A Stabilized Approach for the Chebyshev-Tau Method

Iuliu Sorin Pop

Abstract: We consider a different approach for the Chebyshev-tau spectral method by a modification of the basis for the test function space. This leads to sparse matrices, which are better conditioned than those generated by the usual method, as being pointed out by some numerical examples.

1. Introduction

Spectral methods have been studied intensively in the last two decades because of their good approximation properties. This advantage was shadowed by some difficulties generated by this discretization. Thus, the matrices which arize in the spectral discretization of differential equations are generally full and their condition number increases strongly with the number of shape-functions. Therefore, it is quite difficult to get efficient iterative solvers, mainly for the Galerkin or the tau variant of these methods. Morover, especially for fourth order problems, stability and numerical accuracy of the computation can be strongly affected when a discretization using a large number of shape-functions is applied, and the theoretical accuracy of these methods can be lost. There are several works concerned with the problems mentioned above in any of the three existing types of spectral methods (see, for example [4], [8] for the tau method, [7], [9], [10], [11], [12] for the collocation variant or [15], [16] for the Galerkin approach – all of these cited only in conjunction with Chebyshev polynomials).

The type of the spectral method is dictated by the application. For example, collocation methods are suited to nonlinear problems or complicated coefficients, while Galerkin ones have the advantage of a more convenient analysis and optimal error estimates. The tau method can be appropriate in the case of complicated (nonlinear) boundary conditions, where a Galerkin approach would be impossible and the collocation extremely tedious.

Our work is focused on the tau spectral method using Chebyshev polynomials. We try to present a slightly different approach by a modification of the test-function basis. This leads to better conditioned matrices which, in case of linear equations having constant coefficients, are also sparse (banded). These features are exemplified on some model problems, where the applicability of the Bi-CGSTAB [17] algorithm is studied. The paper is organized as follows: In section 2, some basic properties of Chebyshev polynomials are provided. The following section deals with the convergence of the Chebyshev-tau method. Next, the new approach for the tau method is presented, together with some details regarding the discretization matrices. Finally, we give some numerical examples.

2. Chebyshev polynomials

In the following we will denote by $\mathcal{L}^2_\omega(-1,1)$, $\mathcal{H}^k_\omega(-1,1)$, $\mathcal{H}^k_{\omega,0}(-1,1)$, $(\cdot,\cdot)_{k,\omega}$, $\|\cdot\|_{k,\omega}$, $|\cdot|_{k,\omega}$ the corresponding weighted Sobolev spaces, scalar products, norms and seminorms on $(-1,1)$, where $\omega(x) = \frac{1}{\sqrt{(1-x^2)}}$ is the Chebyshev weight. Let $\mathbb{P}_N$ be the space of (real) polynomials of maximal order N and

$$T_k(x) = \cos(k \arccos(x)),\ k \in \mathbb{N}$$

be the k^{th} order Chebyshev polynomial of the first kind. The following properties can be found for example in [5] or [6]

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x),\ k > 0, \tag{2.1}$$

$$T_k(x)T_p(x) = \frac{1}{2}(T_{k+p} + T_{|k-p|}), \tag{2.2}$$

$$(2.3)\qquad T_k(\pm 1) = (\pm 1)^k,\ T_k'(\pm 1) = (\pm 1)^k k^2,$$

$$(2.4)\qquad (T_n, T_m)_{0,\omega} = \frac{\pi}{2} c_n \delta_{n,m},$$

where $\delta_{n,m}$ represents the Kronecker symbol and $c_i := \begin{cases} 2, & \text{if } i = 0 \\ 1, & \text{if } i > 0 \end{cases}$. The properties above are the starting point for the development of Chebyshev spectral methods for differential eqations. The idea is to approximate the unknown function by a Chebyshev series and to make use of the differentiation rule which can be deduced from (2.1). Thus, if

$$(2.5)\qquad u(x) = \sum_{k=0}^{\infty} a_k T_k(x),$$

its derivative can be expressed in the form ([5])

$$(2.6)\qquad u'(x) = \sum_{k=0}^{\infty} a_k^{(1)} T_k(x),$$

where

$$(2.7)\qquad a_k^{(1)} = \frac{2}{c_k} \sum_{p=k+1,\ p+k-odd}^{\infty} p a_p.$$

Similar relations can be deduced for other operators. After a spectral discretization is done, one has to project the initial spaces onto finite dimensional ones. Therefore, one would consider a truncated series up to an order N in (2.5) and (2.6) in the Galekin or tau case, or an interpolant of the same order for the collocative methods. Then the result will be projected on a finite dimensional space in order to get a finite system, where the in the collocative approach Dirac distributions can be considerred as the "test functions".

