combines the fix point theorem and a maximum priciple regularization.

1. Introduction

In this work we show existence and uniqueness of a solution for a quasilinear elliptic problem. Under the assumptions below, a maximum principle is also obtained. Let Ω be a bounded domain in $\mathbb{R}^d (d \geq 1)$ with a Lipschitz continuous boundary. We deal here with the following nonlinear elliptic problem

Problem P.

$$\begin{aligned} \beta(u) - \Delta u + V \cdot \nabla u = f, &\qquad \text{in } \Omega, \\ u = 0, &\qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

$\beta : \mathbb{R} \to \mathbb{R}$ being a strictly increasing function. Throughout this paper we assume, without loss of generality, that $\beta(0) = 0$. However, there are no growth conditions imposed on β in the vicinity of 0, where its derivative may become infinite.

Such problems can be investigated through the theory of monotone operators (see, e.g. [5]). Other possible approaches are the construction of sub- an supersolutions ([3]). We give here a different proof, based on the fix point theorem. First, the original problem is approximated by a sequence of problems, for which the differential operator is bounded on a closed convex subset. For the approximating problems, the Banach contraction principle guarantees the existence and uniqueness of a solution. Finally, the sequence of solutions are shown to converge to the unique solution of Problem P.

About the problem in (1.1), we make the following assumptions.

(A1) β is continuous and differentiable a.e. (almost everywhere), $\beta(0) = 0$, $\beta'(u) \geq l_\beta > 0$ and β' may be infinite in 0.

(A2) V is $L^\infty(\Omega)$ and divergence free, or it satisfies the condition

$$\|V\|_\infty < 4l_\beta.$$

(A3) $f \in L^\infty(\Omega)$ is positive a.e..

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The nonlinearity function β is strongly monotone. Its smoothness may be weakened, but the above assumption makes the presentation easier. However, we do not impose any growth condition in 0, therefore the corresponding operator may be unbounded. The assumption in (A1) implies that the derivative of β is finite almost everywhere, excluding 0. Then there are two positive constants such that

$$0 < l_\beta \leq \beta'(u) \leq C(\varepsilon) < \infty, \tag{1.2}$$

for any real $u \geq \varepsilon$,

The speed of the first order term is essentially bounded. It is either divergence free, or controlled by the inequality in (A2). We are interested here in positive solutions, so the right hand side of the problem in (1.1) is positive (a. e. in Ω). Moreover, we assume f essentialy bounded, so a constant $M > 0$ can be found s.t. $f(x) \leq M$ a.e. in Ω.

The Laplace operator in (1.3) can be replaced by $\nabla A \nabla u$, A being a symmetric, positive definite matrix. There would be no changes in what follows, excepting the essential bounds for V, if this is not divergence free. In this case the inequality in (A2) becomes $\|V\|_\infty < 4 l_\beta \lambda_{min}$, where λ_{min} stands for the minimal eigenvalue of A. Analogous, other boundary conditions may also be considered here, but only if they provide a positive solution. This restriction is fulfilled in many of the cases of practical interest.

A usual definition of the solution for (1.1) is

Definition 1.1. u is called a solution of the problem in (1.1) iff $u \in H_0^1(\Omega)$ and for all $\varphi \in H_0^1(\Omega)$ the equation holds true

$$(\beta(u), \varphi) + (\nabla u, \nabla \varphi) + (V \cdot \nabla u, \varphi) = (f, \varphi). \tag{1.3}$$

In this paper we show that the problem in (1.3) has a unique solution, which is positive and essentially bounded. This is stated in the following

Theorem. Under the assumptions (A1), (A2) and (A3), the problem in (1.1) has a unique weak solution satisfying

$$0 \leq u(x) \leq M/l_\beta$$

almost everywhere in Ω.