3. The Chebyshev-tau method

In this section the tau varaint is described generally and the convergence of the Chebyshev-tau method for a 4^{th} order Dirichlet boundary value model-problem is given. There are several methods for proving the convergence of the Chebyshev-tau method (see, for example [4] or [14]) but we will respect the approach in [5], chapter 10.

Let us consider the following examples

$$(3.1)\qquad \begin{cases} L_1 u \equiv -u''(x) + \lambda^2 u(x) = f_1(x),\ x \in [-1, 1], \\ u(\pm 1) = 0, \end{cases}$$

respectively

$$(3.2)\qquad \begin{cases} L_2 u \equiv u^{(IV)}(x) + \lambda^2 u(x) = f_2(x),\ x \in [-1, 1], \\ u(\pm 1) = u'(\pm 1) = 0. \end{cases}$$

Both problems can be treated in a similar manner, therefore we will consider a unified approach. Let us denote them by $(\mathcal{P}_k)$, where $k = 1$ in the first case and $k = 2$ in the second one. In the following the index k appearing in any of the relation below should be replaced with the corresponding value for the problems above. In both case the existence and uniqueness of a (variational) solution can be obtained for any f in $\mathcal{H}^{-k}(-1, 1)$ (the dual space of $\mathcal{H}^k_{\omega,0}(-1, 1)$). For working in weighted Sobolev spaces the following bilinear forms should be considerred

$$(3.3)\qquad a_k : \mathcal{H}^k_\omega(-1, 1) \times \mathcal{H}^k_{\omega,0}(-1, 1) \longrightarrow \mathbb{R},\ \ a_k(u, v) = \int_{-1}^{1} u^{(k)} (v\omega)^{(k)}\, dx + \int_{-1}^{1} \lambda^2 u v \omega\, dx,\ \ k = 1, 2.$$

Then, for any f in $\mathcal{H}^{-k}_\omega(-1, 1)$ the problem $(\mathcal{P}_k)$ is equivalent to the following variational one

$$(3.4)\qquad \begin{cases} u_k \in \mathcal{H}^k_{\omega,0}(-1, 1), \\ \forall v \in \mathcal{H}^k_{\omega,0}(-1, 1),\ a_k(u_k, v) = (f, v)_{0,\omega}. \end{cases}$$

Continuity and ellipticity for a_k are proven, e.g., in [5], chapter 11 ($k=1$) and [3], lemma III.1 ($k=2$). Therefore, existence and uniqueness for the solution u_k is assured in both cases and one gets

$$\|u_k\|_{k,\omega} \leq \|f\|_{-k,\omega}. \tag{3.5}$$

The corresponding discrete problems obtained by a Chebyshev-tau discretization, assuming $f \in \mathcal{L}^2_\omega(-1,1)$, are

$$(\mathcal{P}_{k,N}) \quad \begin{cases} u_{k,N} \in \mathbb{P}_N \cap \mathcal{H}^k_{\omega,0}(-1,1), \\ \forall v \in \mathbb{P}_{N-2k},\ a_k(u_k, v) = (f, v)_{0,\omega}, \end{cases} \tag{3.5.a}$$

or their equivalent strong form

$$(\mathcal{P}_{k,N}) \quad \begin{cases} u_{k,N} \in \mathbb{P}_N, \\ (L_k u_k, v)_{0,\omega} = (f, v)_{0,\omega},\ \forall v \in \mathbb{P}_{N-2k}, \\ u^{(l)}_{k,N}(\pm 1) = 0,\ l = \overline{0, k-1}. \end{cases} \tag{3.5.b}$$

Theorem 3.1:*For any $f \in \mathcal{L}^2_\omega(-1,1)$, the problem $(\mathcal{P}_{k,N})$ has a unique solution $u_{k,N} \in \mathbb{P}_N \cap \mathcal{H}^{2k}_{\omega,0}(-1,1)$ which converges to the solution u_k of the problem $(\mathcal{P}_k)$ as N tends to infinity. Moreover, if $u_k \in \mathcal{H}^\alpha_\omega(-1,1)$ with $\alpha \geq 2k$, then the error is bounded by*

$$\|u_k - u_{k,N}\|_{2k,\omega} \leq C N^{2k-\alpha} \|u\|_{\alpha,\omega}. \tag{3.6}$$

Proof: This result is obtained for $(\mathcal{P}_1)$ in [5], chapter 10. A similar approach can be considerred for $(\mathcal{P}_2)$. The continuous "inf-sup" condition ([1], theorem 5.2.1) is fulfilled for L_2 because of the ellipticity and boundedness of a_2. In the discrete case, for any $u_N \in \mathbb{P}_N$, $u_N^{(IV)} \in \mathbb{P}_{N-4}$ can be taken as test function. Notice that, if $u_N \in \mathbb{P}_N \cap \mathcal{H}^2_{\omega,0}(-1,1)$ and $u_N^{(IV)} \equiv 0$, then $u_N \equiv 0$, hence in this case the condition is trivially satisfied. Therefore we obtain

$$\sup_{v\neq 0, v\in \mathbb{P}_{N-4}} \frac{(L_2 u_N, v)_{0,\omega}}{\|v\|_{0,\omega}} \geq \frac{(L_2 u_N, u_N^{(IV)})_{0,\omega}}{\|u_N^{(IV)}\|_{0,\omega}} = \frac{|u_N|^2_{4,\omega} + \lambda^2 \int_{-1}^{1} (u_N \omega)'' u_N''}{|u_N|_{4,\omega}} \geq \frac{|u_N|^2_{4,\omega} + C\lambda^2 |u_N|_{2,\omega}}{|u_N|_{4,\omega}} \geq |u_N|_{4,\omega}.$$

Now, applying Poincaré's inequality (Appendix of [5]) successively, we get

$$|u_N|_{4,\omega} \geq C \|u_N\|_{4,\omega}$$

and by theorem 6.2.1 in [1] the first part of the proof is shown. In order to get the error estimate we have to remind that the discrete stability condition yields

$$\|u_2 - u_{2,N}\|_{4,\omega} \leq C \inf_{p \in \mathbb{P}_N \cap \mathcal{H}^2_{\omega,0}(-1,1)} \|u - p\|_{4,\omega}. \tag{3.7}$$

Theorem 4.1 in [13] provides the existence of an operator $\Pi^{2,0}_{4,N} : \mathcal{H}^\alpha_\omega(-1,1) \cap \mathcal{H}^2_{\omega,0}(-1,1) \to \mathbb{P}_N \cap \mathcal{H}^2_{\omega,0}(-1,1)$ such that for any $u \in \mathcal{H}^\alpha_\omega(-1,1) \cap \mathcal{H}^2_{\omega,0}(-1,1)$ the following holds

$$\forall \beta,\ 0 \leq \beta \leq 4,\ \|u - \Pi^{2,0}_{4,N} u\|_{\beta,\omega} \leq C N^{\beta-\alpha} \|u\|_{\alpha,\omega}. \tag{3.8}$$

The last two relations with $\beta = 4$ in (3.8) leads to the desired estimate.

4. A different approach

As pointed out in the introduction, we will suggest a different approach for the tau method. The basic idea is to preserve the spaces in which this method was formulated originally, while considerring a different basis for the projection space (test functions). We will restrict ourselves to ordinary differential equations, but these ideas can be applied also in the multidimensional case. Let us define the funtions ($k \in \mathbb{N}$)

$$\Phi^{(0)}_k(x) := \frac{2}{\pi} T_k(x),\ \ \Phi^{(i)}_{k+i}(x) := \frac{\Phi^{(i-1)}_{k+i-1}(x) - \Phi^{(i-1)}_{k+i+1}(x)}{2(k+i)},\ i > 0 \tag{4.1}$$

and

$$\tilde{\Phi}_k^{(0)}(x) := d_k \Phi_k^{(0)}(x), \quad \tilde{\Phi}_{k+i}^{(i)}(x) := \frac{\tilde{\Phi}_{k+i-1}^{(i-1)}(x) - \tilde{\Phi}_{k+i+1}^{(i-1)}(x)}{2(k+i)}, \; i > 0, \tag{4.2}$$

where c_k wass given in (2.4), $d_k := d_k(N, \beta) := \begin{cases} 1, & \text{if } k = \overline{0, N-\beta} \\ 0, & \text{otherwise} \end{cases}$ and β stands for the order of the differential operator. The following lemma justifies the choice of $\left\{ \tilde{\Phi}_{k+i}^{(i)}, k = \overline{0, N-\beta} \right\}$ as test function basis for the Chebyshev-tau method.