Here and below we use the standard notation. In particular, we let $(\cdot,\cdot)$ stand for the inner product on $L^2(\Omega)$ and $\|\cdot\|$ for the norm in $L^2(\Omega)$. Correspondingly, $\|\cdot\|_1$ denotes the norm in $H^1(\Omega)$. If necessary, $\|\cdot\|_{k,p,\Omega}$ denotes the norm in $H^{k,p}(\Omega)$.

The paper is organized as follows. In the next section we prove a maximum principle for the perturbed nonlinear problems approximating the one in (1.3). Section 3 is devoted to the proof of existence of a solution for the above mentioned problems. Finally the main result is obtained.

2. A maximum principle

In this section we consider a perturbation of the nonlinear problems in (1.1). Concretely, let ε be an artificial positive small number and consider the perturbed problem

Problem $\mathbf{P}_\varepsilon$.

$$\begin{aligned} \beta(u+\varepsilon) - \Delta u + V \cdot \nabla u = f + \beta(\varepsilon), &\qquad \text{in } \Omega, \\ u = 0, &\qquad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

In fact, Problem P_ε is similar to Problem P, but the boundary data are perturbed by ε. This idea has been applied in the analysis of some class of degenerate equations (see, e.g., [2]). The resulting Dirichlet problem is reduced to P_ε, with homogeneous boundary conditions.

As for the original problem in (1.1), a weak form reads

Definition 2.1. u is called a solution of the problem in (2.1) iff $u \in H_0^1(\Omega)$ and for all $\varphi \in H_0^1(\Omega)$ the equation holds true

$$(\beta(u+\varepsilon),\varphi) + (\nabla u, \nabla\varphi) + (V\cdot\nabla u,\varphi) = (f+\beta(\varepsilon),\varphi). \tag{2.2}$$

A solution for Problem P_ε is sought in the following set

$$W = \left\{\varphi \in H_0^1(\Omega) : 0 \le \varphi \le \frac{M}{l_\beta}, \quad a.e.\right\}. \tag{2.3}$$

In this case u is positive a.e. in Ω and the derivative of β is bounded, as given in (1.2). It is clear that, W is closed and convex. In fact, by considering the above set, a maximum principle has been stated. This is shown in the following

Theorem 2.1. *Assume (A1), (A2) and (A3). Then, if a weak solution of Problem P_ε exists, it belongs to W.*

Proof. We start with the proof of the lower bound because this is essential in our approach. This is done by reductio ad absurdum, following the ideas used in [1] for the proof of the weak maximum principle. Assume that

$$\inf_\Omega u < 0,$$

(here the infimum should be understood as the essential one).

Let φ be a positive (a.e.) function in $H_0^1(\Omega)$. Since $f \ge 0$, for any $\delta < 0$, the monotonicity of β leads to the inequality

$$0 \le (\beta(\varepsilon) - \beta(\varepsilon+\delta),\varphi) + (f,\varphi).$$

Thus, if u solves Problem P_ε, the following holds true

$$(\beta(u+\varepsilon) - \beta(\varepsilon+\delta),\varphi) + (\nabla u,\nabla\varphi) + (V\cdot\nabla u,\varphi) \ge 0$$

for all $\varphi \ge 0$ in $H_0^1(\Omega)$. Moreover, $\varphi_\delta = [\delta - u]_+$ is an $H_0^1(\Omega)$ function, which is also positive a.e. in Ω (here $[x]_+ = x$ if $x \ge 0$, otherwise $[x]_+ = 0$). Taking φ_δ in the above inequality yields

$$\int_{u<\delta}(\beta(u+\varepsilon) - \beta(\varepsilon+\delta))(\delta-u) + \int_{u<\delta}\nabla u\cdot\nabla(\delta-u) + \int_{u<\delta}(\delta-u)V\cdot\nabla u \ge 0.$$