L e m m a 4.1: *For any $i \in \mathbb{N}$ the following relations hold*
a) $\tilde{\Phi}_{k+i}^{(i)} \equiv 0 \;\; \forall k > N - \beta$;
b) span $\left\{ \tilde{\Phi}_{k+i}^{(i)}, k = \overline{0, N-\beta} \right\} = \mathbb{P}_{N-\beta}$.

Proof: The case $i = 0$ is obvious. Then, both *a)* and *b)* can be proven by mathematical induction after i.

R e m a r k 4.1: If the Chebyshev-tau discretization matrices have already been constructed, then testing with the functions described above is similar to an algebraic transformation of the resulting system. This transformation can be described in an iterative way and it refers only to the part of the system corresponding to the differential operator. Let us assume that we have written first the equations for the boundary conditions, and then those resulting form the test with T_k, $k = \overline{0, N-\beta}$. At step j, $1 \le j \le i$, one has to subtract the equation number $k + \beta + 2$ from $k + \beta$ and divide the result with $2(k + j)$. When $k + 2 \ge N - \beta$ only the division should be performed. The resulting system will be identical to the one which arizes when $\left\{ \tilde{\Phi}_{k+i}^{(i)}, k = \overline{0, N-\beta} \right\}$ are considered as test functions. However, this does not take advantage of the "sparsity potential" of this approach and the reasons will be seen below.

L e m m a 4.2: *Let $u(x) = \sum_{k=0}^{\infty} a_k T_k(x)$. Then the following relations are satisfied ($\forall k \in \mathbb{N}$)*

$$(u, \Phi_k^{(0)})_{0,\omega} = c_k a_k, \; (f, \Phi_{k+1}^{(1)})_{0,\omega} = \frac{1}{(k+1)}[c_k a_k - a_{k+2}], \tag{4.3a}$$

$$(u, \Phi_{k+2}^{(2)})_{0,\omega} = \frac{c_k a_k}{4(k+1)(k+2)} - \frac{a_{k+2}}{2(k+1)(k+3)} + \frac{a_{k+4}}{4(k+2)(k+3)}, \tag{4.3b}$$

$$(u, \Phi_{k+3}^{(3)})_{0,\omega} = \frac{c_k a_k}{8(k+1)(k+2(k+3)} - \frac{3a_{k+2}}{8(k+1)(k+3)(k+4)} + \frac{3a_{k+4}}{8(k+2)(k+3)(k+5)} - \frac{a_{k+6}}{8(k+3)(k+4)(k+5)}, \tag{4.3c}$$

$$(u, \Phi_{k+4}^{(4)})_{0,\omega} = \frac{c_k a_k}{16(k+1)(k+2(k+3)(k+4)} - \frac{4a_{k+2}}{16(k+1)(k+3)(k+4)(k+5)} + \frac{6a_{k+4}}{16(k+2)(k+3)(k+5)(k+6)} - \frac{4a_{k+6}}{16(k+3)(k+4)(k+5)(k+7)} + \frac{a_{k+8}}{16(k+4)(k+5)(k+6)(k+7)}. \tag{4.3d}$$

Proof: The above relations can be obtained by a direct computation using the definition of $\Phi_{k+i}^{(i)}$, $i = \overline{0, 4}$.

R e m a r k 4.2: The relations in (4.3a-d) also hold if $\left\{ \tilde{\Phi}_{k+i}^{(i)}, k = \overline{0, N-\beta} \right\}$ are considered as test functions. In this case one has to replace a_{k+j} by $d_{k+j} a_{k+j}$, but only when $j > 0$.

L e m m a 4.2: *Let $u(x) = \sum_{k=0}^{\infty} a_k T_k(x)$. Then, for any $i > 0$, $k \in \mathbb{N}$*

$$(u', \Phi_{k+i}^{(i)})_{0,\omega} = (u, \Phi_{k+i}^{(i-1)})_{0,\omega}. \tag{4.4}$$

Proof: We use again the mathematical induction after i. The case $i=0$ can be obtained directly from the relations in (2.5) – (2.7) and (4.1). Assuming that (4.4) holds for $j=\overline{0,i-1}$, (4.1) gives us

$$(u', \Phi^{(i)}_{k+i})_{0,\omega} = \frac{1}{2(k+i)}\left[(u', \Phi^{(i-1)}_{k+i-1})_{0,\omega} - (u', \Phi^{(i-1)}_{k+i+1})_{0,\omega}\right]. \tag{4.5}$$

Now, because of the assumption we have made, we get

$$(u', \Phi^{(i-1)}_{k+i-1})_{0,\omega} = (u, \Phi^{(i-2)}_{k+i-1})_{0,\omega},\ (u', \Phi^{(i-1)}_{k+i+1})_{0,\omega} = (u, \Phi^{(i-2)}_{k+i+1})_{0,\omega}, \tag{4.6}$$

and by (4.1)

$$\Phi^{(i-1)}_{k+i-1} = \frac{\Phi^{(i-2)}_{k+i-1} - \Phi^{(i-2)}_{k+i+1}}{2(k+i)}. \tag{4.7}$$

Putting together the relations in (4.5) – (4.7) we get the desired result.

R e m a r k 4.3: Lemma 4.2 remains true also in the case defined in (4.2).

R e m a r k 4.4: Lemmas 4.1 and 4.2 give an iterative way to build the differentiation matrices for the desired differential operator. Lemma 4.2 also suggests that, in the case of a differential operator of order β, a reasonable choice for the test functions would be $\left\{\tilde{\Phi}^{(\beta)}_{k+i}, k=\overline{0,N-\beta}\right\}$. This reduces all the differentiation matrices to banded ones. The numerical examples are performed in this manner.

R e m a r k 4.5: Similar differentiation matrices are obtained in the integral formulation of the Chebyshev-tau method ([4], [8]). The difference could appear in the case of nonconstant coefficient problems or nonlinear ones. But the approach proposed above has also a stabilization effect, in the sense that the discretization matrices have elements with a reduced order of magnitude. More, in comparizon to the classsical approach, the complexity of the computations involved in this discretization is decreased. The following lemma, which can be useful in the case of nonlinear problems (e.g. Burger's or the Navier-Stokes equations), sustaines the former statement.

L e m m a 4.2: *Let $u(x)=\sum_{k=0}^{\infty} a_k T_k(x)$ and $v(x)=\sum_{k=0}^{\infty} b_k T_k(x)$. Then, for any $k\in\mathbb{N}$ we have*

$$(v'u, \Phi^{(1)}_{k+1})_{0,\omega} = \frac{1}{4(k+1)}\left\{A_k a_0 + \sum_{j=1}^{\infty}\left[2(k+1+j)b_{|k+1-j|} + 2(k+1+j)b_{k+j+1}\right]a_j\right\}, \tag{4.8a}$$

$$(u'u, \Phi^{(1)}_{k+1})_{0,\omega} = \frac{1}{4(k+1)}\begin{cases}\sum_{p=1,\, p-odd}^{\infty} p a_0 a_p + 2\sum_{j=0}^{\infty} a_j a_{j+1}, & \text{if } k=0,\\ a_0 a_1 + \sum_{j=1}^{k} 2(k+1-j)a_j a_{k+1-j} + 2(k+1)\sum_{j=1}^{\infty} a_j a_{k+1+j}, & \text{otherwise,}\end{cases} \tag{4.8b}$$

where $A_k := \begin{cases} 3b_1 + \sum_{p=3,\, p+k-odd}^{\infty} p b_p, & \text{if } k=0,\\ b_1, & \text{otherwise.}\end{cases}$

Proof: Both relations can be obtained from (2.2), (2.6) and (4.1), but the calculus is quite tedious and therefore it is skipped.

R e m a r k 4.6: Similar features are obtained for other bases defined in (4.1) or (4.2).