Denoting by Ω_δ the support of $[\delta-u]_+$ and applying the Cauchy inequality, the above relation becomes

$$\int_{\Omega_\delta}(\beta(u+\varepsilon) - \beta(\varepsilon+\delta))(u-\delta) + \|\nabla u\|^2_{0,2,\Omega_\delta} \le \|V\|_\infty \|\nabla u\|_{0,2,\Omega_\delta}\|u-\delta\|_{0,2,\Omega_\delta\cap supp\{\nabla u\}},$$

where $supp\{\nabla u\}$ is the support of ∇u. Now, the monotonicity of β yields

$$\|\nabla\varphi_\delta\|^2_{0,2,\Omega_\delta} \le \|V\|_\infty\|\varphi_\delta\|_{0,2,\Omega_\delta\cap supp\{\nabla u\}}\|\nabla\varphi_\delta\|_{0,2,\Omega_\delta}.$$

Hence, the following holds true (for any $\delta < 0$)

$$\|\nabla\varphi_\delta\|_{0,2,\Omega_\delta} \le \|V\|_\infty\|\varphi_\delta\|_{0,2,\Omega_\delta\cap supp\{\nabla u\}}.$$

Next, the Sobolev embedding theorem is applied. If $d > 2$ (where d stands for the dimension of the domain Ω), because φ_δ lies in $H_0^1(\Omega)$, we get

$$\|\varphi_\delta\|_{0,\frac{2d}{d-2},\Omega} \leq C\|\nabla\varphi_\delta\|_{0,2,\Omega} \leq \|V\|_\infty \|\varphi_\delta\|_{0,2,\Omega_\delta \cap supp\{\nabla u\}}.$$

Now, the Hölder inequality for the last term yields

$$\|\varphi_\delta\|_{0,\frac{2d}{d-2},\Omega} \leq \|V\|_\infty \left(meas\{\Omega_\delta \cap supp\{\nabla u\}\}\right)^{\frac{1}{d}} \|\varphi_\delta\|_{0,\frac{2d}{d-2},\Omega},$$

where the inclusion $\Omega_\delta \subset \Omega$ was used. Therefore

$$meas\{\Omega_\delta \cap supp\{\nabla u\}\} \geq C, \tag{2.4}$$

where $C > 0$ does not depend on δ.

Analogous, if $d = 2$, the same property of the above set can be obtained. This shows that the essential infimum of u is finite. Moreover, since the constant in (2.4) does not depend on δ, the inequality must hold as δ tends to $\inf\limits_\Omega u$. That is, the function u attains its infimum in Ω on a set of positive measure, where at the same time its gradient vanishes (since the function is constant almost everywhere there). This contradicts the inequality in (2.4) and therefore the assumption on the infimum of u is false.

A similar argument shows that u has a finite essential supremum. The equality in (2.1) can be rewritten as

$$(\beta(u+\varepsilon) - \beta(\delta+\varepsilon), \varphi) + (\nabla u, \nabla\varphi) + (V \cdot \nabla u, \varphi) = (f + \beta(\varepsilon) - \beta(\delta+\varepsilon), \varphi),$$

for any real number δ. Assume now that the essential supremum of u lies above $\dfrac{M}{l_\beta}$, taking an arbitrary $\delta > M/l_\beta$ and $\varphi_\delta = [u-\delta]_+ \in H_0^1(\Omega)$, everything follows as before. In this case we have

$$\beta(\delta+\varepsilon) - \beta(\varepsilon) \geq l_\beta \delta \geq M,$$

and therefore

$$(f + \beta(\varepsilon) - \beta(\delta+\varepsilon), \varphi_\delta) \leq 0.$$

The rest of the proof relies on similar arguments as for the lower bound. □

Remark 2.1. Everywhere in the proof ε can be taken 0 since the upper bound for β' in (1.2) is never used in the proof. Therefore the above maximum principle remains valid also for the problem in (1.1), in its weak form.