It is worth to complete the approach with a treatement of the algebraic equations corresponding to the boundary conditions. By the method described above, the elements of the part of the discretization matrix corresponding to the differential operator are scaled to $O(1)$ (in fact the order of magnitude for the coefficients of the equation); this results from the discretization of the highest order derivative. Therefore, it is natural to modify the equations arizing from the boundary conditions similarily. The simplest way is to divide any of the equations mentioned above by a number of order $O(N^{2k})$, where k is the highest order of the derivatives appearing in the corresponding boundary condition. Although it seems trivial, using this trick a sensible improvement of the condition number of the discretization matrix can be obtained.

5. Numerical examples

In this part some results obtained with the Chebyshev-tau method in both approaches are given. All the computation are performed in double precision on an IBM-RS/6000 computer, and NAG routines are used to compute the necessary eigenvalues. At first, we compare the condition numbers of the resulting Chebyshev-tau discretization matrices for the problems in (3.1) and (3.2) in the classical respectively the modified approach. This number plays a determinant role in the convergence behaviour of iterative methods and represents an important source of roundoff errors. The results are presented in tables 5.1a and b for the problem in (3.1) and in tables 5.2a and b for the one in (3.2). For the first problem, the condition number is of order $O(N^4)$ in the classical tau method, while in the modified variant decreases to $O(N)$. Similar features are obtained for the second problem. In this case, the matrices generated by the classical method have a condition number proportional to $O(N^8)$, while in the modified approach the same characteristic has been reduced to $O(N^3)$.

The increased stability of the modified approach is shown through the results obtained for the examples mentioned before. Here $\mathbf{f}$ was taken such that $u(x) = \sin^2 \pi x$ is the exact solution. The discrete problems are solved using the (unpreconditioned) Bi-CGSTAB algorithm [17]. The stopping criterion is set to $\|\mathbf{r}^{(\mathbf{k})}\|_2/\|\mathbf{b}\|_2 \leq \epsilon$, where $\epsilon = 10^{-8}$ for the problem in (3.1) and $\epsilon = 10^{-6}$ for the one in (3.2) ($\mathbf{r}^{(\mathbf{k})}$ stands for the residual of the k-th iterate of the linear problem $\mathbf{Au} = \mathbf{b}$; $\mathbf{r}^{(\mathbf{k})} = \mathbf{b} - \mathbf{Au}^{(\mathbf{k})}$). From Tables 5.3a and b, a gain in accuracy can be observed in the modified version. We had no problems in achieving the stopping criterion in a moderate number of iterations, even in the cases when the classical tau method failed. In the tables '*' indicates divergence, or non-convergence in 250 steps. The failure of the method in the classical approach is due to roundoff errors, the algorithm being finite. In the modified approach, the number of iterations tends to remain stable with respect to the discretization order, while a significant increasal with N can be seen in the classical method. Therefore, we can affirm that the proposed variant is more robust.

Concluding remarks. We have proposed a different approach for the Chebyshev-tau method, which removes partially some of the inconvenients of the classical one. More sparsity and better conditioned matrices are provided, and therefore an improved stability and converging properties are obtained. The modification is easy to implement. However, a good preconditioner for this method is still to be found.

Acknowledgements This work was done at the Interdisciplinary Center for Scientific Computing (IWR) of the University of Heidelberg and was supported partially by the DAAD organization. This financial support, as well as, the guidance of Prof.Dr.Dr.h.c. Willi Jäger is gratefully acknowledged. The author also expresses his gratitude to Prof. Dr. C. I. Gheorghiu for his generous suggestions. Special thanks to Dr. M. Hiegemann for the carefully reading of the manuscript and for his kind remarks.

Table 5.1a Problem (3.1), classical-tau

N	λ	*cond*	$cond \cdot N^{-4}$
8	0.0	575.17	0.876E-01
	10.0	310.21	0.472E-01
32	0.0	0.115E+06	0.971E-01
	10.0	0.630E+05	0.531E-01
128	0.0	0.273E+08	0.987E-01
	10.0	0.149E+08	0.540E-01
256	0.0	0.432E+09	0.990E-01
	10.0	0.236E+09	0.542E-01
512	0.0	0.686E+10	0.991E-01
	10.0	0.376E+10	0.542E-01

Table 5.1b The same, modified-tau

N	λ	*cond*	$cond \cdot N^{-1}$
8	0.0	7.64	0.849
	10.0	36.40	4.045
32	0.0	24.70	0.748
	10.0	73.77	2.235
128	0.0	92.62	0.717
	10.0	148.32	1.149
256	0.0	183.13	0.712
	10.0	209.82	0.816
512	0.0	364.15	0.709
	10.0	296.80	0.578

Table 5.2a Problem (3.2), classical-tau

N	λ	*cond*	$cond \cdot N^{-8}$
8	0.0	0.625E+05	0.145E-02
	10.0	0.564E+05	0.131E-02
32	0.0	0.326E+10	0.232E-02
	10.0	0.295E+10	0.209E-02
128	0.0	0.186E+15	0.242E-02
	10.0	0.168E+15	0.219E-02
256	0.0	0.464E+17	0.244E-02
	10.0	0.420E+17	0.220E-02
512	0.0	0.117E+20	0.244E-02
	10.0	0.106E+20	0.221E-02

Table 5.2b The same, modified-tau

N	λ	*cond*	$cond \cdot N^{-3}$
8	0.0	105.68	0.144
	10.0	31.96	0.438E-01
32	0.0	0.465E+03	0.129
	10.0	0.423E+03	0.456E-01
128	0.0	0.261E+06	0.121
	10.0	0.970E+05	0.451E-01
256	0.0	0.204E+07	0.119
	10.0	0.764E+06	0.450E-01
512	0.0	0.161E+08	0.119
	10.0	0.607E+07	0.449E-01

Table 5.3a Number of iterations and absolut error for (3.1)

	classical-tau			modified-tau		
	λ			λ		
N	0.0	10.0	100.0	0.0	10.0	100.0
8	9 ($0.381 \cdot 10^{-1}$)	14 ($0.329 \cdot 10^{-1}$)	18 ($0.200 \cdot 10^{-1}$)	5 ($0.381 \cdot 10^{-1}$)	10 ($0.329 \cdot 10^{-1}$)	12 ($0.200 \cdot 10^{-1}$)
16	20 ($0.347 \cdot 10^{-6}$)	32 ($0.320 \cdot 10^{-6}$)	100 ($0.761 \cdot 10^{-6}$)	5 ($0.347 \cdot 10^{-6}$)	11 ($0.322 \cdot 10^{-6}$)	28 ($0.195 \cdot 10^{-6}$)
32	63 ($0.106 \cdot 10^{-6}$)	51 ($0.172 \cdot 10^{-6}$)	$*$ ($*$)	5 ($0.999 \cdot 10^{-15}$)	11 ($0.166 \cdot 10^{-7}$)	51 ($0.164 \cdot 10^{-4}$)
64	196 ($0.409 \cdot 10^{-7}$)	225 ($0.451 \cdot 10^{-6}$)	$*$ ($*$)	5 ($0.166 \cdot 10^{-14}$)	11 ($0.237 \cdot 10^{-7}$)	44 ($0.131 \cdot 10^{-4}$)
128	$*$ ($*$)	$*$ ($*$)	$*$ ($*$)	5 ($0.255 \cdot 10^{-14}$)	11 ($0.301 \cdot 10^{-7}$)	42 ($0.744 \cdot 10^{-5}$)
256	$*$ ($*$)	$*$ ($*$)	$*$ ($*$)	5 ($0.999 \cdot 10^{-15}$)	11 ($0.308 \cdot 10^{-7}$)	44 ($0.727 \cdot 10^{-5}$)
512	$*$ ($*$)	$*$ ($*$)	$*$ ($*$)	5 ($0.878 \cdot 10^{-14}$)	11 ($0.333 \cdot 10^{-7}$)	46 ($0.296 \cdot 10^{-4}$)

Table 5.3b Number of iterations and absolut error for (3.2)