3. Existence and uniqueness

In this part existence and uniqueness of a solution for Problem P_ε is obtained. This is done by a fix point argument. To do so let K be a constant which will be given below and define, for $\psi, \varphi \in H_0^1(\Omega)$ and $\chi \in W$,

$$\begin{aligned} a_K(\psi,\varphi) &= K(\psi,\varphi) + (\nabla\psi, \nabla\varphi) + (V \cdot \nabla\psi, \varphi), \\ f_{K,\varepsilon}(\chi;\varphi) &= K(\chi,\varphi) - (\beta(\chi+\varepsilon) - \beta(\varepsilon), \varphi) + (f,\varphi), \end{aligned}$$

which are linear. Clearly, a_K is also bounded. The same holds for $f_{K,\varepsilon}$ since χ is positive a.e. in Ω and therefore we can get

$$\begin{aligned} |f_{K,\varepsilon}(\chi;\varphi)| &\leq \Big(K\|\chi\| + \|(\beta(\chi+\varepsilon)\| + \beta(\varepsilon)\ meas\{\Omega\}^{\frac{1}{2}} + \|f\|\Big)\|\varphi\| \\ &\leq \Big[(K + C(\varepsilon))\|\chi\| + \beta(\varepsilon)\ meas\{\Omega\}^{\frac{1}{2}} + \|f\|\Big]\|\varphi\|, \end{aligned}$$

where $C(\varepsilon)$ is the constant in (1.2).

Recalling the notation in (2.3), an iteration can be induced through $T : W \to W$, the operator giving the solution of the following problem

Problem Aux: Let $\psi \in W$. Find $T\psi \in W$ such that

$$a_K(T\psi, \varphi) = f_{K,\varepsilon}(\psi; \varphi) \tag{3.1}$$

for all $\varphi \in H_0^1(\Omega)$.

Now the iteration can be defined as

$$\psi^{i+1} = T\psi^i \tag{3.2}$$

for $i \geq 0$ and $\psi^0 \in W$ arbitarily choosen.

This will help us in the proof of existence and uniqueness for the solution of Problem P_ε. These properties cannot be obtained as a direct consequence of a nonlinear Lax-Milgram lemma since the differential operator appearing here is bounded only on a subset of $H_0^1(\Omega)$.

If K satisfies

$$K \geq C(\varepsilon), \qquad \text{and} \qquad K \geq l_\beta \tag{3.3}$$

(the last inequality being implied by the first one if ε is small), the coercivity of a_K follows as a consequence of the Cauchy inequality

$$\begin{aligned} a_K(\psi, \psi) =& K\|\psi\|^2 + \tau\|\nabla\psi\|^2 - \|V\|_\infty \|\nabla\psi\| \|\psi\| \\ \geq& \left(K - \frac{\eta}{2}\right)\|\psi\|^2 + \left(1 - \frac{\|V\|_\infty^2}{2\eta}\right)\|\nabla\psi\|^2, \end{aligned} \tag{3.4}$$

where $\eta = \dfrac{4l_\beta + \|V\|_\infty^2}{4}$. Clearly, this choice of η satisfies the inequality

$$\|V\|_\infty^2 < 2\eta < 4l_\beta, \tag{3.5}$$

and therefore (A2) and (3.3) ensures the coercivity of a_K.

Remark 3.1. If V is divergence free, applying the Gauß divergence theorem, a simple calculation shows that, for any H_0^1 function φ we get

$$(V \cdot \nabla\varphi, \varphi) = (\nabla \cdot (V\varphi), \varphi) = 0 - (V\varphi, \nabla\varphi) = -(\varphi, V \cdot \nabla\varphi),$$

and therefore $(V \cdot \nabla\varphi, \varphi) = 0$. In this case the coercivity of a_K holds true without imposing any bounds for $\|V\|_\infty$, namely

$$a_K(\psi, \psi) \geq K\|\psi\|^2 + \|\nabla\psi\|^2.$$

If $\psi \in W$, the Lax-Milgram lemma can be applied to get a unique solution $T\psi \in H_0^1(\Omega)$ of Problem Aux, which is linear. In this way, we have defined an operator T from W to $H_0^1(\Omega)$. In fact, a similar proof as the one given for Theorem 2.1 leads to the following result

Lemma 3.1. *Assume (A1), (A2), (A3) and $\psi^0 \in W$. Then $TW \subset W$.*

Having defined the iterative scheme, a convergence result for the corresponding sequence is necessary. Below we will show that, under the restrictions in (3.3), T is a contraction mapping on the convex and closed set W with an appropriate norm, so u_ε can be taken as

$$u_\varepsilon = \lim_{i\to\infty} \psi^i.$$

In this case, assuming the above limit makes sense, it is easy to see that we have obtained a solution in W of Problem P_ε.