	classical-tau			modified-tau		
	λ			λ		
N	0.0	10.0	100.0	0.0	10.0	100.0
8	16 (0.801)	18 (0.400)	17 (0.103)	5 (0.801)	8 (0.400)	12 (0.103)
16	62 ($0.142 \cdot 10^{-3}$)	71 ($0.494 \cdot 10^{-3}$)	58 ($0.360 \cdot 10^{-4}$)	6 ($0.183 \cdot 10^{-5}$)	9 ($0.175 \cdot 10^{-5}$)	13 ($0.484 \cdot 10^{-5}$)
32	$*$ ($*$)	$*$ ($*$)	$*$ ($*$)	6 ($0.322 \cdot 10^{-10}$)	9 ($0.394 \cdot 10^{-7}$)	17 ($0.851 \cdot 10^{-6}$)
64	$*$ ($*$)	$*$ ($*$)	$*$ ($*$)	6 ($0.189 \cdot 10^{-6}$)	11 ($0.274 \cdot 10^{-8}$)	16 ($0.332 \cdot 10^{-3}$)
128	$*$ ($*$)	$*$ ($*$)	$*$ ($*$)	6 ($0.189 \cdot 10^{-6}$)	10 ($0.668 \cdot 10^{-5}$)	12 ($0.442 \cdot 10^{-2}$)
256	$*$ ($*$)	$*$ ($*$)	$*$ ($*$)	8 ($0.384 \cdot 10^{-4}$)	11 ($0.188 \cdot 10^{-5}$)	13 ($0.952 \cdot 10^{-3}$)
512	$*$ ($*$)	$*$ ($*$)	$*$ ($*$)	10 ($0.208 \cdot 10^{-7}$)	18 ($0.360 \cdot 10^{-7}$)	13 ($0.316 \cdot 10^{-3}$)

6. References

1 Babuška, I.; Aziz, A.K.: Survey lectures on the mathematical foundations of the finite element method, in *The Mathematical Foundation of the Finite Element Method*, A. K Aziz (ed.), Academic Press, London, New York, 1972, 3–359.

2 Bernardi, C.; Maday, Y.: Properties of some weighted Sobolev spaces and application to spectral approximations SIAM J. Numer. Anal. **26** (1989), 769–829.

3 Bernardi, C.; Maday, Y.: Some spectral approximations of one-dimensional fouth-order problems, in *Progress in Approximation Theory*, P. Nevai and A. Pinkus (eds.), Academic Press, Boston, MA, 43–116.

4 Cabos, Ch.: A preconditioning of the tau operator for ordinary differential equations, ZAMM **74** (1994), 521–532.

5 Canuto, C.; Hussaini, M.Y.; Qarteroni, A.; Zang, T.A.: *Spectral Methods in Fluid Dynamics*, Springer-Verlag, Berlin, Heidelberg, New-York, 1988.

6 Fox, L.; Parker, I.B.: *Chebyshev Polynomials in Numerical Analysis*, Oxford University Press, London, 1968.

7 Funaro, D.; Heinrichs, W.: Some results about the pseudospectral approximation of one-dimensional fourth-order problems, Numer. Math. **58** (1990), 399–418.

8 Hiegemann, M.: Chebyshev matrix operator method for the solution of integrated forms of linear ordinary differential equations, Acta Mech. (to appear).

9 Heinrichs, W.: Improved condition number for spectral methods, Math. Comput. **53** (1989), 103–109.

10 Heinrichs, W.: Stabilization techniques for spectral methods, J. Sci. Comp. **6** (1991), 1–19.

11 Heinrichs, W.: A stabilized treatment of the biharmonic operator with spectral methods, SIAM J. Sci. Stat. Comput. **12** (1991), 1162–1172.

12 Huang, W.; Sloan, D.M.: The pseudospectral method for solving differential eigenvalue problems, J. Comput. Phys. **111** (1994), 399–409.

13 Maday, Y: Analysis of spectral projectors in one-dimensional domains, Math. Comput. **55** (1990), 537–562.

14 Roos, H.G.; Pfeiffer, E.: A convergence result for the tau method, Computing **42** (1989), 81–84.

15 Shen, J.: Efficient spectral-Galerkin method II. Direct solvers of second and fourth order equations by using Chebyshev polynomials, SIAM J. Sci. Stat. Comput. **16** (1995), 74–87.

16 Shen, J.: Effcient Chebyshev-Legendre Galerkin methods for elliptic problems, Houston J. Math. (to appear).

17 van der Vorst, H.A.: Bi-CCGSTAB: a fast and smoothly converging variant of Bi-CG for the solution of non-symmetric linear systems, SIAM J. Sci. Stat. Comput. **13** (1992), 631–644.

Address: Iuliu Sorin Pop, "Babeş-Bolyai" University, Faculty of Mathematics, M. Kogalniceanu 1, 3400 Cluj Napoca, Romania, e-mail: ispop@cs.ubbcluj.ro