The existence of $\lim_{i\to\infty}\psi^i$ in W can be immediately seen by applying the fixed point theorem to T. This statement is supported by the following lemma

Lemma 3.2. *Assume (A1), (A2), (A3) and $\psi^0 \in W$. If K satisfies the inequalities in (3.3), then there is a norm on $H_0^1(\Omega)$ equivalent to the usual one, such that T is contractive on the closed set W.*

Proof. We consider here only the case V is not divergence free. Lemma 3.1 shows that W is preserved by T. Note that

$$\|\varphi\|_K^2 \equiv \left(K - \frac{\eta}{2}\right)\|\varphi\|^2 + \left(1 - \frac{\|V\|_\infty^2}{2\eta}\right)\|\nabla\varphi\|^2$$

is a norm on $H_0^1(\Omega)$, where η was choosen in (3.4). Moreover, $\|\varphi\|_K \le \sqrt{a_K(\varphi,\varphi)}$. With this new norm, T is a contraction mapping on the closed subset W, as follows from (3.1)

$$\begin{aligned}
|a_K(T\psi_1 - T\psi_2, \varphi)| = \;& |K(\psi_1 - \psi_2, \varphi) - (\beta(\psi_1+\varepsilon) - \beta(\psi_2+\varepsilon), \varphi)| \\
\le \;& |((K - \beta'(\theta))(\psi_1 - \psi_2), \varphi)| \\
\le \;& (K - l_\beta)\|\psi_1 - \psi_2\|\|\varphi\|,
\end{aligned}$$

for any $\psi_1, \psi_2 \in W$ and θ between $\psi_1 + \varepsilon$ and $\psi_2 + \varepsilon$. In the above inequalities the mean value theorem, the inequalities in (3.3), (1.2), and the positiveness of ψ_1, ψ_2 have been used. Hence, since

$$\|\varphi\|_K \ge \sqrt{K - \frac{\|V\|_\infty^2}{2\eta}}\|\varphi\|,$$

recalling the inequalities in (3.5), by taking $\varphi = T\psi_1 - T\psi_2$ the proof is completed. $\square$

Remark 3.2. If V is divergence free, the norm $\|\cdot\|_K$ can be defined as

$$\|\varphi\|_K^2 \equiv K\|\varphi\|^2 + \|\nabla\varphi\|^2.$$

Then, in the above proof we get

$$\|T\psi_1 - T\psi_2\|_K^2 \le (K - l_\beta)\|\psi_1 - \psi_2\|\|T\psi_1 - T\psi_2\| \le \frac{K - l_\beta}{K}\|\psi_1 - \psi_2\|_K\|T\psi_1 - T\psi_2\|_K.$$

Remark 3.3. The above iteration relies on the operator T, which has been used in [3], pp. 96. But we show that T is a contraction mapping, at least for the present problems, in the setting defined in the proof of Lemma 3.2. This approach was considerd in [4]. The monotonicity of T can lead to an alternative proof similar to the one in [3].

The above results show that Problem P_ε has a unique solution u_ε in W. But in the frame set in Lemma 3.2, the uniqueness holds in the whole $H_0^1(\Omega)$.

Lemma 3.3. *Assume (A1), (A2), (A3) and $\psi^0 \in W$. If K satisfies the inequalities in (3.3), Problem P_ε has at most one solution $u_\varepsilon \in H_0^1(\Omega)$.*

Proof. Assume first V is not of divergence 0. Let $\psi, \theta \in H_0^1(\Omega)$, then $\psi - \theta \in H_0^1(\Omega)$. Assuming that both solve Problem P_ε with the same ψ^0, testing with $\varphi = \psi - \theta$ and subtracting the corresponding equalities for both solutions yields

$$\begin{aligned} 0 = \ & (\beta(\psi+\varepsilon) - \beta(\theta+\varepsilon), \psi-\theta) + \|\nabla(\psi-\theta)\|^2 + (V\cdot\nabla(\psi-\theta), \psi-\theta) \\ \geq \ & l_\beta\|\psi-\theta\|^2 + \|\nabla(\psi-\theta)\|^2 - \|V\|_\infty\|\nabla(\psi-\theta)\|\|\psi-\theta\| \\ \geq \ & \left(1 - \tfrac{\|V\|_\infty^2}{4l_\beta}\right)\|\nabla(\psi-\theta)\|^2, \end{aligned}$$

where (1.2) has been used. Assumption (A2) on V guarantees that the multiplication constant above is strictly positive, therefore $\|\nabla(\psi-\theta)\| = 0$, so ψ and θ coincide a.e. in Ω.

The case $\nabla \cdot V = 0$ can be handled as before. In this situation the first order term $(V \cdot \nabla(\psi - \theta), \psi - \theta)$ vanishes as shown in Remark 3.1. □

Remark 3.4. A similar approach shows that the original Problem P admits at most one solution in $H_0^1(\Omega)$.

Up to now we have considered Problem P_ε, which, as seen before, admits a unique solution u_ε satisfying a maximum principle. Moreover, $\|u_\varepsilon\|_1$ is uniformly bounded as ε goes to 0, as resulting from

$$\begin{aligned} (\beta(u_\varepsilon+\varepsilon), u_\varepsilon) + \|\nabla u_\varepsilon\|^2 = \ & -(V\cdot\nabla u_\varepsilon, u_\varepsilon) + (f + \beta(\varepsilon), u_\varepsilon) \\ \leq \ & \|V\|_\infty\|\nabla u_\varepsilon\|\|u_\varepsilon\| + (M + \beta(\varepsilon))\ meas\{\Omega\}^{\frac{1}{2}}\|u_\varepsilon\| \\ \leq \ & \frac{1}{2}\|\nabla u_\varepsilon\|^2 + C, \end{aligned}$$

where the constant C depends on M, l_β, Ω, but can be choosen the same for all ε less than a given one. The uniform boundedness results immediately since the first term of the left-hand side is positive. Therefore the sequence $\{u_\varepsilon, \varepsilon > 0\}$ is compact in H_0^1 and admits a weakly convergent subsequence, denoted - for simplicity - $\{u_\varepsilon\}$. If we denote the weak limit by u, the subsequence converges strongly in L^2 and a.e. in Ω to u. Thus, by the continuity of β, we get

$$\begin{aligned} (\beta(u_\varepsilon+\varepsilon), \varphi) &\longrightarrow (\beta(u), \varphi) \\ (\nabla u_\varepsilon, \nabla\varphi) &\longrightarrow (\nabla u, \nabla\varphi) \\ (\nabla u_\varepsilon, V\varphi) &\longrightarrow (\nabla u, V\varphi) \\ (f + \beta(\varepsilon), \varphi) &\longrightarrow (f, \varphi), \end{aligned} \tag{3.6}$$

for all $\varphi \in H_0^1(\Omega)$. Therefore the limit u is a solution of Problem P in the sense of Definition 1.1. This, together with the maximum principle and the uniqueness result mentioned in Remarks 2.1 and 3.4 demonstrates the main result stated in the introduction

Theorem 3.4. *Under the assumptions (A1), (A2) and (A3), Problem P has a unique weak solution satisfying*

$$0 \leq u(x) \leq M/l_\beta$$

almost everywhere in Ω.

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